Differential
Equations
& Applications
Volume 6, Number 1 (2014), 21–31
doi:10.7153/dea-06-02

MAXIMAL AND MINIMAL POSITIVE SOLUTIONS OF A NONLINEAR QUADRATIC INTEGRAL EQUATION

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(*Communicated by Michal Fečkan*)

Abstract. We are concerned here with the existence of at least one continuous positive solution of the nonlinear quadratic integral equation

$$x(t) \; = \; a(t) \; + \lambda \int_0^t k_1(t,s) f(s,x(s)) ds \int_0^t k_2(t,s) g(s,x(s)) ds, \quad t \in [0,T].$$

where f and g are L^1- Carathéodory functions. The maximal and minimal solutions are also proved.

1. Introduction and preliminaries

Quadratic integral equations have received increasing attention during recent years due to its applications in numerous diverse fields of science and engineering for example, the theory of radiative transfer, kinetic theory of gases, the theory of neutron transport and the traffic theory. Many authors have studied quadratic integral equations (see[1]-[16], [17] and [21]-[23]).

Consider the quadratic integral equation

$$x(t) = 1 + \lambda \int_t^1 x(s)\, x(s-t)\, ds, \quad 0 \leqslant t \leqslant 1. \tag{1}$$

This equation was presented by Dr. M. S. Wertheim, a physicist at the Los Alamos Scientific Laboratory, as a simplified model of a certain equations arising in Statistical Mechanics. This equation was studied by H. George and Jr. Pimbley [17].

The quadratic integral equation

$$x(t) = a(t) + \int_0^t f(s,x(s))\, ds \int_0^t g(s,x(s))\, ds \tag{2}$$

has been studied in [10]. The existence of continuous solution was proved, also the existence of the maximal and minimal solutions was proved.

The quadratic integral equations of fractional order

Mathematics subject classification (2010): 45G10, 45D05, 45M20.
Keywords and phrases: Carathéodory functions, quadratic integral equations, Schauder fixed point theorem, maximal and minimal solutions.

Paper DEA-06-02

$$x(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds, \quad \alpha,\beta \in (0,1) \tag{3}$$

has been studied in [12]. The existence of continuous solution, maximal and minimal solutions was proved. Also, the existence of a unique positive continuous solution for the quadratic integral of fractional order

$$x(t) = a(t) + \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds, \quad \alpha,\beta \in (0,1) \tag{4}$$

was proved where f and g are Carathéodory functions [15].

The existence of unique continuous solution of the quadratic integral equation

$$x(t) = a(t) + \lambda \int_0^t k_1(t,s) f(s,x(s)) ds \int_0^t k_2(t,s) g(s,x(s)) ds, \quad t \in [0,T]. \tag{5}$$

was studied in [14] by using Banach fixed point Theorem, where f and g are L^1-Carathéodory functions and satisfy the Lipschitz condition with respect to the second argument.

The quadratic integral equation (5)(with $\lambda = 1$) is solved by using Adomian Decomposition method and the maximum absolute truncated error of Adomian series solution is estimated [23]. In the paper [21], an efficient numerical technique based on the fixed point method and quadrature rules to approximate a solution for quadratic Volterra integral equation was prepared and the convergence of numerical scheme was proved by some theorems.

Here we are concerned with the existence of at least one continuous positive solution $x \in C[0,T]$ of the nonlinear quadratic integral equation (5), where f and g are L^1-Carathéodory functions.

The existence of the maximal and minimal positive continuous solutions of the nonlinear quadratic integral equation (5) will be proved.

2. Existence of solutions

Consider the nonlinear quadratic integral equation (5) under the following assumptions

(i) $a : I = [0,T] \to R_+$ is continuous

(ii) $f, g : I \times R_+ \to R_+$ are L^1-Carathèodory functions i.e f, g are measurable in t for all $x \in R_+$ and continuous in x for almost all $t \in [0,T]$, and there exist two functions $m_1, m_2 \in L^1[0,T]$ such that

$$|\, f(t,x) \,| \leqslant m_1(t)$$

$$|\, g(t,x) \,| \leqslant m_2(t)$$

(iii) $k_i : [0,T] \times [0,T] \to R_+$ are continuous in $t \in [0,T]$ for every $s \in [0,T]$ and measurable in $s \in [0,T]$ for all $t \in [0,T]$ such that

$$\int_0^t |k_i(t,s)| m_i(s) ds \leqslant K_i, \qquad i = 1,2, \quad t \in [0,T].$$

Now for the existence of at least one positive continuous solution of the nonlinear quadratic integral equation (5) we have the following theorem.

THEOREM 2.1. *If the assumptions (i)-(iii) are satisfied, then the nonlinear quadratic integral equation (5) has at least one positive solution $x \in C[0,T]$.*

Proof. Let $C = C[0,T]$ and define the set S by

$$S = \{x \in C : 0 \leqslant x(t) \leqslant r, \; t \in [0,T]\}$$

where $r = a + |\lambda| K_1 K_2$ and $a = \sup_{t \in [0,T]} |a(t)|$.

It is clear that S is nonempty (at least contains 0) and S is bounded and closed.

Let $x_1, x_2 \in S$ and $\delta \in [0,1]$, then we have

$$\begin{aligned} \delta x_1(t) + (1-\delta) x_2(t) &\leqslant | \delta x_1(t) + (1-\delta) x_2(t) | \\ &\leqslant \delta | x_1(t) | + (1-\delta) | x_2(t) | \end{aligned}$$

and

$$\begin{aligned} \| \delta x_1(t) + (1-\delta) x_2(t) \| &\leqslant \delta \| x_1 \| + (1-\delta) \| x_2 \| \\ &\leqslant \delta \, r + (1-\delta) \, r = r. \end{aligned}$$

Then $\delta x_1(t) + (1-\delta) x_2(t) \in S$, which means that S is convex set.

Define the operator F associated with the quadratic integral equation (5) by

$$Fx(t) = a(t) + \lambda \int_0^t k_1(t,s) f(s,x(s)) \, ds \int_0^t k_2(t,s) g(s,x(s)) \, ds.$$

To show that $F : S \to S$, let $x \in S$, then

$$\begin{aligned} |Fx(t)| &= | a(t) + \lambda \int_0^t k_1(t,s) f(s,x(s)) \, ds \int_0^t k_2(t,s) \, g(s,x(s)) \, ds | \\ &\leqslant | a(t) | + |\lambda| \int_0^t |k_1(t,s)| \, |f(s,x(s))| ds \int_0^t |k_2(t,s)| \, |g(s,x(s))| \, ds \\ &\leqslant | a(t) | + |\lambda| \int_0^t |k_1(t,s)| \, m_1(s) ds \int_0^t |k_2(t,s)| \, m_2(s) \, ds \\ &\leqslant |a(t)| + |\lambda| K_1 K_2 \leqslant a + |\lambda| K_1 K_2 = r, \end{aligned}$$

then $Fx \in S$.

This proves that $F : S \to S$ and the class of functions $\{F(x)\}$ is uniformly bounded.

Let $t_1, t_2 \in [0,T]$, $t_1 < t_2$ and $|t_2 - t_1| \leqslant \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| = \Big| & a(t_2) - a(t_1) \\
& + \lambda \int_0^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds \int_0^{t_2} k_2(t_2,s)\, g(s,x(s))\, ds \\
& - \lambda \int_0^{t_1} k_1(t_1,s)\, f(s,x(s))\, ds \int_0^{t_1} k_2(t_1,s)\, g(s,x(s))\, ds \Big| \\
= \Big| & a(t_2) - a(t_1) \\
& + \lambda \int_0^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds \int_0^{t_2} k_2(t_2,s)\, g(s,x(s))\, ds \\
& - \lambda \int_0^{t_1} k_1(t_1,s)\, f(s,x(s))\, ds \int_0^{t_1} k_2(t_1,s)\, g(s,x(s))\, ds \\
& + \lambda \int_0^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds \int_0^{t_1} k_2(t_1,s)\, g(s,x(s))\, ds \\
& - \lambda \int_0^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds \int_0^{t_1} k_2(t_1,s)\, g(s,x(s))\, ds \Big| \\
\leqslant\ & |a(t_2) - a(t_1)| \\
& + \Big| \lambda \int_0^{t_1} k_2(t_1,s)\, g(s,x(s)) ds [\int_0^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds \\
& - \int_0^{t_1} k_1(t_1,s)\, f(s,x(s))\, ds] \\
& + \lambda \int_0^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds [\int_0^{t_2} k_2(t_2,s)\, g(s,x(s))\, ds \\
& - \int_0^{t_1} k_2(t_1,s)\, g(s,x(s))\, ds] \Big| \\
\leqslant\ & |a(t_2) - a(t_1)| \\
& + \Big| \lambda \int_0^{t_1} k_2(t_1,s)\, g(s,x(s))\, ds [\int_0^{t_1} k_1(t_2,s)\, f(s,x(s))\, ds \\
& + \int_{t_1}^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds - \int_0^{t_1} k_1(t_1,s)\, f(s,x(s))\, ds] \\
& + \lambda \int_0^{t_2} k_1(t_2,s)\, f(s,x(s))\, ds [\int_0^{t_1} k_2(t_2,s)\, g(s,x(s))\, ds \\
& + \int_{t_1}^{t_2} k_2(t_2,s)\, g(s,x(s))\, ds - \int_0^{t_1} k_2(t_1,s)\, g(s,x(s))\, ds] \Big| \\
\leqslant\ & |a(t_2) - a(t_1)| \\
& + |\lambda| \int_0^{t_1} |k_2(t_1,s)|\, |g(s,x(s))|\, ds \\
& \qquad \times \int_0^{t_1} |k_1(t_2,s) - k_1(t_1,s)|\, |f(s,x(s))|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+|\lambda| \int_0^{t_1} |k_2(t_1,s)|\ |g(s,x(s))|\ ds \int_{t_1}^{t_2} |k_1(t_2,s)|\ |f(s,x(s))|\ ds \\
&+|\lambda| \int_0^{t_2} |k_1(t_2,s)|\ |f(s,x(s))|\ ds \\
&\qquad \times \int_0^{t_1} |k_2(t_2,s)-k_2(t_1,s)|\ |g(s,x(s))|\ ds \\
&+|\lambda| \int_0^{t_2} |k_1(t_2,s)|\ |f(s,x(s))|\ ds \int_{t_1}^{t_2} |k_2(t_2,s)|\ |g(s,x(s))|\ ds \\
\leqslant\ & |a(t_2)-a(t_1)| \\
&+|\lambda| \int_0^{t_1} |k_2(t_1,s)|\ m_2(s)ds\ \int_0^{t_1} |k_1(t_2,s)-k_1\ (t_1,s)|\ m_1(s)\ ds \\
&+|\lambda| \int_0^{t_1} |k_2(t_1,s)|\ m_2(s)ds\ \int_{t_1}^{t_2} |k_1(t_2,s)|\ m_1(s)ds \\
&+|\lambda| \int_0^{t_2} |k_1(t_2,s)|\ m_1(s)ds\ \int_0^{t_1} |k_2(t_2,s)-k_2\ (t_1,s)|\ m_2(s)ds \\
&+|\lambda| \int_0^{t_2} |k_1(t_2,s)|\ m_1(s)ds\ \int_{t_1}^{t_2} |k_2(t_2,s)|\ m_2(t)ds.
\end{aligned}
$$

This means that the class of functions $\{Fx\}$ is equi-continuous on $[0,T]$. Using Arzela-Ascoli Theorem (see[19]), we fined that F is compact.

Now we prove that $F : S \to S$ is continuous.

Let $\{x_n\} \subset S$, $x \in S$ and $x_n \to x$, then

$$
Fx_n(t) = a(t) + \lambda \int_0^t k_1\ (t,s)\ f(s,x_n(s))\ ds \int_0^t k_2\ (t,s)\ g(s,x_n(s))\ ds
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= \lim_{n\to\infty} a(t) \\
&\quad + \lim_{n\to\infty} \left\{ \lambda \int_0^t k_1\ (t,s)\ f(s,x_n(s))\ ds \int_0^t k_2\ (t,s)\ g(s,x_n(s))\ ds \right\}.
\end{aligned}
$$

Now

$$
f(s,x_n) \to f(s,x) \ \Rightarrow k_1(t,s)f(s,x_n) \to k_1(t,s)f(s,x)
$$

and

$$
g(s,x_n) \to g(s,x) \ \Rightarrow k_2(t,s)g(s,x_n) \to k_2(t,s)g(s,x).
$$

Also

$$
k_1(t,s)f(s,x_n) \leqslant \ |k_1(t,s)||f(s,x_n)| \ \leqslant |k_1(t,s)|m_1(s) \in L^1[0,T].
$$

Similarly

$$
|k_2(t,s)g(s,x_n)| \ \leqslant |k_2(t,s)|m_2(s) \in L^1[0,T].
$$

Then by using Lebesgue dominated convergence Theorem (see[19]), we have

$$Fx(t) = \lim_{n\to\infty} Fx_n(t) = a(t) + \lambda \int_0^t k_1\,(t,s)\,\lim_{n\to\infty} f(s,x_n(s))\,ds \int_0^t k_2\,(t,s)\,\lim_{n\to\infty} g(s,x_n(s))\,ds$$

and

$$Fx(t) \;=\; a(t)\; + \lambda \int_0^t k_1\,(t,s)\,f(s,x(s))\,ds \int_0^t k_2\,(t,s)\,g(s,x(s))\,ds.$$

Then $Fx_n(t) \to Fx(t)$. Which means that the operator F is continuous.

Since all conditions of Schauder fixed point Theorem (see[18]) are satisfied, then the operator F has at least one fixed point $x \in C[0,T]$, which completes the proof. □

Now let $k_1(t,s) = k_2(t,s) = 1$ and $\lambda = 1$ in equation (5), then we have the following corollary;

COROLLARY 2.2. *Let the assumptions (i)-(ii) of Theorem 2.1 be satisfied, then the quadratic integral equation (2) has at least one continuous solution* $x \in C[0,T]$.

REMARK 2.3. Corollary 2.2 is the main result in [10]. This proves the generality of our result.

3. Existence of the maximal and minimal solutions

DEFINITION 3.1. Let $q(t)$ be a solution of the quadratic integral equation (5). Then $q(t)$ is said to be a maximal solution of (5) if every solution $x(t)$ of (5) satisfies the inequality (see[20]).

$$x(t) < q(t), \qquad t \in [0,T]. \tag{6}$$

A minimal solution $l(t)$ can be defined by similar way by reversing the above inequality i.e

$$x(t) > l(t), \qquad t \in [0,T]. \tag{7}$$

Consider the following lemma

LEMMA 3.2. *Let* $\lambda > 0$. *Let* $f(t,x)$, $g(t,x)$ *be* L^1*-Carathèodory and* x, y *are two continuous functions on* $[0,T]$ *satisfying*

$$x(t) \leqslant\; a(t)\; + \lambda \int_0^t k_1\,(t,s)\,f(s,x(s))ds \int_0^t k_2\,(t,s)\,g(s,x(s))ds, \qquad t \in [0,T]$$

$$y(t) \geqslant\; a(t)\; + \lambda \int_0^t k_1\,(t,s)\,f(s,y(s))\,ds \int_0^t k_2\,(t,s)\,g(s,y(s))\,ds, \quad t \in [0,T]$$

and one of them is strict.

If f, g *are monotonic nondecreasing in* x, *then*

$$x(t) < y(t), \quad t \in [0,T]. \tag{8}$$

Proof. Let the conclusion (8) be false, then there exists t_1 such that

$$x(t_1) = y(t_1), \quad t_1 > 0$$

and

$$x(t) < y(t), \quad 0 < t < t_1, t \in [0,T].$$

From the monotonicity of f, g in x, we get

$$\begin{aligned} x(t_1) &\leqslant a(t_1) + \lambda \int_0^{t_1} k_1\,(t_1,s)\, f(s,x(s))\, ds \int_0^{t_1} k_2\,(t_1,s)\, g(s,x(s))\, ds, \;\; t \in [0,T] \\ &< a(t_1) + \lambda \int_0^{t_1} k_1\,(t_1,s)\, f(s,y(s))\, ds \int_0^{t_1} k_2\,(t_1,s)\, g(s,y(s))\, ds, \;\; t \in [0,T] \\ x(t_1) &< y(t_1) \end{aligned}$$

which contradicts the fact that $x(t_1) = y(t_1)$.

Then

$$x(t) < y(t). \; \square$$

Now, for the existence of the continuous maximal and minimal solutions of the quadratic integral equation (5) we have the following theorem.

THEOREM 3.3. *Let the assumptions (i)-(iii) of Theorem 2.1 are satisfied. If $f(t,x)$ and $g(t,x)$ are monotonic nondecreasing in x for each $t \in [0,T]$, then the quadratic integral equation (5) has maximal and minimal positive continuous solutions.*

Proof. Firstly we shall prove the existence of the maximal solution of (5).

Let $\varepsilon > 0$ be given, and consider the quadratic integral equation

$$x_\varepsilon(t) = a(t) + \lambda \int_0^t k_1\,(t,s)\, f_\varepsilon(s,x_\varepsilon(s))\, ds \int_0^t k_2\,(t,s)\, g_\varepsilon(s,x_\varepsilon(s))\, ds, \;\; t \in [0,T] \quad (9)$$

where

$$f_\varepsilon(t,x_\varepsilon(t)) = f(t,x_\varepsilon(t)) + \varepsilon, \; g_\varepsilon(t,x_\varepsilon(t)) = g(t,x_\varepsilon(t)) + \varepsilon.$$

Clearly the function $f_\varepsilon(t,x_\varepsilon(t))$, $g_\varepsilon(t,x_\varepsilon(t))$ are L^1 - Carathèodory functions, therefore the equation (9) has a solution on $C[0,T]$.

Let ε_1, ε_2 be such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$, then

$$\begin{aligned} x_{\varepsilon_2}(t) &= a(t) + \lambda \int_0^t k_1\,(t,s)\, f_{\varepsilon_2}(s,x_{\varepsilon_2}(s))\, ds \int_0^t k_2\,(t,s)\, g_{\varepsilon_2}(s,x_{\varepsilon_2}(s))\, ds \\ &= a(t) + \lambda \int_0^t k_1(t,s)\, (f(s,x_{\varepsilon_2}(s)) + \varepsilon_2)\, ds \int_0^t k_2(t,s)\, (g(s,x_{\varepsilon_2}(s)) + \varepsilon_2)\, ds \end{aligned} \quad (10)$$

also

$$x_{\varepsilon_1}(t) = a(t) + \lambda \int_0^t k_1(t,s)\, f_{\varepsilon_1}(s,x_{\varepsilon_1}(s))\, ds \int_0^t k_2(t,s)\, g_{\varepsilon_1}(s,x_{\varepsilon_1}(s))\, ds$$

$$= a(t)+\lambda \int_0^t k_1(t,s)\ (f(s,x_{\varepsilon_1}(s))+\varepsilon_1)\ ds \int_0^t k_2(t,s)\ (g(s,x_{\varepsilon_1}(s))+\varepsilon_1)\ ds$$

$$x_{\varepsilon_1}(t) > a(t)+\lambda \int_0^t k_1\ (t,s)\ (f(s,x_{\varepsilon_1}(s))+\varepsilon_2)\ ds \int_0^t k_2(t,s)\ (g(s,x_{\varepsilon_1}(s))+\varepsilon_2)\ ds \quad (11)$$

Applying Lemma 3.2 to (10) and (11) we have $x_{\varepsilon_2}(t) < x_{\varepsilon_1}(t), \quad t \in [0,T]$.

As shown before, the family of functions $x_\varepsilon(t)$ is equi-continuous and uniformly bounded. Hence, by Arzela-Ascoli Theorem (see[19]), there exists a decreasing sequence ε_n such that $\varepsilon_n \to 0$ an $n \to \infty$, and $\lim_{n\to\infty} x_{\varepsilon_n}(t)$ exists uniformly in $[0,T]$ and denote the limit by $q(t)$. From the continuity of the functions $f_\varepsilon(t,x_\varepsilon(t)), g_\varepsilon(t,x_\varepsilon(t))$ in the second argument, we get

$$f_\varepsilon(t,x_\varepsilon(t)) \to f(t,q(t)) \qquad \text{as} \quad n \to \infty$$

$$g_\varepsilon(t,x_\varepsilon(t)) \to g(t,q(t)) \qquad \text{as} \quad n \to \infty$$

and

$$q(t) = \lim_{n\to\infty} x_{\varepsilon_n}(t) = a(t) + \lambda \int_0^t k_1(t,s) f(s,q(s))ds \int_0^t k_2(t,s) g(s,q(s))ds$$

which implies that $q(t)$ is a solution of the quadratic integral equation (5).

Finally we shall show that $q(t)$ is the maximal solution of (5). To do this let $x(t)$ be any solution of (5), then

$$x(t) \ = \ a(t)+\lambda \int_0^t k_1(t,s) f(s,x(s))ds \int_0^t k_2(t,s) g(s,x(s))ds \qquad (12)$$

also

$$x_\varepsilon(t) \ = \ a(t)+\lambda \int_0^t k_1(t,s) f_\varepsilon(s,x_\varepsilon(s))ds \int_0^t k_2(t,s) g_\varepsilon(s,x_\varepsilon(s))\ ds$$

$$x_\varepsilon(t) \ = \ a(t)+\lambda \int_0^t k_1(t,s)(f(s,x_\varepsilon(s))+\varepsilon)ds \int_0^t k_2(t,s) g((s,x_\varepsilon(s))+\varepsilon)\ ds$$

$$x_\varepsilon(t) > \ a(t)+\lambda \int_0^t k_1(t,s) f(s,x_\varepsilon(s))ds \int_0^t k_2(t,s) g(s,x_\varepsilon(s))\ ds \qquad (13)$$

Applying Lemma 3.2 to (12) and (13) we get

$$x(t) \ < \ x_\varepsilon(t), \qquad \text{for } t \in \ [0,T].$$

From the uniqueness of the maximal solution (see[20]), it is clear that $x_\varepsilon(t)$ tends to $q(t)$ uniformly in $[0,T]$ as $\varepsilon \to 0$.

By similar way we can prove the existence of the minimal solution. □

Now let $k_1(t,s) = k_2(t,s) = 1$ and $\lambda = 1$ in equation (5), then we have the following corollary;

COROLLARY 3.4. *Let the assumptions of Theorem (2.1) be satisfied, then the quadratic integral equation*

$$x(t) \ = \ a(t) \ + \int_0^t f(s,x(s)) \ ds \int_0^t g(s,x(s)) \ ds.$$

has a maximal and minimal positive continuous solutions, which is the same result obtained in [10].

4. Remarks

The quadratic integral equation of fractional order (3) has been studied in [12]. The authors proved the existence of at least one positive solution $x \in C[0,T]$ of (3) under the following assumptions;

(i) $a : I = [0,T] \to R_+$ is continuous function.

(ii) $f,g : I \times R_+ \to R_+$ such that f,g are measurable in t for all $x \in R_+$ and continuous in x for each fixed $t \in [0,T]$, and there exist two functions m_1, $m_2 \in L^1(I)$ such that

$$|\ f(t,x)\ | \ \leqslant \ m_1(t) \quad \text{and} \quad |\ g(t,x)\ | \ \leqslant \ m_2(t)$$

Also they proved the existence of the maximal and minimal solutions when $f(t,x)$ and $g(t,x)$ are monotonic nondecreasing in x for each $t \in [0,T]$.

It must be noticed that the quadratic integral equation (3) is a spacial case of the quadratic integral equation (5), with

$$k_1(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \quad \text{and} \quad k_2(t,s) = \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}, \qquad \alpha,\ \beta \in (0,1). \tag{14}$$

But in [12] the two kernel k_1 and k_2 give by (14) are not continuous.

This shows that our assumption (iii) of Theorem 2.1 is sufficient condition only. □

Now,let us give an example which illustrate the main results in Theorems 2.1, but such an example that is not covered by the main results of [12] and [14].

EXAMPLE 4.1. Consider the quadratic integral equation

$$x(t) = 1 + \int_0^t \frac{t}{t+s}\left(s + \frac{|x(s)|}{1+|x(s)|}\right) ds \int_0^t \frac{t}{t+s}\left(\frac{|x(s)|}{1+3|x(s)|}\right) ds, \ \ t \in [0,1]. \tag{15}$$

where

$$k_1(t,s) = k_2(t,s) = \frac{t}{t+s}$$

is the well known Chandrasekhar kernel.

$$f(t,x) = t + \frac{|x(t)|}{1+|x(t)|}, \qquad g(t,x) = \frac{|x(t)|}{1+3|x(t)|}$$

We can easily verify that f,g,k_1 and k_2 satisfy all the assumptions of Theorem 2.1.

Acknowledgements. The authors are thankful to the referee for his/ her useful comments and suggestions that helps to improve this manuscript.

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(Received July 23, 2013)
(Revised December 17, 2013)

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