Differential
Equations
& Applications
Volume 6, Number 1 (2014), 59–72
doi:10.7153/dea-06-04

EXISTENCE RESULT OF POSITIVE SOLUTION FOR BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER WITH INTEGRO–DIFFERENTIAL BOUNDARY CONDITIONS

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(*Communicated by Sotiris K. Ntouyas*)

Abstract. In this paper we study the following fractional boundary value problem with integro-differential boundary conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) - f(t,u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{1-\alpha}u(t)) = 0,\ t \in [0,T],\ n-1 \leqslant \alpha < n, \\ u^{(j)}(0) = 0,\ D_{0+}^{\alpha-1}u(T) + \int_0^T u(\omega)d\omega + \sum_{i=1}^{m-2} \beta_i u(\xi_i) = 0,\ j = 0,\dots,n-2, \\ 0 < \xi_i < \xi_{i+1} < T,\ \beta_i \in [0,\infty),\ i = 1,2,\dots,m-2,\ n \in \mathbb{N}\setminus\{1\},\ T > 0, \end{cases}$$

where $D_{0+}^{\alpha}, D_{0+}^{\alpha-1}$ represent the standard Riemann-Liouville fractional derivative of order α. The main result includes some interesting fixed point and functional analysis techniques to obtain claimed existence result.

1. Introduction

In the last two decades, differential equations of fractional order has introduced as a wonderful branch of fractional calculus that is applicable in sciences such as medicine, economy, basic sciences, engineering and so on (see the monographes [7], [10]). Also FDE's can describes many phenomenons in natural sciences significantly. Therefore many researchers interested to investigate about various applications of fractional differential equations. One of the most popular fields between this variety, is investigation about existence of positive solutions for boundary value problems of fractional order ([1]-[5], [9], [11]-[15]). Xiankui Zhao and Weigao Ge in [13], investigated the existence of an unbounded solution for the following fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t,u(t)) = 0,\ t \in (0,\infty),\ \alpha \in (1,2), \\ u(0) = 0, \quad \lim_{t\to\infty} D_{0+}^{\alpha-1}u(t) = \beta u(\xi),\ \ 0 < \xi < \infty, \end{cases}$$

where D_{0+}^{α} denotes the standard Riemann-Liouville fractional derivative of order α.

Mathematics subject classification (2010): fractional derivative, generalized boundary conditions, positive solutions, fixed point theory.

Keywords and phrases: 34A08, 34B18, 47H10.

Kazem Ghanbari and Yousef Gholami in [3] studied the existence of multiple positive solutions for the boundary value problem of fractional order

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda a(t)f(t,u(t))=0, \quad t\in(0,\infty),\ \alpha\in(2,3)\\ u(0)+u^{'}(0)=0, \quad \lim_{t\to\infty} D_{0+}^{\alpha-1}u(t)=\sum_{i=1}^{m-2}\beta_i u^{'}(\xi_i),\\ 0<\xi_1<\xi_2<\ldots<\xi_{m-2}<\infty,\ \beta_i\in\mathbb{R}^+\cup\{0\},\ i=1,2,\ldots,m-2, \end{cases}$$

where D_{0+}^{α} represent the fractional Riemann-Liouville derivative of order α.

Bashir Ahmad and Sotiris K. Ntouyas in [1], obtained existence of solution for the following multi strip fractional order boundary value problem

$$\begin{cases} {}^{c}D_{0+}^{q}x(t)=f(t,x(t)),\ t\in(0,1),\ m-1\leqslant q<m, m\in\mathbb{N}_{\geqslant 2},\\ x^{(j)}(0)=0, \quad x(1)=\sum_{i=1}^{n-2}\alpha_i\int_{\xi_i}^{\eta_i}x(s)ds,\ j=0,i,\ldots,m-2,\\ 0<\xi_i<\eta_i<1,\ i=1,2,\ldots,n-2, \end{cases}$$

such that ${}^{c}D_{0+}^{q}$ is the Caputo fractional derivative of order q.

Motivated by papers mentioned above, we consider the FBVP:

$$\begin{cases} D_{0+}^{\alpha}u(t)-f(t,u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{1-\alpha}u(t))=0,\ t\in[0,T], n-1\leqslant\alpha<n,\\ u^{(j)}(0)=0,\ D_{0+}^{\alpha-1}u(T)+\int_0^T u(\omega)d\omega+\sum_{i=1}^{m-2}\beta_i u(\xi_i)=0,\ j=0,\ldots,n-2,\\ 0<\xi_i<\xi_{i+1}<T,\ \beta_i\in[0,\infty), \quad i=1,2,\ldots,m-2,\ n\in\mathbb{N}\setminus\{1\}, \end{cases} \tag{1.1}$$

that equipped with fractional Riemann-Liouville derivatives D_{0+}^{α}, $D_{0+}^{\alpha-1}$.

In order to represent some sufficient conditions for existence of positive solutions for FBVP (1.1), assume that throughout this paper the following necessary condition hold:

(H) $f: C([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R}^+)$ and $f(t,0,0,0)$ dos not vanish identically on $[0,T]$.

2. Preliminaries

We will divide this section to two steps that, in first step we introduce some standard definitions and lemmas from fractional calculus and in second step, we represent some arguments and theorems from fixed point theory that will be needed in the next sections.

2.1. Step 1. Some Arguments About Fractional Calculus

DEFINITION 1. Assume that $u \in L^1(0,\infty)$. The fractional *Riemann – Liouville* primitive of order α for u is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds, \quad \alpha > 0.$$

DEFINITION 2. The fractional Riemann-Liouville derivative of order α for a given real valued function u on $(0,\infty)$ is defined by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)ds, \quad \alpha > 0,\ n = [\alpha]+1,$$

provided that the right hand side is point-wise defined on $(0,\infty)$.

DEFINITION 3. [7] Let $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$. The partial *Riemann-Liouville* fractional derivative with respect to k-th variable x_k for real valued function $u : \mathbb{R}^n \to \mathbb{R}^+$ is defined by

$$\frac{\partial_{0+}^{\alpha}}{\partial x_k} u(X) = \frac{1}{\Gamma(J-\alpha)} \left(\frac{\partial}{\partial x_k}\right)^J \int_0^{x_k} (x_k - s)^{J-\alpha-1} u(x_1, \dots, x_{k-1}, s, x_{k+1}, \dots, x_n)ds,$$

where $J = [\alpha]+1$.

LEMMA 1. [10] *Let* $\alpha > 0$.

(i) *If* $\mu > -1$, $\mu \neq \alpha - i$ *with* $i = 1, 2, ..., [\alpha]+1$ *and* $t > 0$, *then*

$$D_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.$$

(ii) $D_{0+}^{\alpha} t^{\alpha-i} = 0$, *for* $i = 1, 2, ..., [\alpha]+1$.

(iii) *If* $t \in (0,\infty)$, $u \in L^1(0,T)$, $I_{0+}^{\alpha} u \in C(0,T)$, $D_{0+}^{\alpha} u \in C(0,T) \cap L(0,T)$, *then*

$$D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t),\ I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-i}\ ,\ c_i \in \mathbb{R}, n = [\alpha]+1.$$

(iv) $D_{0+}^{\alpha} u(t) = 0$ *iff* $u(t) = \sum_{i=1}^{n} c_i t^{\alpha-i}$, $c_i \in \mathbb{R}$, $n = [\alpha]+1$.

LEMMA 2. *Suppose that* $h \in L^1[0,T]$. *Then the fractional boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) - h(t) = 0,\ t \in [0,T], n-1 \leqslant \alpha < n, n \in \mathbb{N} \setminus \{1\}, \\ u^{(j)}(0) = 0,\ D_{0+}^{\alpha-1} u(T) + \int_0^T u(\omega)d\omega + \sum_{i=1}^{m-2} \beta_i u(\xi_i) = 0,\ j = 0, ..., n-2, \\ 0 < \xi_i < \xi_{i+1} < T,\ \beta_i \in [0,\infty), \quad i = 1, 2, ..., m-2, \end{cases} \tag{2.1}$$

has the unique solution as

$$u(t) = \int_0^T K(t,s)h(s)ds, \tag{2.2}$$

where

$$K(t,s) = G(t,s) + \frac{1}{\Delta}\sum_{i=1}^{m-2} \beta_i t^{\alpha-1} G(\xi_i, s) + \frac{t^{\alpha-1}}{\Delta\Gamma(\alpha+1)}\{T^\alpha + (T-s)^\alpha\}, \tag{2.3}$$

such that

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} + (t-s)^{\alpha-1}; & 0 \leqslant s \leqslant t < T \\ t^{\alpha-1} & ; \quad 0 \leqslant t \leqslant s < T \end{cases} \tag{2.4}$$

and

$$\Delta = \Gamma(\alpha) + \frac{T^\alpha}{\alpha} + \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}. \tag{2.5}$$

Proof. Using Lemma 2.4, we can reduce fractional differential equation (2.1) as follows

$$u(t) = -c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - \dots - c_{n-1} t^{\alpha-n+1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.$$

Applying boundary conditions $u^{(j)}(0) = 0$ for $j = 0,1,\dots,n-2$, with simple calculating shows that $c_{n-1} = c_{n-2} = \dots = c_2 = 0$ respectively.

Implementing the last boundary condition

$$D_{0^+}^{\alpha-1} u(T) + \int_0^T u(\omega)d\omega + \sum_{i=1}^{m-2} \beta_i u(\xi_i) = 0,$$

we conclude that

$$c_1 = \frac{1}{\Delta}\Bigg[\sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^T h(s)ds + \int_0^T \int_0^\omega \frac{(\omega-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)dsd\omega\Bigg].$$

Thus we have

$$u(t) = \frac{t^{\alpha-1}}{\Delta}\Bigg[\int_0^T h(s)ds + \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \frac{(\xi^i - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^T \int_0^\omega \frac{(\omega-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)dsd\omega\Bigg] + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

$$
\begin{aligned}
&= \frac{t^{\alpha-1}}{\Delta}\left[\int_0^T h(s)ds + \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} \frac{(\xi^i - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right. \\
&\qquad\qquad\qquad\qquad \left. + \int_0^T \int_0^{\omega} \frac{(\omega - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)dsd\omega\right] \\
&\quad - \int_0^T \frac{t^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^T G(t,s)h(s)ds \\
&= \frac{1}{\Delta}\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}\int_0^T \frac{\xi_i^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{1}{\Delta}\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}\int_0^{\xi_i} \frac{(\xi^i - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\
&\quad + \frac{t^{\alpha-1}}{\Delta}\int_0^T \frac{T^\alpha}{\alpha\Gamma(\alpha)} h(s)ds + \frac{t^{\alpha-1}}{\Delta}\int_0^T\int_0^{\omega} \frac{(\omega - s)^{\alpha}}{\Gamma(\alpha)} h(s)dsd\omega \\
&\quad + \int_0^T G(t,s)h(s)ds \\
&= \int_0^T \left[G(t,s) + \frac{1}{\Delta}\sum_{i=1}^{m-2}\beta_i t^{\alpha-1} G(\xi_i, s)\right] h(s)ds \\
&\quad + \underbrace{\frac{t^{\alpha-1}}{\Delta}\int_0^T\int_0^{\omega} \frac{(\omega - s)^{\alpha}}{\Gamma(\alpha)} h(s)dsd\omega}_{Replacemaent\ Order\ of\ Variables} + \frac{t^{\alpha-1}}{\Delta}\int_0^T \frac{T^\alpha}{\alpha\Gamma(\alpha)} h(s)ds \\
&= \int_0^T \left[G(t,s) + \frac{t^{\alpha-1}}{\Delta}\left\{\sum_{i=1}^{m-2}\beta_i G(\xi_i,s) + \frac{1}{\Gamma(\alpha+1)}\{T^\alpha + (T-s)^\alpha\}\right\}\right] h(s)ds \\
&= \int_0^T K(t,s)h(s)ds.
\end{aligned}
$$

Unique coefficients $c_1, c_2, ..., c_{n-1}$ imply that, (2.2) is the unique solution of boundary value problem of fractional order (2.1). This completes the proof. □

LEMMA 3. *Let $p > 1$ be fixed. Considering $K(t,s)$, we claim that there exists a positive constant ρ such that*

$$
\begin{aligned}
&\min_{t\in[p,q]}\left\{K(t,s) + \frac{\partial_{0^+}^{\alpha-1}}{\partial t}K(t,s) + \frac{\partial_{0^+}^{1-\alpha}}{\partial t}K(t,s)\right\} \\
&\qquad \geqslant \rho \max_{t\in[0,T]}\left\{K(t,s) + \frac{\partial_{0^+}^{\alpha-1}}{\partial t}K(t,s) + \frac{\partial_{0^+}^{1-\alpha}}{\partial t}K(t,s)\right\}, \quad p>0, q<T. \qquad (2.6)
\end{aligned}
$$

Proof. Indeed considering Lemma 2.4 and replacement α with $-\alpha$ we can calculate the fractional primitive for power functions. Thus we have

$$
D_{0^+}^{\alpha-1} t^{\alpha-1} = \Gamma(\alpha), \qquad D_{0^+}^{1-\alpha} t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)} t^{2(\alpha-1)}. \qquad (2.7)
$$

Now by definition 3 and (2.4), we have

$$\frac{\partial_{0^+}^{\alpha-1}}{\partial t}G(t,s)=\begin{cases}2; & 0\leqslant s\leqslant t\leqslant T,\\ 1; & 0\leqslant t\leqslant s\leqslant T,\end{cases} \tag{2.8}$$

and

$$\frac{\partial_{0^+}^{1-\alpha}}{\partial t}G(t,s)=\frac{1}{\Gamma(2\alpha-1)}\begin{cases}t^{2(\alpha-1)}+(t-s)^{2(\alpha-1)}, \ 0\leqslant s\leqslant t\leqslant T,\\ t^{2(\alpha-1)}, \ 0\leqslant t\leqslant s\leqslant T.\end{cases} \tag{2.9}$$

Obviously we can observe that both of $G(t,s)$, $\frac{\partial_{0^+}^{1-\alpha}}{\partial t}G(t,s)$ are increasing with respect to first variable t and decreasing with respect to second variable s. Thus we have

$$\min_{[p,q]}G(t,s)=\frac{p^{\alpha-1}}{\Gamma(\alpha)},$$
$$\max_{t\in[0,T]}G(t,s)=\frac{1}{\Gamma(\alpha)}\left(T^{\alpha-1}+(T-s)^{\alpha-1}\right), \quad s\in[0,T].$$

Setting

$$\gamma(s)=\frac{\min_{[p,q]}G(t,s)}{\max_{t\in[0,T]}G(t,s)}=\frac{p^{\alpha-1}}{T^{\alpha-1}+(T-s)^{\alpha-1}}, \quad s\in[0,T],$$

and choosing $s=0$, we have

$$\gamma(s)\geqslant\rho_1=\left(\frac{p}{2T}\right)^{\alpha-1}. \tag{2.10}$$

Hence

$$\min_{t\in[p,q]}G(t,s)\geqslant\rho_1\max_{t\in[0,T]}G(t,s). \tag{2.11}$$

Similarly by direct calculation on $\frac{\partial_{0^+}^{1-\alpha}}{\partial t}G(t,s)$, we deduce that

$$\min_{t\in[p,q]}\frac{\partial_{0^+}^{1-\alpha}}{\partial t}G(t,s)\geqslant\rho_2\max_{t\in[0,T]}\frac{\partial_{0^+}^{1-\alpha}}{\partial t}G(t,s), \tag{2.12}$$

where

$$\rho_2=\left(\frac{p}{2T}\right)^{2(\alpha-1)}. \tag{2.13}$$

Now using (2.8),(2.11),(2.12), we conclude that

$$\begin{aligned}&\min_{t\in[p,q]}K(t,s)\geqslant\lambda_1\max_{t\in[0,T]}K(t,s),\\ &\min_{t\in[p,q]}\frac{\partial_{0^+}^{\alpha-1}}{\partial t}K(t,s)\geqslant\lambda_2\max_{t\in[0,T]}\frac{\partial_{0^+}^{\alpha-1}}{\partial t}K(t,s),\\ &\min_{t\in[p,q]}\frac{\partial_{0^+}^{1-\alpha}}{\partial t}K(t,s)\geqslant\lambda_3\max_{t\in[0,T]}\frac{\partial_{0^+}^{1-\alpha}}{\partial t}K(t,s),\end{aligned} \tag{2.14}$$

where

$$\lambda_1 = \rho_1 \left\{ 1 + \frac{\sum_{i=1}^{m-2} \beta_i}{\Delta} \left(\frac{p}{q}\right)^{\alpha-1} \right\} + \frac{T^{\alpha-1}}{\Delta\Gamma(\alpha+1)} \left(\frac{p}{q}\right)^{\alpha-1},$$

$$\lambda_2 = \frac{1}{2} + \frac{\Gamma(\alpha)}{\Delta} \left\{ \sum_{i=1}^{m-2} \beta_i \rho_1 + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right\}, \tag{2.15}$$

$$\lambda_3 = \rho_2 + \frac{\Gamma(\alpha)}{\Delta} \left(\frac{p}{q}\right)^{2(\alpha-1)} \left\{ \sum_{i=1}^{m-2} \beta_i \rho_1 + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right\}.$$

At last considering (2.15), we conclude that

$$\min_{t\in[p,q]} \left\{ K(t,s) + \frac{\partial_{0^+}^{\alpha-1}}{\partial t} K(t,s) + \frac{\partial_{0^+}^{1-\alpha}}{\partial t} K(t,s) \right\}$$
$$\geqslant \rho \max_{t\in[0,T]} \left\{ K(t,s) + \frac{\partial_{0^+}^{\alpha-1}}{\partial t} K(t,s) + \frac{\partial_{0^+}^{1-\alpha}}{\partial t} K(t,s) \right\},$$

where

$$\rho = \min\{\lambda_1, \lambda_2, \lambda_3\}.$$

The proof is complete. □

2.2. Step 2. Technical Requirements of Functional Analysis

In this step in order to design the desirable field for implementing our claimed results, firstly we proceed to build the relevant *Banach* space as follows

$$B = \left\{ u(t) \middle| \; u \in C[0,T], \; \sup_{t\in[0,T]} |u(t)| < \infty \right\},$$

$$E = \left\{ u(t) \in B \middle| \; D_{0^+}^{\alpha-1} u, D_{0^+}^{1-\alpha} u \in C[0,T], \; \sup_{t\in[0,T]} |D_{0^+}^{\alpha-1} u(t)|, \; \sup_{t\in[0,T]} |D_{0^+}^{1-\alpha} u(t)| < \infty \right\},$$

where E, B endowed with the norms $||u||_B, ||u||_E$ respectively as below

$$||u||_B = \sup_{t\in[0,T]} |u(t)|, \quad ||u||_E = ||u||_B + \sup_{t\in[0,T]} |D_{0^+}^{\alpha-1} u(t)| + \sup_{t\in[0,T]} |D_{0^+}^{1-\alpha} u(t)|.$$

$(B, ||.||_B)$ and $(E, ||.||_E)$ are *Banach* spaces and our favorite *Banach* space in this investigation is $(E, ||.||_E)$.

Now we introduce a nonvoid, closed convex subset of E as follows

$$C = \Big\{ u \in E \Big| \; u(t) \geqslant 0,$$
$$\min_{t\in[p,q]} \Big\{ u(t) + |D_{0^+}^{\alpha-1}u(t)| + |D_{0^+}^{1-\alpha}u(t)| \Big\} \geqslant \rho ||u||_E \Big\}. \quad (2.16)$$

Try to obtain positive solutions for FBVP (1.1), implies that why we introduced the cone C mentioned above.

We define the integral operator $A : C \to E$ by

$$A(u(t)) = \int_0^T K(t,s) f(s,u(s), D_{0^+}^{\alpha-1}u(s), D_{0^+}^{1-\alpha}u(s))ds, \quad t \in [0,T]. \quad (2.17)$$

DEFINITION 4. [6] Let Y be a normed space and suppose $S \subset Y$. A finite set of N balls $B(y_n, \varepsilon)$ with $y_n \in Y$ and $\varepsilon > 0$ is said to be a *finite* ε*-covering* of S, provided that every element of S lies inside one of the balls $B(y_n, \varepsilon)$, i.e.

$$S \subset \bigcup_{n=1}^{N} B(y_n, \varepsilon).$$

The set of centers $\{y_n\}$ of a finite ε-covering is called a *finite* ε*-net for* S.

DEFINITION 5. [6] Let Y be a normed space. A set $S \subset Y$ is said to be a *Totally Bounded* iff it has a *finite* ε*-covering* for every $\varepsilon > 0$.

THEOREM 1. [6] *Assume that Y be a normed space. A set $S \subset Y$ is compact iff it is closed and totally bounded.*

THEOREM 2. [8] *Let Y be a Banach space. Assume that $A : Y \longrightarrow Y$ is a completely continuous mapping. If $L : Y \longrightarrow Y$ be a linear bounded mapping such that* 1 *is not an eigenvalue of L and*

$$\lim_{||u|| \to \infty} \frac{||Au - Lu||}{||u||} = 0, \quad (2.18)$$

then A has a fixed point in Y.

3. Main Results

In the beginning of this section, first of all we shall show that if we consider the operator $A : C \longrightarrow C$ defined by (2.17), then $AC \subset C$. By (2.3)-(2.5) and (2.14), clearly we find that *Hammerstein* operator A is nonnegative. Also we have

$$\begin{aligned}\min_{t\in[p,q]}(Au)(t) &\geqslant \int_0^T \min_{\in[p,q]} K(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{1-\alpha}u(s))ds \\ &\geqslant \lambda_1 \int_0^T \max_{t\in[0,T]} K(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{1-\alpha}u(s))ds \\ &\geqslant \lambda_1 \max_{t\in[0,T]} \int_0^T K(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{1-\alpha}u(s))ds \\ &= \lambda_1 \max_{\in[0,T]}(Au)(t).\end{aligned} \tag{3.1}$$

Similarly the following results can be derived:

$$\min_{t\in[p,q]} D_{0^+}^{\alpha-1}(Au)(t) \geqslant \lambda_2 \max_{t\in[0,T]} D_{0^+}^{\alpha-1}(Au)(t), \tag{3.2}$$

and

$$\min_{t\in[p,q]} D_{0^+}^{1-\alpha}(Au)(t) \geqslant \lambda_3 \max_{t\in[0,T]} D_{0^+}^{1-\alpha}(Au)(t). \tag{3.3}$$

So therefore using (3.1)-(3.3), we have the following:

$$\min_{t\in[p,q]}\left\{(Au)(t)+\mid D_{0^+}^{\alpha-1}(Au)(t)\mid+\mid D_{0^+}^{1-\alpha}(Au)(t)\mid\right\} \geqslant \rho||Au||_E.$$

Thus we have proved that $AC \subset C$.

LEMMA 4. *Assume that condition* (H) *be satisfied. Then* $A: C \to C$ *is completely continuous.*

Proof. We know that the cone C is closed subset of E. Thus if we prove that C is totally bounded, according to Theorem 2.9 we can deduce that C is compact and consequently we can derive that because the integral operator A is continuous, hence A is completely continuous. So we perform above operations as below:
let recall the cone C,

$$C = \left\{u\in E\,\middle|\, u(t)\geqslant 0,\ t\in[0,T],\right. \\ \left.\min_{t\in[p,q]}\left\{u(t)+|D_{0^+}^{\alpha-1}u(t)|+|D_{0^+}^{1-\alpha}u(t)|\right\}\geqslant\rho||u||_E\right\}.$$

We want to prove that C is totally bounded. In order to prove this claim, consider the following:

$$\begin{aligned}E_u &= \{u\in E|u(t)\geqslant 0, t\in[0,T]\}, \\ E_{D_{0^+}^{\alpha-1}u} &= \left\{D_{0^+}^{\alpha-1}u \mid u\in E_u\right\}, \\ E_{D_{0^+}^{1-\alpha}u} &= \left\{D_{0^+}^{1-\alpha}u \mid u\in E_u\right\}.\end{aligned} \tag{3.4}$$

Clearly $E_u, E_{D_{0^+}^{\alpha-1}u}, E_{D_{0^+}^{1-\alpha}u}$ are three closed subsets of E. Thus all of them are complete spaces. Obviously $u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{1-\alpha}u(t)$ are equicontinuous on $[0,T]$. Hence using Arzela-Ascoli theorem we conclude that $E_u, E_{D_{0^+}^{\alpha-1}u}$ $, E_{D_{0^+}^{1-\alpha}u}$ are relatively compact. So observing the Hausdorff compactness criterion in Theorem 2.9, we deduce that $E_u, E_{D_{0^+}^{\alpha-1}u}, E_{D_{0^+}^{1-\alpha}u}$ are totally bounded.

Thus there exist three finite ε-coverings as

$$B_\varepsilon(u_i), B_\varepsilon\left(D_{0^+}^{\alpha-1}v_j\right), B_\varepsilon\left(D_{0^+}^{1-\alpha}w_k\right),\ i=1,...,l_1,\ j=1,...,l_2,\ k=1,...,l_3,$$

such that

$$\begin{aligned} E_u &\subset \bigcup_{i=1}^{l_1} B_\varepsilon(u_i), \\ E_{D_{0^+}^{\alpha-1}u} &\subset \bigcup_{j=1}^{l_2} B_\varepsilon\left(D_{0^+}^{\alpha-1}(v_j)\right), \\ E_{D_{0^+}^{1-\alpha}u} &\subset \bigcup_{k=1}^{l_3} B_\varepsilon\left(D_{0^+}^{1-\alpha}(w_k)\right), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} B_\varepsilon(u_i) &= \{u \in E_u \mid \|u-u_i\|_B < \varepsilon\}, \\ B_\varepsilon\left(D_{0^+}^{\alpha-1}(v_j)\right) &= \left\{D_{0^+}^{\alpha-1}u \in E_{D_{0^+}^{\alpha-1}u} \mid \|D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}v_j\|_B < \varepsilon\right\}, \\ B_\varepsilon\left(D_{0^+}^{1-\alpha}(w_k)\right) &= \left\{D_{0^+}^{1-\alpha}u \in E_{D_{0^+}^{1-\alpha}u} \mid \|D_{0^+}^{1-\alpha}u - D_{0^+}^{1-\alpha}w_k\|_B < \varepsilon\right\}. \end{aligned} \tag{3.6}$$

Now define

$$E_{ijk} = \left\{u \in E_u \;\middle|\; u \in B_\varepsilon(u_i), D_{0^+}^{\alpha-1}u \in B_\varepsilon\left(D_{0^+}^{\alpha-1}(v_j)\right), D_{0^+}^{1-\alpha}u \in B_\varepsilon\left(D_{0^+}^{1-\alpha}(w_k)\right)\right\}.$$

It is easy to see that $C \subset E_u \subset \bigcup_{1\leqslant i\leqslant l_1, 1\leqslant j\leqslant l_2, 1\leqslant k\leqslant l_3} E_{ijk}$. Indeed if we take $u_{ijk} \in E_{ijk}$, then E_u can be covered by finite 9ε-covering

$$B_{9\varepsilon}(u_{ijk}) = \left\{u \in E_u \;\middle|\; \|u-u_{ijk}\|_E < 9\varepsilon\right\}.$$

In other words for every $u \in C$ there exist indexes i,j,k such that

$$u \in B_\varepsilon(u_i)\ ,\ D_{0^+}^{\alpha-1}u \in B_\varepsilon\left(D_{0^+}^{\alpha-1}(v_j)\right)\ ,\ D_{0^+}^{1-\alpha}u \in B_\varepsilon\left(D_{0^+}^{1-\alpha}(w_k)\right).$$

Thus we have

$$|u-u_{ijk}| \leqslant |u-u_i| + |u_i - u_{ij}| + |u_{ij} - u_{ijk}| < \varepsilon+\varepsilon+\varepsilon = 3\varepsilon. \tag{3.7}$$

Similarly

$$\left| D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}u_{ijk} \right| < 3\varepsilon \quad , \quad \left| D_{0^+}^{1-\alpha}u - D_{0^+}^{1-\alpha}u_{ijk} \right| < 3\varepsilon. \tag{3.8}$$

Therefore relations (3.7),(3.8) ensure that $||u-u_{ijk}||_E < 9\varepsilon$. Hence we have managed to show that cone C has a finite 9ε-covering $B_{9\varepsilon}(u_{ijk})$. Thus C is totally bounded and closed. Hence according to Theorem2.9 we conclude that C is compact. On the other hand continuity of $D_{0^+}^{\alpha-1}u, D_{0^+}^{1-\alpha}u$ and *Lebesgue* dominated convergence theorem, imply that the Hammerstein operator A defined by (2.17) is continuous and $AC \subset C$. Thus operator A is completely continuous on C. This completes the proof. □

Moreover stipulation (H) we impose the following hypothesis;

$$\text{(S)} \quad \lim_{|u(t)|\to\infty} \frac{f(t,u(t),D_{0^+}^{\alpha-1}u(t),D_{0^+}^{1-\alpha}u(t))}{\frac{u(t)}{2}+aD_{0^+}^{\alpha-1}u(t)+bD_{0^+}^{1-\alpha}u(t)} = \psi(t),$$

where $a,b \in \mathbb{R}$ and

$$\frac{u(t)}{2}+aD_{0^+}^{\alpha-1}u(t)+bD_{0^+}^{1-\alpha}u(t) \neq 0 \;\text{ for } t \in [0,T]$$

and

$$a||D_{0^+}^{\alpha-1}u||+b||D_{0^+}^{1-\alpha}u|| \leqslant \frac{||u||}{2}, \qquad (||.|| \equiv ||.||_E).$$

THEOREM 3. *Let the hypothesis $(H),(S)$ be satisfied and assume that*

$$\int_0^T K(t,s)ds < ||\psi||^{-1}. \tag{3.9}$$

Then the FBVP (1.1) has a positive solution.

Proof. Let us define the linear bounded mapping $L: C \to C$ as follows;

$$Lu(t) = \int_0^T K(t,s)\left\{\frac{u(s)}{2}+aD_{0^+}^{\alpha-1}u(s)+bD_{0^+}^{1-\alpha}u(s)\right\}\psi(s)ds, \quad t\in[0,T].$$

Let take $\lambda = 1$ as an eigenvalue of operator L (i.e. $Lu = u$). As a result of (3.9), we have

$$\begin{aligned}
||Lu|| &\leqslant \int_0^T K(t,s)||\psi(s)||ds\frac{||u||}{2}+\int_0^T K(t,s)||\psi(s)||ds \\
&\qquad\qquad \times\left\{a||D_{0^+}^{\alpha-1}u||+b||D_{0^+}^{1-\alpha}u||\right\} \\
&\leqslant \int_0^T K(t,s)||\psi(s)||ds\frac{||u||}{2}+\int_0^T K(t,s)||\psi(s)||ds\frac{||u||}{2} \\
&< \frac{||u||}{2}+\frac{||u||}{2} = ||u||.
\end{aligned}$$

Thus we conclude that $||Lu|| < ||u||$, which is contradiction with this assumption that $\lambda = 1$ is an eigenvalue of L. Hence $\lambda = 1$ can not be an eigenvalue of L.

Considering standard definition of limit at infinity in (S) we deduce that for $\varepsilon > 0$, there exist positive constant M such that for every $||u|| > M$:

$$\left|\left| \frac{f(t,u(t),D_{0^+}^{\alpha-1}u(t),D_{0^+}^{1-\alpha}u(t))}{\frac{u(t)}{2}+aD_{0^+}^{\alpha-1}u(t)+bD_{0^+}^{1-\alpha}u(t)} - \psi(s) \right|\right| < \varepsilon, \quad s\in[0,T]. \tag{3.10}$$

Thus if $||u|| > M$, using (3.10) and (2.3) we have

$$\begin{aligned}
||Au-Lu|| &\leqslant \int_0^T K(t,s)\left|\left| f(t,u(t),D_{0^+}^{\alpha-1}u(t),D_{0^+}^{1-\alpha}u(t)) - \right.\right.\\
&\qquad \left.\left. \left\{\frac{u(t)}{2}+aD_{0^+}^{\alpha-1}u(t)+bD_{0^+}^{1-\alpha}u(t)\right\}\psi(s)\right|\right| ds\\
&< \left(\int_0^T K(t,s)ds\right)\varepsilon\left|\left|\frac{u(t)}{2}+aD_{0^+}^{\alpha-1}u(t)+bD_{0^+}^{1-\alpha}u(t)\right|\right|\\
&\leqslant \left(\int_0^T K(t,s)ds\right)\varepsilon||u||\\
&\leqslant \frac{2T^\alpha}{\Gamma(\alpha)}\left\{1+\frac{T^{\alpha-1}}{\Delta}\left\{\sum_{i=1}^{m-2}\beta i+\frac{1}{\alpha}\right\}\right\}\varepsilon||u||.
\end{aligned}$$

Hence (2.18) is verified. Then according to the Theorem 2.10, the Integral operator A has a fixed point in cone C, which this fixed point is equivalently positive solution of FBVP (1.1). The proof is complete. □

EXAMPLE 1. Let us consider the following FBVP:

$$\begin{cases} D_{0^+}^{\frac{3}{2}}u(t)-f(t,u(t),D_{0^+}^{\frac{1}{2}}u(t),D_{0^+}^{-\frac{1}{2}}u(t))=0,\ t\in[0,10^{-2}],\\ u(0)=0,\ D_{0^+}^{\alpha-1}u(t)\Big|_{t=10^{-2}}+\int_0^{10^{-2}}u(\omega)d\omega+\sum_{i=1}^{3}\beta_i u(\xi_i)=0, \end{cases} \tag{3.11}$$

where

$$m=5,\ T=10^{-2},\ \beta_1=10^{-1},\ \beta_2=10^{-2},\ \beta_3=10^{-3},$$
$$\xi_1=\frac{1}{64},\ \xi_2=\frac{1}{16},\ \xi_3=\frac{1}{4},$$

and

$$f(t,u(t),D_{0^+}^{\frac{1}{2}}u(t),D_{0^+}^{-\frac{1}{2}}u(t))=(1+t^2)\left\{\frac{u(t)}{2}+aD_{0^+}^{\frac{1}{2}}u(t)+bD_{0^+}^{-\frac{1}{2}}u(t)\right\}.$$

Applying (S), we deduce that $\psi(t) = (1+t^2)$. Direct calculation shows that $||\psi|| \approx 6.7593$. Equivalently we have

$$||\psi||^{-1} \approx 0.1479. \tag{3.12}$$

On the other hand, during the proof of Theorem 3.2, we used the following inequality:

$$\int_0^T K(t,s)ds \leqslant \frac{2T^\alpha}{\Gamma(\alpha)}\left\{1+\frac{T^{\alpha-1}}{\Delta}\left\{\sum_{i=1}^{m-2}\beta i+\frac{1}{\alpha}\right\}\right\}.$$

Simple computation implies that

$$\int_0^T K(t,s)ds \leqslant 0.0042. \tag{3.13}$$

Now comparing results (3.12) and (3.13), we have demonstrated that

$$\int_0^T K(t,s)ds < ||\psi||^{-1}.$$

Hence using Theorem 3.2, we conclude that FBVP (3.11) has a positive solution.

Acknowledgements. The author is very grateful to the anonymous reviewers for their valuable comments and suggestions that led to improvement of present article.

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(Received August 22, 2013)
(Revised November 14, 2013)

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