Differential
Equations
& Applications
Volume 5, Number 4 (2013), 577–594
doi:10.7153/dea-05-34

EXISTENCE AND CONCENTRATION OF GROUND STATE SOLUTION TO A CRITICAL p–LAPLACIAN EQUATION

XUDONG SHANG AND JIHUI ZHANG

(*Communicated by Claudianor C. Alves*)

Abstract. In this paper, we consider the existence and concentration behavior of positive ground state solution to the following problem

$$\begin{cases} -h^p\Delta_p u+V(x)|u|^{p-2}u=K(x)|u|^{q-2}u+|u|^{p^*-2}u, \;\; x\in\mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N),\; u>0,\; x\in\mathbb{R}^N,\end{cases}$$

where h is a small positive parameter, $1<p<N$, $\max\{p,p^*-\frac{p}{p-1}\}<q<p^*$, $p^*=\frac{Np}{N-p}$ is the critical Sobolev exponent, $V(x)$ and $K(x)$ are positive smooth functions. Under some necessary restrictions, we show that for small $h>0$, the equation has a positive ground state solution. Furthermore, we establish the concentration property of such solutions when h tends to zero.

1. Introduction and main results

In present paper, we consider the existence and concentration behavior of positive ground state solution to the following problem

$$\begin{cases} -h^p\Delta_p u+V(x)|u|^{p-2}u=K(x)|u|^{q-2}u+|u|^{p^*-2}u, \;\; x\in\mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N),\; u>0,\; x\in\mathbb{R}^N,\end{cases} \tag{1.1}$$

where h is a small positive parameter, $\Delta_p u=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian,

$$1<p<N,\;\; \max\{p,p^*-\frac{p}{p-1}\}<q<p^* \quad\text{and}\quad p^*=\frac{Np}{N-p}$$

is the critical Sobolev exponent, $V(x)$ and $K(x)$ are positive smooth functions with $V(x)$ bounded below by a positive constant and $K(x)$ bounded.

We note that problem (1.1) with $p=2$ arise when one seeks for the standing wave solutions of the following nonlinear Schrödinger equation

$$ih\frac{\partial\varphi}{\partial t}=\frac{-h^2}{2m}\Delta\varphi+W(x)\varphi-\gamma|\varphi|^{q-2}\varphi, \qquad x\in\mathbb{R}^N.$$

Mathematics subject classification (2010): 35J92, 35J35.
Keywords and phrases: p-Laplacian, ground state, critical growth, positive solution.
This research was supported by Project of Graduate Education Innovation of Jiangsu Province (No.CXZZ13-0389) and Soft Science Fund of Jiangsu Province (BR2013039).

We can see [15] and the references therein for more background.

The nonlinear Schrödinger equations have been extensively studied in recent years, obtained numerous results on existence, multiplicity and concentration behavior of the positive solutions, see for example[2, 6, 7, 8, 9, 10, 16, 17, 20, 21]. Wang and Zeng in [26] studied the following nonlinear Schrödinger equation with competing potential functions

$$-h^2\Delta u+V(x)u=K(x)|u|^{p-2}u+Q(x)|u|^{q-2}u, \qquad x\in\mathbb{R}^N, \tag{1.2}$$

where $2<q<p<2N/N-2$, $V(x)$ has a positive lower bound, $K(x)$ is positive and bounded, $Q(x)$ is bounded and allowed to change sign. By min-max argument, they proved existence of a ground state solution and studied the concentration behavior of such solutions, obtained a necessary condition for location of concentration of positive bound state solution of (1.2).

Very recently, several papers have appeared about the p-Laplacian problems, we can see[3, 4, 5, 11, 12, 14] and there references. Alves and Figueiredo in [3] studied the following class of problem

$$-h^p\Delta_p u+V(x)|u|^{p-2}u=f(u), \qquad x\in\mathbb{R}^N. \tag{1.3}$$

$V(x)$ is a continuous function satisfying

$$V_\infty=\liminf_{|x|\to\infty}V(x)>\inf_{x\in\mathbb{R}^N}V(x)=V_0>0.$$

Under some assumptions on f, they established the existence, multiplicity of solutions to (1.3). Moreover, they proved that solutions of (1.3) which concentrate around a global minimum point of V. In [11], do Ó considered the quasilinear critical problem

$$-h^p\Delta_p u+V(x)u^{p-1}=f(u)+u^{p^*-1}, \qquad x\in\mathbb{R}^N, \tag{1.4}$$

where f is a C^1 function and satisfying some necessary conditions, $V\in C(\mathbb{R}^N,\mathbb{R})$ and there exists an open bounded subset $\Omega\subset\mathbb{R}^N$ such that

$$\inf_{\partial\Omega}V(x)>\inf_{\Omega}V(x)=V_0>0.$$

Using the penalization method, the author studied the existence of bounded state which concentrate around a local minima of V as $h\to 0$. Furthermore, Figueiredo and Furtado in [14] using Ljusternik-Schnirelmann theory obtained multiplicity result of problem (1.4). In [12], the author established the multiplicity and concentrations of positive solutions for the supercritical problem

$$-h^p\Delta_p u+V(x)|u|^{p-2}u=|u|^{q-2}u+\lambda|u|^{s-2}u, \qquad x\in\mathbb{R}^N,$$

where $1<p<N$, $p<q<Np/N-p\leqslant s$, and V is a positive continuous function.

Based on the above reviews and observations, we know that the existence of positive ground states along with the concentration behavior of solutions to the problem (1.1) has not been studied. More precisely, motivated by the argument used in [26, 8], we will consider the existence and concentration behavior of positive ground state solutions of quasilinear problem with competing functions. We will prove that solutions of (1.1) which concentrate around a global minimum points of a ground energy function $G(\xi)$, which is defined to be the ground energy associated with the equation

$$-\Delta_p u+V(\xi)|u|^{p-2}u=K(\xi)|u|^{q-2}u+|u|^{p^*-2}u, \qquad x\in\mathbb{R}^N, \tag{1.5}$$

where $\xi\in\mathbb{R}^N$ is regard as a parameter instead of an independent variable.

In this paper, we assume

(V) $V(x)$ is a continuous function in $\mathbb{R}^N$ and

$$\inf_{x\in\mathbb{R}^N} V(x)=V_0>0.$$

(K) $K(x)$ is a positive and bounded continuous function in $\mathbb{R}^N$.

Now, we consider the following equation with constant coefficients

$$-\Delta_p u+\lambda|u|^{p-2}u=\mu|u|^{q-2}u+|u|^{p^*-2}u, \qquad x\in\mathbb{R}^N. \tag{1.6}$$

The functional is defined as

$$I(u)=\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p+\lambda|u|^p)dx-\frac{1}{q}\int_{\mathbb{R}^N}\mu|u|^q dx-\frac{1}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}dx. \tag{1.7}$$

Define $c(\lambda,\mu)=\inf_{u\in\mathscr{N}}I(u)$, where $\mathscr{N}^*$ is the Nehari manifold with

$$\mathscr{N}^*=\left\{u\in W^{1,p}(\mathbb{R}^N)\backslash\{0\}:\int_{\mathbb{R}^N}(|\nabla u|^p+\lambda|u|^p)dx=\int_{\mathbb{R}^N}(\mu|u|^q+|u|^{p^*})dx.\right\} \tag{1.8}$$

By the Sobolev embedding theorem, $c(\lambda,\mu)$ is finite and positive. Furthermore, using the similar proof of Lemma 2.2 of [26], we obtain that $c(\lambda,\mu)$ satisfy a monotonicity property: If $\lambda_1\leqslant\lambda_2$, $\mu_1\geqslant\mu_2$, then $c(\lambda_1,\lambda_2)\leqslant c(\mu_1,\mu_2)$.

Next, for each $\xi\in\mathbb{R}^N$, we consider the functional associated to problem (1.5) given by

$$I^\xi(u)=\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p+V(\xi)|u|^p)dx-\frac{1}{q}\int_{\mathbb{R}^N}K(\xi)|u|^q dx-\frac{1}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}dx. \tag{1.9}$$

Define the ground energy function of by

$$G(\xi)=c(V(\xi),K(\xi))=\inf_{u\in\mathscr{M}^\xi}I^\xi(u),$$

where $\mathscr{M}^\xi$ is the Nehari manifold defined as (1.8). By the continuity of V, K and Sobolev embedding theorem , we know that G is a continuous, positive map. By [13], we know that for each $\xi\in\mathbb{R}^N$, problem (1.5) possesses a ground state solution.

Define

$$c_\infty=\inf_{u\in\mathscr{N}^\infty}I^\infty(u),$$

where

$$I^\infty(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V_\infty |u|^p)dx - \frac{1}{q}\int_{\mathbb{R}^N} K_\infty |u|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx,$$

$$\liminf_{|x|\to\infty} V(x) = V_\infty, \qquad \liminf_{|x|\to\infty} K(x) = K_\infty,$$

and always assume $V_\infty < \infty$. The main result of this paper is stated as follows:

THEOREM 1. *Assume*

$$\inf_{\xi\in\mathbb{R}^N} G(\xi) < c_\infty, \tag{1.10}$$

then problem (1.1) has a positive ground state solution u_h for small $h > 0$. Moreover, if $x_h \in \mathbb{R}^N$ is maximum point of u_h, then

$$\lim_{h\to 0} G(x_h) = G(x_0) = \inf_{\xi\in\mathbb{R}^N} G(\xi).$$

The plan of this paper is as follows. In Section 2, we present some notation and some technical results, including the estimates for the critical levels and the geometric hypotheses of the Mountain Pass Theorem. In Section 3, we show that the corresponding energy functional satisfies the Palais-Smale condition and the existence of ground state solution. Finally we establish the concentration property of the ground state solution when h tends to zero.

2. Notation and preliminaries

In this section, we will use the following notation frequently. The letters $C, C_1, C_2, \cdots$ denote positive constants, $B_R(0)$ denotes the ball centered at the origin with radius $R > 0$, $\|\cdot\|_\infty$ denotes the norm in L^∞. Let us consider the energy functional associated with problem (1.1)

$$J_h(u) = \frac{1}{p}\int_{\mathbb{R}^N}(h^p|\nabla u|^p + V(x)|u|^p)dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx, \tag{2.1}$$

which is well defined on the Banach space X_h, where

$$X_h = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} h^p|\nabla u|^p + V(x)|u|^p dx < \infty \right\}$$

endowed with the norm

$$\|u\|_X^p = \int_{\mathbb{R}^N}(h^p|\nabla u|^p + V(x)|u|^p)dx.$$

We can always assume that critical points of J_h are nonnegative functions since J_h is even. Furthermore, let us define the Nehari manifold associated to J_h by

$$\mathcal{N}_h = \{u \in X_h\backslash\{0\} : \int_{\mathbb{R}^N} h^p|\nabla u|^p + V(x)|u|^p dx = \int_{\mathbb{R}^N} |u|^{p^*} dx + \int_{\mathbb{R}^N} K(x)|u|^q dx\}.$$

Now, we start recalling that the functional J_h satisfies the mountain-pass geometry conditions and its proof is standard.

LEMMA 1. *The functional J_h satisfies the following conditions:*
(i) There exist $\beta, \rho > 0$, such that $J_h(u) \geqslant \beta$ if $\|u\|_{X_h} = \rho$.
(ii) There exists an $e \in X_h$ with $\|e\|_{X_h} > \rho$ such that $J_h(e) < 0$.

From the Lemma above, by virtue of the mountain pass theorem without the Palais-Smale condition ([25]), there exists a sequence $\{u_n\} \subset X_h$ satisfying $J_h(u_n) \to c_h$ and $J_h^{'}(u_n) \to 0$ in X_h^{-1} at the minimax level

$$c_h = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_h(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C^1([0,1], X_h) : \gamma(0) = 0, \gamma(1) = e\}$. By the same proof of Lemma 2.1 in [26], We have the following lemma

LEMMA 2. *For any $u \in X_h \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathscr{N}_h$. Moreover, $J_h(t_u u) = \max_{t \geqslant 0} J_h(tu)$.*

In order to show the existence of ground state solution, we first define the ground state energy associated with J_h by

$$c_h^* = \inf_{u \in \mathscr{N}_h} J_h(u).$$

Next, we define another minimax value

$$c_h^{**} = \inf_{u \in E_h \setminus \{0\}} \sup_{t \geqslant 0} J_h(tu).$$

As in Proposition 3.11 of [22], we shall have the following equivalent characterization of c_h.

LEMMA 3. $c_h = c_h^* = c_h^{**}$.

We denote by S the Sobolev constant, that is

$$S = \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{\frac{p}{p^*}}} : u \in W^{1,p}(\mathbb{R}^N), u \neq 0 \right\}, \tag{2.2}$$

and S is attained by the functions

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-p}{p^2}}}{\left(\varepsilon + |x|^{p/(p-1)}\right)^{\frac{N-p}{p}}},$$

for any $x \in \mathbb{R}^N$, $\varepsilon > 0$. Now, we recall the concentration-compactness principle due to Lions.

LEMMA 4. *([18]) Let $\{u_n\}$ converge weakly to u in $W^{1,p}(\mathbb{R}^N)$ such that $|u_n|^{p^*}$ and $|\nabla u_n|^p$ converge weakly to nonnegative measures ν and μ on $\mathbb{R}^N$ respectively. Then, for some at most countable set J, we have*
(i) $\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}$,
(ii) $\mu \geqslant |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j}$,
(iii) $S\nu_j^{\frac{p}{p^*}} \leqslant \mu_j$*, where* $x_j \in \mathbb{R}^N$, δ_{x_j} *is the Dirac measure at* x_j*, and* ν_j *and* μ_j *are constants.*

Making the change of variable $x \mapsto hx$, we can rewrite (1.1) as the following equivalent equation

$$-\Delta_p u + V(hx)|u|^{p-2}u = K(hx)|u|^{q-2}u + |u|^{p^*-2}u, \qquad x \in \mathbb{R}^N. \tag{2.3}$$

We know that its solution are the critical points of the functional given by

$$I_h(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V(hx)|u|^p)dx - \frac{1}{q}\int_{\mathbb{R}^N} K(hx)|u|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}dx, \tag{2.4}$$

which is well defined on the Banach space E_h, where

$$E_h = \left\{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(hx)|u|^p dx < \infty\right\}$$

endowed with the norm

$$\|u\|_h^p = \int_{\mathbb{R}^N}(|\nabla u|^p + V(hx)|u|^p)dx.$$

Next we state the result which provides an appropriate estimate on the minimax level.

LEMMA 5. *Assume the assumptions (V) and (K) hold. Then the number c_h satisfies*

$$0 < c_h < \frac{h^N}{N}S^{N/p}.$$

Proof. Given $\varepsilon > 0$, we consider the function

$$w_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^{p/p-1})^{N-p/p}} \qquad \text{and} \qquad v_\varepsilon(x) = \frac{w_\varepsilon(x)}{\|w_\varepsilon(x)\|_{p^*}},$$

where $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $0 \leqslant \phi(x) \leqslant 1$ and $\phi(x) \equiv 1$ on $B(0,1)$, $\phi(x) \equiv 0$ in $\mathbb{R}^N \setminus B(0,2)$. Then, as $\varepsilon \to 0$, we obtain the following estimates inspired by [1].

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p dx = S + O(\varepsilon^{N-p/p}). \tag{2.5}$$

$$\int_{\mathbb{R}^N} |v_\varepsilon|^p dx \leqslant \begin{cases} C\varepsilon^{p-1}, & \text{if } N > p^2, \\ C\varepsilon^{p-1}\log(1/\varepsilon), & \text{if } N = p^2, \\ C\varepsilon^{N-p/p}, & \text{if } N < p^2. \end{cases} \tag{2.6}$$

$$\int_{\mathbb{R}^N} |v_\varepsilon|^q dx = \begin{cases} O\big(\varepsilon^{(p-1)/p\cdot(N-q(N-p)/p)}\big), & \text{if } q > p^*(1-1/p), \\ O\big(\varepsilon^{(N-p)q/p^2}\log(1/\varepsilon)\big), & \text{if } q = p^*(1-1/p), \\ O\big(\varepsilon^{(N-p)q/p^2}\big), & \text{if } q < p^*(1-1/p). \end{cases} \tag{2.7}$$

Define

$$g(t) = I_h(tv_\varepsilon) = \frac{t^p}{p}\int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^p + V(hx)|v_\varepsilon|^p)dx - \frac{t^q}{q}\int_{\mathbb{R}^N} K(hx)|v_\varepsilon|^q dx - \frac{t^{p^*}}{p^*}.$$

It's easy to see that $g(t)$ attains its maximum at some $t_\varepsilon \in (0,\infty)$ with $g^{'}(t_\varepsilon) = 0$. That is

$$0 = g^{'}(t_\varepsilon) = t_\varepsilon^{p-1}\left(\int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^p + V(hx)|v_\varepsilon|^p)dx - t_\varepsilon^{q-p}\int_{\mathbb{R}^N} K(hx)|v_\varepsilon|^q dx - t_\varepsilon^{p^*-p}\right),$$

it implies that,

$$\int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^p + V(hx)|v_\varepsilon|^p)dx = t_\varepsilon^{q-p}\int_{\mathbb{R}^N} K(hx)|v_\varepsilon|^q dx + t_\varepsilon^{p^*-p}. \tag{2.8}$$

Thus, we have

$$t_\varepsilon^{p^*-p} \leqslant \int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^p + V(hx)|v_\varepsilon|^p)dx, \tag{2.9}$$

by Sobolev embedding theorem and (2.5), there exist an $A > 0$ independent of ε such that

$$t_\varepsilon < A. \tag{2.10}$$

Thus, by (2.8) and (2.9), we obtain

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p dx \leqslant t_\varepsilon^{p^*-p} + \|K\|_\infty \left[\int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^p + V(hx)|v_\varepsilon|^p)dx\right]^{\frac{q-p}{p^*-p}} \int_{\mathbb{R}^N} |v_\varepsilon|^q dx.$$

Choosing ε small, by combining (2.5), (2.7) and (2.10), it follows that

$$t_\varepsilon^{p^*-p} \geqslant \frac{S}{2}.$$

Hence,

$$\begin{aligned} g(t_\varepsilon) \leqslant{}& \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N}(|\nabla v_\varepsilon|^p + V(hx)|v_\varepsilon|^p)dx \\ &- \frac{\min_{x\in B(0,2)} K(x)}{q}\left(\frac{S}{2}\right)^{\frac{q}{p^*-p}}\int_{\mathbb{R}^N} |v_\varepsilon|^q dx - \frac{t_\varepsilon^{p^*}}{p^*} \\ &\leqslant \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p dx + C\int_{\mathbb{R}^N} |v_\varepsilon|^p dx - C\int_{\mathbb{R}^N} |v_\varepsilon|^q dx - \frac{t_\varepsilon^{p^*}}{p^*}. \end{aligned} \tag{2.11}$$

Consider the function

$$s(t)=\frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx-\frac{t^{p^*}}{p^*},$$

we know that $\bar{t}=(\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx)^{\frac{1}{p^*-p}}$ is an maximum point of $s(t)$ and

$$s(\bar{t})=\frac{1}{N}\left(\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^p dx\right)^{\frac{N}{p}}.$$

Thus, by (2.9), (2.11), $q>\max\{p,p^*-\frac{p}{p-1}\}$ and inequality

$$|a+b|^p\leqslant\begin{cases}(1+\varepsilon)^{p-1}|a|^p+(1+1/\varepsilon)^{p-1}|b|^p, & \text{for } 1\leqslant p<\infty,\\ |a|^p+|b|^p \quad \text{for } 0<p<1,\end{cases}$$

$\forall a,b\in\mathbb{R}$ and $\varepsilon>0$, we have

$$g(t_\varepsilon)\leqslant\frac{1}{N}S^{\frac{N}{p}}+\begin{cases}C_2\varepsilon^{\frac{N-p}{p}}+C_2\varepsilon^{p-1}-C_1\varepsilon^{\frac{p-1}{p}(N-\frac{q(N-p)}{p})}, & \text{if } N>p^2,\\ C_2\varepsilon^{\frac{N-p}{p}}+C_4\varepsilon^{p-1}|\log\varepsilon|-C_1\varepsilon^{\frac{p-1}{p}(N-\frac{q(N-p)}{p})}, & \text{if } N=p^2,\\ C_5\varepsilon^{\frac{N-p}{p}}-C_1\varepsilon^{\frac{p-1}{p}(N-\frac{q(N-p)}{p})}, & \text{if } N<p^2.\end{cases}$$

We conclude from the above that, for $\varepsilon>0$ small enough, $I_h(t_\varepsilon v_\varepsilon)<\frac{1}{N}S^{\frac{N}{p}}$. By the relation (2.1) and (2.4), we conclude the result. □

By the proof of Lemma 3 and 5, it is easy to see that $0<c_\infty<\frac{1}{N}S^{N/p}$ and $0<G(\xi)<\frac{1}{N}S^{N/p}$.

3. Existence of a ground state solution to (1.1)

In this section, we shall study the existence of ground state solution. To begin with, we first show some compactness results for the functional J_h. Let $\{u_n\}\subset X_h$ be a $(PS)_c$ sequence of J_h, i.e

$$J_h(u_n)\to c \text{ and } J_h'(u_n)\to 0, \tag{3.1}$$

with $c<\frac{h^N}{N}S^{N/p}$. We have

$$\begin{aligned}c+o(1)&=J_h(u_n)-\frac{1}{q}\langle J_h'(u_n),u_n\rangle\\ &=\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|_X^p+\left(\frac{1}{q}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}|u_n|^{p^*}dx\\ &\geqslant\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|_X^p.\end{aligned}$$

Hence $\{u_n\}$ is bounded X_h, then there exists a $u \in X_h$ such that

$$u_n \rightharpoonup u \text{ weakly in } X_\mu \quad \text{and} \quad u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N. \tag{3.2}$$

In order to prove that $u_n \to u$ strongly in X_h, we first show the following lemma.

LEMMA 6. *For $h > 0$ small, if $c < h^N c_\infty$, then*

$$\int_{\mathbb{R}^N} K(x)|u_n|^q dx \to \int_{\mathbb{R}^N} K(x)|u|^q dx, \qquad p < q < p^*.$$

Proof. By the boundedness of $\{u_n\}$ and Sobolev embedding theorem, we have $u_n \to u$ in $L^s_{loc}(\mathbb{R}^N)$ for $p \leqslant s < p^*$. Hence, we only consider

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{|x|\geqslant R} K(x)|u_n|^q dx = 0. \tag{3.3}$$

We claim for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\limsup_{n\to\infty}\int_{|x|\geqslant R} h^p|\nabla u_n|^p + V(x)|u_n|^p dx < \varepsilon.$$

Otherwise, for some subsequence $\{u_{n_k}\}$ and some $\delta > 0$ such that

$$\int_{|x|\geqslant R} h^p|\nabla u_{n_k}|^p + V(x)|u_{n_k}|^p dx \geqslant \delta. \tag{3.4}$$

Take $\rho > 0$ such that $c < h^N c(V_\infty - \rho, K_\infty - \rho) = c_\rho$. Let $R(\rho) > 0$ be such that

$$V(x) \geqslant V_\infty - \rho, \quad K(x) \geqslant K_\infty - \rho \ \text{ for } |x| \geqslant R(\rho). \tag{3.5}$$

It is not difficult to show that there exists $r > R(\rho)$ satisfying

$$\int_{r\leqslant|x|\leqslant r+1} (h^p|\nabla u_{n_k}|^p + V(x)|u_{n_k}|^p)dx < \rho, \tag{3.6}$$

for any k. Let $\eta \in C^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta = 1$ for $|x| \leqslant r$, $\eta = 0$ for $|x| \geqslant r+1$ and $0 \leqslant \eta \leqslant 1$, $|\nabla\eta| \leqslant \frac{C}{r}$. Define $w_k = \eta u_{n_k}$ and $v_k = (1-\eta)u_{n_k}$. A direct computation shows

$$\left|\langle J_h'(u_{n_k}), w_k\rangle - \langle J_h'(w_k), w_k\rangle\right| \leqslant C\int_{r\leqslant|x|\leqslant r+1} (h^p|\nabla u_{n_k}|^p + V(x)|u_{n_k}|^p)dx \tag{3.7}$$

and

$$\left|\langle J_h'(u_{n_k}), v_k\rangle - \langle J_h'(v_k), v_k\rangle\right| \leqslant C\int_{r\leqslant|x|\leqslant r+1} (h^p|\nabla u_{n_k}|^p + V(x)|u_{n_k}|^p)dx. \tag{3.8}$$

Using $\langle J_h'(u_{n_k}), w_k\rangle = o(1)$ and (3.5)-(3.8), we have

$$\langle J_h'(w_k), w_k\rangle = O(\rho) + o(1), \quad \langle J_h'(v_k), v_k\rangle = O(\rho) + o(1). \tag{3.9}$$

Thus, by (3.1), (3.9) and Sobolev embedding theorem, we obtain

$$c+o(1)=J_h(u_{n_k})=J_h(w_k)+J_h(v_k)+O(\rho)\geqslant J_h(v_k)+O(\rho). \tag{3.10}$$

From (3.4), we have

$$\begin{aligned}\int_{\mathbb{R}^N}(h^p|\nabla v_k|^p+V(x)|v_k|^p)dx &= \int_{|x|\geqslant r+1}(h^p|\nabla u_{n_k}|^p+V(x)|u_{n_k}|^p)dx \\ &\quad+\int_{r\leqslant|x|\leqslant r+1}(h^p|\nabla v_k|^p+V(x)|v_k|^p)dx \\ &\geqslant \delta.\end{aligned} \tag{3.11}$$

By Lemma 2, there exists $\theta_k>0$ such that $\theta_k v_k\in\mathcal{N}_h$, it follows from (3.11) that θ_k has a positive lower bound. Hence, by (3.9), we have

$$J_h(\theta_k v_k)=J_h(v_k)+O(\rho)+o(1). \tag{3.12}$$

Let $\overline{v}_k(x)=\theta_k v_k(hx)$ and let t_k be such that $t_k\overline{v}_k\in\mathcal{N}^\rho$, where $\mathcal{N}^\rho$ denote the solution manifold defined as (1.8) by $V_\infty-\rho$ and $K_\infty-\rho$. Hence, we have

$$\begin{aligned}h^N\int_{\mathbb{R}^N}(|\nabla\overline{v}_k|^p+(V_\infty-\rho)|\overline{v}_k|^p)dx &= \int_{\mathbb{R}^N}(h^p|\nabla(\theta_k v_k)|^p+(V_\infty-\rho)|\theta_k v_k|^p)dx\\ &\leqslant \int_{\mathbb{R}^N}(h^p|\nabla(\theta_k v_k)|^p+V(x)|\theta_k v_k|^p)dx\\ &= \int_{\mathbb{R}^N}(K(x)|\theta_k v_k|^q+|\theta_k v_k|^{p^*})dx\\ &\leqslant \int_{\mathbb{R}^N}((K_\infty+\rho)|\theta_k v_k|^q+|\theta_k v_k|^{p^*})dx\\ &= h^N\int_{\mathbb{R}^N}((K_\infty+\rho)|\overline{v}_k|^q+|\overline{v}_k|^{p^*})dx.\end{aligned}$$

Thus, by $t_k\overline{v}_k\in\mathcal{N}^\rho$, we obtain

$$t_k^{q-p}\int_{\mathbb{R}^N}(K_\infty+\rho)|\overline{v}_k|^q dx+t_k^{p^*-p}\int_{\mathbb{R}^N}|\overline{v}_k|^{p^*}dx\leqslant\int_{\mathbb{R}^N}((K_\infty+\rho)|\overline{v}_k|^q+|\overline{v}_k|^{p^*})dx,$$

it follows that $t_k\leqslant 1$. Furthermore, by (3.1), (3.10), (3.12) and $\theta_k v_k\in\mathcal{N}_h$, we get

$$\begin{aligned}h^N c_\rho &\leqslant h^N\bigg[\frac{t_k^p}{p}\int_{\mathbb{R}^N}(|\nabla\overline{v}_k|^p+(V_\infty-\rho)|\overline{v}_k|^p)dx\\ &\qquad -\frac{t_k^q}{q}\int_{\mathbb{R}^N}(K_\infty+\rho)|\overline{v}_k|^q dx-\frac{t_k^{p^*}}{p^*}\int_{\mathbb{R}^N}|\overline{v}_k|^{p^*}dx\bigg]\\ &= \frac{t_k^p}{p}\int_{\mathbb{R}^N}h^p|\nabla\theta_k v_k|^p+(V_\infty-\rho)|\theta_k v_k|^p dx\\ &\qquad -\frac{t_k^q}{q}\int_{\mathbb{R}^N}(K_\infty+\rho)|\theta_k v_k|^q dx-\frac{t_k^{p^*}}{p^*}\int_{\mathbb{R}^N}|\theta_k v_k|^{p^*}dx\end{aligned}$$

$$\leqslant \frac{t_k^p}{p}\int_{\mathbb{R}^N} h^p|\nabla\theta_k v_k|^p + V(x)|\theta_k v_k|^p dx$$
$$-\frac{t_k^q}{q}\int_{\mathbb{R}^N} K(x)|\theta_k v_k|^q dx - \frac{t_k^{p^*}}{p^*}\int_{\mathbb{R}^N}|\theta_k v_k|^{p^*}dx$$
$$= \frac{t_k^p}{p}\left(\int_{\mathbb{R}^N}(K(x)|\theta_k v_k|^q + |\theta_k v_k|^{p^*})dx\right)$$
$$-\frac{t_k^q}{q}\int_{\mathbb{R}^N} K(x)|\theta_k v_k|^q dx - \frac{t_k^{p^*}}{p^*}\int_{\mathbb{R}^N}|\theta_k v_k|^{p^*}dx$$
$$\leqslant \frac{1}{p}\left(\int_{\mathbb{R}^N}(K(x)|\theta_k v_k|^q + |\theta_k v_k|^{p^*})dx\right)$$
$$-\frac{1}{q}\int_{\mathbb{R}^N} K(x)|\theta_k v_k|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N}|\theta_k v_k|^{p^*}dx$$
$$= J_h(\theta_k v_k) = J_h(v_k) + O(\rho) + o(1)$$
$$\leqslant c + O(\rho) + o(1).$$

Letting $k \to \infty$ and $\rho \to 0$, we have $h^N c_\infty \leqslant c$. This is a contradiction. Hence, by the Gagliardo-Nirenberg inequality, we have (3.3). Thus,

$$\int_{\mathbb{R}^N} K(x)|u_n|^q dx \to \int_{\mathbb{R}^N} K(x)|u|^q dx. \tag{3.13}$$

Thus the Lemma is proved. □

LEMMA 7. $\int_{\mathbb{R}^N}|u_n|^{p^*}dx \to \int_{\mathbb{R}^N}|u|^{p^*}dx$.

Proof. Let $v_n(x) = u_n(hx)$, then $J_h(u_n) = h^N I_h(v_n)$. By the boundedness of u_n, we may suppose that $v_n \rightharpoonup v$ weakly in E_h. By Lemma 4, we assume that

$$\begin{cases} |v_n|^{p^*} \rightharpoonup \nu = |v|^{p^*} + \sum_{j\in J}\nu_j\delta_{x_j}, \\ |\nabla v_n|^p \rightharpoonup \mu \geqslant |\nabla v|^p + \sum_{j\in J}\mu_j\delta_{x_j}. \end{cases} \tag{3.14}$$

Take $x_j \in \{x_j \in \mathbb{R}^N, j \in \Lambda\}$, $\phi_j \in C_0^\infty(\mathbb{R}^N)$, for $\varepsilon > 0$, such that

$$\phi_j \equiv 1 \text{ on } B(x_j,\varepsilon),\ \phi_j \equiv 0 \text{ on } \mathbb{R}^N\setminus B(x_j,2\varepsilon) \text{ and } \nabla\phi_j \leqslant \tfrac{2}{\varepsilon}.$$

From (3.1), we have $\lim_{n\to\infty}\langle I_h'(u_n),\phi_j u_n\rangle = 0$, that is

$$\int_{\mathbb{R}^N}|\nabla v_n|^{p-2}v_n\langle\nabla v_n,\nabla\phi_j\rangle dx + \int_{\mathbb{R}^N}|\nabla u_n|^p\phi_j dx + \int_{\mathbb{R}^N}V(hx)|v_n|^p\phi_j dx$$
$$= \int_{\mathbb{R}^N}K(hx)|u_n|^q\phi_j dx + \int_{\mathbb{R}^N}|u_n|^{p^*}\phi_j dx + o(1).$$

By (3.14) and Sobolev embedding theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^{p-2}v_n\langle\nabla v_n,\nabla\phi_j\rangle dx+\int_{\mathbb{R}^N}V(hx)|v|^p\phi_j dx+\int_{\mathbb{R}^N}\phi_j d\mu$$
$$=\int_{\mathbb{R}^N}K(hx)|v|^q\phi_j dx+\int_{\mathbb{R}^N}\phi_j d\nu. \quad (3.15)$$

It follows from the Hölder inequality and the boundedness of $\{v_n\}$ that

$$\begin{aligned}0&\leqslant\left|\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^{p-2}v_n\langle\nabla v_n,\nabla\phi_j\rangle dx\right|\\&\leqslant C\left(\int_{\mathbb{R}^N}|\nabla v_n|^p dx\right)^{(p-1)/p}\cdot\left(\int_{B(x_j,2\varepsilon)}|v_n|^p dx\right)^{1/p^*}\to 0\end{aligned}$$

as $\varepsilon\to 0$. It hence follows from (3.14) and (3.15) that, as $\varepsilon\to 0$,

$$\nu_j\geqslant\mu_j.$$

We thus conclude from Lemma 4 that

$$\nu_j=0\quad\text{or}\quad\nu_j\geqslant S^{N/p}.$$

Assume $\nu_j\neq 0$ for some $j\in\Lambda$, by (3.1), (3.14) and Lemma 4,

$$\begin{aligned}h^{-N}c&=\lim_{n\to\infty}(I_h(v_n)-\frac{1}{p}\langle I_h'(v_n),v_n\rangle)\\&=\lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N}K(hx)|v_n|^q dx+\frac{1}{N}\int_{\mathbb{R}^N}|v_n|^{p^*}dx\right]\\&\geqslant\left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N}K(hx)|v|^q dx+\frac{1}{N}\int_{\mathbb{R}^N}|v|^{p^*}dx+\frac{1}{N}S^{N/p}\\&\geqslant\frac{1}{N}S^{N/p},\end{aligned}$$

which is impossible. Thus, $\nu_j=0$ for all $j\in\Lambda$. Hence,

$$\int_{\mathbb{R}^N}|v_n|^{p^*}dx\to\int_{\mathbb{R}^N}|v|^{p^*}dx.$$

Thus the Lemma is proved. □

LEMMA 8. *For $h>0$ small, the functional J_h satisfies the $(PS)_c$ condition provided $c<h^N c_\infty$.*

Proof. The weak convergence of (3.2) implies that $J_h'(u)=0$, by Lemma 6 and Lemma 7, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h^p|\nabla u_n|^p+V(x)|u_n|^p dx=\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}K(x)|u_n|^q dx+\int_{\mathbb{R}^N}|u_n|^{p^*}dx\right)$$
$$=\int_{\mathbb{R}^N}K(x)|u|^q dx+\int_{\mathbb{R}^N}|u|^{p^*}dx$$

$$= \int_{\mathbb{R}^N} h^p|\nabla u|^p + V(x)|u|^p dx.$$

Hence, $u_n \to u$ strongly in X_h. □

LEMMA 9. *For $h > 0$ small,*

$$\limsup_{h\to 0} c_h \leqslant h^N \inf_{\xi\in\mathbb{R}^{\mathbb{N}}} G(\xi).$$

Proof. For any $\xi \in \mathbb{R}^N$, let $\omega(x) = \omega(V(\xi), K(\xi), x)$ be a ground state solution of (1.5). Define $\varphi \in C_0^\infty(\mathbb{R}^+, [0,1])$ such that $\varphi(s) \equiv 1$ if $0 \leqslant s \leqslant 1$, and $\varphi(s) \equiv 0$ if $s \geqslant 2$. Let

$$\omega_h(x) = \varphi(|x-\xi|)\omega\Big(\frac{x-\xi}{h}\Big).$$

Then,

$$\begin{aligned} J_h(\omega_h(x)) = h^N\Big(&\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla(\varphi(h|x|)\omega(x))|^p + V(hx+\xi)|\varphi(h|x|)\omega(x)|^p)dx \\ &-\frac{1}{q}\int_{\mathbb{R}^N} K(hx+\xi)|\varphi(h|x|)\omega(x)|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N}|\varphi(h|x|)\omega(x)|^{p^*}dx\Big). \end{aligned}$$

Let $h \to 0$, it is easy to check that

$$\begin{cases} \int_{\mathbb{R}^N}(|\nabla(\varphi(h|x|)\omega(x))|^p + V(hx+\xi)|\varphi(h|x|)\omega(x)|^p)dx \\ \qquad \to \int_{\mathbb{R}^N}(|\nabla\omega|^p + V(\xi)|\omega|^p)dx, \\ \int_{\mathbb{R}^N} K(hx+\xi)|\varphi(h|x|)\omega(x)|^q dx \to \int_{\mathbb{R}^N} K(\xi)|\omega|^q dx, \\ \int_{\mathbb{R}^N}|\varphi(h|x|)\omega(x)|^{p^*}dx \to \int_{\mathbb{R}^N}|\omega|^{p^*}dx. \end{cases}$$

By the definition of ω_h, we can find a $L > 0$ large enough such that $J_h(L\omega_h) < 0$. Hence, we can construct a path $g_h(t) = tL\omega_h$, $t \in [0,1]$. By Lemma 2, we have

$$\begin{aligned} c_h &\leqslant \max_{0\leqslant t\leqslant 1} J_h(g_h(t)) = \max_{0\leqslant t\leqslant 1} J_h(tL\omega_h) \\ &= \max_{0\leqslant t\leqslant 1} h^N\Big(\frac{(tL)^p}{p}\int_{\mathbb{R}^N}(|\nabla\omega|^p + V(\xi)|\omega|^p)dx - \frac{(tL)^q}{q}\int_{\mathbb{R}^N}K(\xi)|\omega|^q dx \\ &\quad -\frac{(tL)^{p^*}}{p^*}\int_{\mathbb{R}^N}|\omega|^{p^*}dx + o(1)\Big) \\ &= h^N(G(\xi) + o(1)). \end{aligned}$$

Since ξ is arbitrary and the smallness of h is independent of ξ. □

Proof of the existence of Theorem 1.1. By Lemma 1, 8 and 9, the standard Mountain pass theorem implies that problem (1.1) has a nontrivial nonegative solution. Then, a Harnack's inequality of [23] implies that it's a positive ground state. □

4. The concentration of the ground state

In this section, u_h is always referred to a positive ground state of (1.1). We see $v_h(x)=u_h(hx)$ is always a positive ground state of

$$-\Delta_p v_h + V(hx)v_h^{p-1} = K(hx)v_h^{q-1} + v_h^{p^*-1}, \quad x\in\mathbb{R}^N.$$

Conversely, $u_h(x)=v_h(\frac{x}{h})$ is a positive ground state of (1.1).

LEMMA 10. *There exist $R,\sigma>0$ and a sequence $\{y_h\}\subset\mathbb{R}^N$ such that*

$$\liminf_{h\to0}\int_{B_R(y_h)}|v_h|^p dx \geqslant \sigma>0.$$

Proof. We observe that since

$$\begin{aligned} h^{-N}c_h &= I_h(v_h)-\frac{1}{q}\langle I_h'(v_h),v_h\rangle \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N}(|\nabla v_h|^p+V(hx)|v_h|^p)dx+\left(\frac{1}{q}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}|v_h|^{p^*}dx. \end{aligned}$$

It follows from Lemma 9, $\{v_h\}$ is bounded as $h\to0$.

If for any $R>0$, there is a sequence $h_n\to0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|v_{h_n}|^p dx=0.$$

Then from Lemma 1.1 in [19], we conclude that

$$v_{h_n}\to0 \ \text{ in } \ L^s(\mathbb{R}^N) \quad \text{for } \ s\in(p,p^*). \tag{4.1}$$

Moreover, it follows from $I_h'(v_{h_n})=0$ that

$$\int_{\mathbb{R}^N}(|\nabla v_{h_n}|^p+V(h_nx)|v_{h_n}|^p)dx=\int_{\mathbb{R}^N}K(h_nx)|v_{h_n}|^q dx+\int_{\mathbb{R}^N}|v_{h_n}|^{p^*}dx. \tag{4.2}$$

Assume that $l\geqslant0$ be such that

$$\int_{\mathbb{R}^N}(|\nabla v_{h_n}|^p+V(h_nx)|v_{h_n}|^p)dx\to l. \tag{4.3}$$

If $l>0$, by (4.1) and (4.2), let $n\to\infty$

$$\int_{\mathbb{R}^N}|v_{h_n}|^{p^*}dx\to l. \tag{4.4}$$

Hence, from (4.3) and (4.4), we have

$$\lim_{n\to\infty}h_n^{-N}c_{h_n}=\lim_{n\to\infty}I_{h_{h_n}}(v_{h_n})=\frac{l}{N}. \tag{4.5}$$

By (2.2), we have

$$l \geqslant S^{\frac{N}{p}}.$$

This combine with (4.5), we have

$$\lim_{n\to\infty} h_n^{-N} c_{h_n} \geqslant \frac{1}{N} S^{N/p}.$$

This contradicts with $G(\xi) < \frac{1}{N} S^{\frac{N}{p}}$ and Lemma 9. Thus, $l = 0$. But $c_{h_n} > 0$ and J_h is continuous, we see that $l \neq 0$. It is a contradiction. The lemma is proved. □

Now, let $w_h(x) = v_h(x+y_n) = u_h(hx+hy_h)$. Then, by Lemma 10, $w_h \not\equiv 0$. Hence, $w_h(x)$ is a positive ground state of

$$-\Delta_p w_h + V(hx+hy_n) w_h^{p-1} = K(hx+hy_n) w_h^{q-1} + w_h^{p^*-1} \qquad x \in \mathbb{R}^N. \tag{4.6}$$

LEMMA 11. *For small $h > 0$, then the sequence $\{hy_h\}$ is bounded.*

Proof. Suppose that there exists a subsequence $h_n \to 0$ such that $|h_n y_{h_n}| \to \infty$. Obviously, $w_n = w_{h_n}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Then, up to a subsequence, $w_n \rightharpoonup w$ weakly in $W^{1,p}(\mathbb{R}^N)$, $w_n \to w$ strongly in $L^s_{loc}(\mathbb{R}^N)$, $p < s < p^*$ and $w_n \to w$ $a.e$ in $\mathbb{R}^N$. From Lemma 10, $w \not\equiv 0$. By (1.10), we choose $\varepsilon > 0$ small such that

$$c^\varepsilon = c(V_\infty - \varepsilon, K_\infty - \varepsilon) > \inf_{\xi\in\mathbb{R}^N} G(\xi). \tag{4.7}$$

According to the assumption $|h_n y_{h_n}| \to \infty$, we have

$$\Delta_p w - (V_\infty - \frac{\varepsilon}{2}) w^{p-1} + (K_\infty + \frac{\varepsilon}{2}) w^{q-1} + |w|^{p^*-1} \geqslant 0 \quad \text{ in } \; H^{-1}.$$

In particular,

$$\int_{\mathbb{R}^N} (|\nabla w|^p + (V_\infty - \varepsilon)|w|^p) dx \leqslant \int_{\mathbb{R}^N} (K_\infty - \varepsilon)|w|^q dx + \int_{\mathbb{R}^N} |w|^{p^*} dx. \tag{4.8}$$

Take $\theta > 0$ such that $\theta w \in \mathscr{N}^\varepsilon$, by (4.8), we have $\theta < 1$. Thus, by Fatou's lemma,

$$\begin{aligned}
c^\varepsilon \leqslant \frac{\theta^p}{p} \int_{\mathbb{R}^N} & (|\nabla w|^p + (V_\infty - \varepsilon)|w|^p) dx \\
&- \frac{\theta^q}{q} \int_{\mathbb{R}^N} (K_\infty + \varepsilon)|w|^q dx - \frac{\theta^{p^*}}{p^*} \int_{\mathbb{R}^N} |w|^{p^*} dx \\
\leqslant \liminf_{h_h\to 0} \Bigg[& \frac{\theta^p}{p} \int_{\mathbb{R}^N} (|\nabla w_n|^p + V(h_n x + h_n y_n)|w_n|^p) dx \\
&- \frac{\theta^q}{q} \int_{\mathbb{R}^N} K(h_n x + h_n y_n)|w_n|^q dx - \frac{\theta^{p^*}}{p^*} \int_{\mathbb{R}^N} |w_n|^{p^*} dx \Bigg]
\end{aligned}$$

$$\leqslant \liminf_{h_h \to 0}\left[\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla w_n|^p + V(h_n x + h_n y_n)|w_n|^p)dx \right.$$
$$\left. - \frac{1}{q}\int_{\mathbb{R}^N} K(h_n x + h_n y_n)|w_n|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N}|w_n|^{p^*}dx\right]$$
$$= \liminf_{h_h \to 0} h_n^{-N} c_{h_n} \leqslant \inf_{\xi \in \mathbb{R}^N} G(\xi).$$

This contradicts with (4.7). Hence, $\{hy_h\}$ is bounded. The lemma is proved. □

Proof of the concentration of Theorem 1.1. By Lemma 11, there exist a subsequence $h_n \to 0$ such that $x_n = h_n y_{h_n} \to x_0$, $w_n = w_{h_n} \rightharpoonup w$ weakly in $W^{1,p}(\mathbb{R}^N)$ and $a.e$ in $\mathbb{R}^N$, where $w \geqslant 0$, $\not\equiv 0$. Applying the regularity result due to [24], we have $w_n \to w$ in $C^{1,\alpha}_{loc}(\mathbb{R}^N)$, where $\alpha \in (0,1)$, and

$$-\Delta_p w + V(x_0)w^{p-1} = K(x_0)w^{q-1} + w^{p^*-1}, \ \ x \in \mathbb{R}^N.$$

Consequently, by Lemma 9 and Fatou's lemma,

$$\inf_{\xi \in \mathbb{R}^N} G(\xi) \leqslant G(x_0)$$
$$\leqslant \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N}(|\nabla w|^p + V(x_0)|w|^p)dx + \left(\frac{1}{q} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|w|^{p^*}dx$$
$$\leqslant \liminf_{n \to \infty}\left[\left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N}(|\nabla w_n|^p + V(x_0)|w_n|^p)dx \right.$$
$$\left. - \left(\frac{1}{q} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|w_n|^{p^*}dx\right]$$
$$\leqslant \limsup_{n \to \infty}\left[\left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N}(|\nabla w_n|^p + V(x_0)|w_n|^p)dx \right.$$
$$\left. - \left(\frac{1}{q} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|w_n|^{p^*}dx\right]$$
$$\leqslant \limsup_{n \to \infty}\left[\left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N}(|\nabla w_n|^p + V(h_n x + x_n)|w_n|^p)dx \right.$$
$$\left. - \left(\frac{1}{q} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|w_n|^{p^*}dx\right]$$
$$= \limsup_{n \to \infty} h_n^{-N} c_{h_n} \leqslant \inf_{\xi \in \mathbb{R}^N} G(\xi).$$

This implies that $G(x_0) = \inf_{\xi \in \mathbb{R}^N} G(\xi)$. Moreover, by the above inequality and Lemma 7, it implies that

$$\int_{\mathbb{R}^N}(|\nabla w_n|^q + V(x_0)|w_n|^p)dx \to \int_{\mathbb{R}^N}(|\nabla w|^q + V(x_0)|w|^p)dx.$$

Hence, $w_n \to w$ strongly in $W^{1,p}(\mathbb{R}^N)$. In particular, as the result of [11, 12], we obtain $w_n \in L^\infty(\mathbb{R}^N)$, $\|w_n\|_\infty \leqslant C$, and

$$\lim_{|x| \to \infty} w_n(x) = 0 \quad \text{uniformly in } n. \tag{4.9}$$

Now, we claim that there exists a $\delta > 0$ such that $\|w_n\|_\infty \geqslant \delta$ for all n. If we assume, contrary to the claim, that $\|w_n\|_\infty \to 0$. Then by (4.6), we have

$$\int_{\mathbb{R}^N} (|\nabla w_n|^p + V(x_0)|w_n|^p)dx \leqslant \|K\|_\infty \int_{\mathbb{R}^N} |w_n|^q dx + \int_{\mathbb{R}^N} |w_n|^{p^*} dx.$$

It follows that $w_n \to 0$ in $W^{1,p}(\mathbb{R}^N)$, which is impossible. Then claim is true.

Let z_n be the global maximum point of w_n, by (4.9) and the claim above, we obtain that $z_n \in B_R(0)$ for some $R > 0$. Thus the global maximum point of $u_n(x) = u_{h_n}(x) = w_n(\frac{x-x_n}{h_n})$ given by $\bar{x}_n = h_n z_n + x_n$. Since $\{z_n\}$ is bounded, we know that $\bar{x}_n \to x_0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} G(\bar{x}_n) = G(x_0) = \inf_{\xi \in \mathbb{R}^N} G(\xi)$. $\square$

Acknowledgements. The authors would like to express sincere thanks to the anonymous referee for his/her carefully reading the manuscript and valuable comments and suggestions.

REFERENCES

[1] J.G. AZORERO, I.P. ALONSO, *Multiplicity of solutions for elliptic problems with critical exponent or with nonsymmetric term*, Trans. Amer. Math. Soc. **323**, 2 (1991), 877–895.

[2] A. AMBROSETTI, M. BADIALE, S. CINGOLANI, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal., **140**, 1 (1997), 285–300.

[3] C.O. ALVES, G.M. FIGUEIREDO, *Existence and multiplicity of positive solutions to a p-Laplacian equation in $\mathbb{R}^N$*, Differential Integral Equations Anal., **19** (2006), 143–162.

[4] C.O. ALVES, G.M. FIGUEIREDO, *On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$*, J. Differential Equations, **246** (2009), 1288–1311.

[5] C.O. ALVES, *Existence and multiplicity of solution for a class of quasilinear equations*, Adv. Nonlinear Studies, **5** (2005), 73–86.

[6] C.O. ALVES, M. S. SOUTO, *Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity, J. Differential Equations*, **254** (2013), 1977–1991.

[7] J. BYEON, Z.Q. WANG, *Standing waves with a critical frequency for nonlinear Schrödinger equations*, Arch. Rational Mech. Anal. **165** (2002), 295–316.

[8] S. CINGOLANI, M. LAZZO, *Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions*, J. Differential Equations, **160** (2000), 118–138.

[9] M. DEL PINO, P. FELMER, *Semi-classical states of nonlinear Schrödinger equations: a variational reduction method*, Math. Ann, **324** (2002), 1–32.

[10] Y. DING, F. LIN, *Solutions of perturbed Schrödinger equations with critical nonlinearity*, Calc. Var. PDE., **30** (2007), 231–249.

[11] J.M. DO Ó, *On existence and concentration of positive bounded states of p-Laplacian equation in $\mathbb{R}^N$ involving critical growth*, Nonlinear Anal., **62** (2005), 777–801.

[12] G.M. FIGUEIREDO, *Multiplicity of solutions for a quasilinear problem with supercritical growth*, Electronic Journal of Differ. Equ., **31** (2006), 1–7.

[13] G.M. FIGUEIREDO, *Existence, multiplicity and concentration of positive solutions for a class of quasilinear problems with critical growth*, Communications on Applied Nonlinear Analysis, **4** (2006), 79–99.

[14] G.M. FIGUEIREDO, M.F. FURTADO, *Positive solutions for a quasilinear Schrödinger equation with critical growth*, J. Dyn. Diff. Equat., **24** (2012), 13–28.

[15] A. FLOER, A. WEINSTEIN, *Nonspreading waves packets for the cubic Schrödinger equations with a bounded potential*, J. Funct. Anal., **69** (1986), 397–408.

[16] M.W. FEI, H.C. YIN, *Existence and concentration of bound states of nonlinear Schrödinger equations with compactly supported competing potentials*, Pacific Journal of Mathematics, **244** (2010), 261–295.

[17] C.F. GUI, *Existence of multi-bumps solutions for nonlinear Schrödinger equations via variational methods*, Comm. Partial Differential Equations, (1996), 787–820.

[18] P.L. LIONS, *The concentration-compcatness principle in the calculus of variations. The limit case. Part I*, Rev. Mat. Iber. **1**, 2 (1985), 109–145.

[19] P.L. LIONS, *The concentration-compcatness principle in the calculus of variations. The locally compact case. Part II*, Ann. Inst. H. Poincare Non Lineaire, **1** (1984), 223–283.

[20] L. JEANJEAN, K. TANAKA, *Singularly perturbed elliptic problems with superlinear or asympotically linear nonlinearities*, Calc. Var. PDE., **21** (2004), 287 – 318.

[21] Y.G. OH, *Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class(V)*, Comm. Partial Differential Equations, **13** (1988), 1499–1519.

[22] P.H. RABINOWITZ, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys., **43** (1992), 270–291.

[23] J. SERRIN, *Local behavior of solutions of quasilinear equations*, Acta Math., **111** (1964), 247–302.

[24] P. TOLKSDORF, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, **51** (1984), 126–150.

[25] M. WILLEM, *Minimax Theorems, Birkhäuser*, Bosten, 1996.

[26] X.F. WANG, B. ZENG, *On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions*, SIAM J. Math. Anal., **28** (1997), 633–655.

(Received May 3, 2013)

Xudong Shang
Jiangsu Key Laboratory for NSLSCS
School of Mathematical Sciences
Nanjing Normal University
Nanjing 210023
PR China
e-mail: xudong-shang@163.com

Xudong Shang
School of Mathematics
Nanjing Normal University Taizhou College
Taizhou 225300
PR China

Jihui Zhang
Jiangsu Key Laboratory for NSLSCS
School of Mathematical Sciences
Nanjing Normal University
Nanjing 210023
PR China
e-mail: Zhangjihui@njnu.edu.cn