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ON THE UNIQUENESS OF WEAK SOLUTIONS FOR THE 3D PHASE FIELD NAVIER–STOKES VESICLE–FLUID INTERACTION MODEL

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Abstract. In this paper, we study a hydrodynamical system modeling the deformation of vesicle membrane in incompressible viscous fluids. In three dimensional case, we establish some uniqueness criteria of weak solutions for this system which reveal that the regularity of velocity field alone controls the uniqueness of weak solutions.

1. Introduction

Recently, there have been many experimental and mathematical studies focusing on the formation and dynamics of elastic vesicle membranes [1, 3, 19, 21, 24]. The single component vesicles are elastic membranes containing a liquid and surrounded by another liquid, which are possibly the simplest models for the biological cells and molecules. Such vesicles can be formed by certain amphiphilic molecules assembled in water to build bilayers, and exhibit a rich set of geometric structures in various mechanical, physical and biological environment [7, 20]. Their equilibrium shapes can be characterized by minimizing the following bending elastic energy of the membranes [11]:

$$E=\int_{\Gamma}\frac{k}{2}(H-c_0)^2dS, \tag{1.1}$$

where Γ is the surface of vesicle membrane, $H=\frac{k_1+k_2}{2}$ is the mean curvature of the membrane surface with k_1 and k_2 as the principal curvatures, c_0 is the spontaneous curvature which arises due to inhomogeneities in the bilayer lipid membrane structure, and k is the bending modulus of the vesicle membrane.

In order to model and understand the formation and dynamics of vesicle membranes and the fluid structure interaction, one approach is to consider equations of elasticity for the vesicle membranes and the Navier-Stokes equations for the fluid. However, the model established in this approach is very difficult to study and numerically

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simulate due to the fact that the description for elasticity is in Lagrangian coordinate (Hooke's law) and for fluids is in Eulerian coordinate. To overcome this difficulty, in [4, 6, 7], the authors established a phase field Navier-Stokes vesicle fluid interaction model for the vesicle shape dynamics in flow fields via the phase field approach. In this model, the vesicle membrane is described by a phase function ϕ, which is a labeling function defined on computational domain Q. The function ϕ takes value $+1$ inside of the vesicle membrane and -1 outside, with a thin transition layer of width characterized by a small (compared to the vesicle size) positive parameter ε. Obviously, the vesicle membrane Γ coincides with the zero level set $\{x:\phi(x)=0\}$. The convergence of the phase field model to the original sharp interface model as the transition width of the diffuse interface $\varepsilon\to 0$ has been carried out in [5]. On the other hand, the viscous fluid is modeled by the incompressible Navier-Stokes equations with unit density and with an external force defined in terms of ϕ.

As in [4], for simplicity, we assume that k is a positive constant and $c_0=0$. The elastic bending energy (1.1) will be approximated by a modified Willmore energy (cf. [7])

$$E_\varepsilon(\phi)=\frac{k}{2\varepsilon}\int_Q |f(\phi)|^2dx \ \text{ with } \ f(\phi)=-\varepsilon\Delta\phi+\frac{1}{\varepsilon}(\phi^2-1)\phi, \tag{1.2}$$

which depends on the interface transitional thickness ε. Moreover, in order to keep the total volume and the surface area of the vesicle membrane are conserved in time, two constraint functionals for the vesicle volume and surface area are prescribed by (cf. [7])

$$A(\phi)=\int_Q \phi\, dx,\ \ B(\phi)=\int_Q\Big(\frac{\varepsilon}{2}|\nabla\phi|^2+\frac{1}{4\varepsilon}(|\phi|^2-1)^2\Big)dx. \tag{1.3}$$

To enforce these constraints, two penalty terms were added to the elastic bending energy $E_\varepsilon(\phi)$, and the approximate elastic bending energy is given by (cf. [8, 9])

$$E(\phi)=E_\varepsilon(\phi)+\frac{1}{2}M_1(A(\phi)-\alpha)^2+\frac{1}{2}M_2(B(\phi)-\beta)^2, \tag{1.4}$$

where M_1 and M_2 are two penalty constants, $\alpha=A(\phi_0)$ and $\beta=B(\phi_0)$ are determined by the initial value of the phase function ϕ_0.

In this paper, we study the three dimensional phase field Navier-Stokes vesicle fluid interaction model with the periodic boundary conditions (i.e., in torus $\mathbb{T}^3$), which reads as follows:

$$\partial_t u+u\cdot\nabla u+\nabla\pi=\mu\Delta u+\frac{\delta E(\phi)}{\delta\phi}\nabla\phi \quad \text{in } [0,T]\times Q, \tag{1.5}$$

$$\nabla\cdot u=0 \quad \text{in } [0,T]\times Q, \tag{1.6}$$

$$\partial_t\phi+u\cdot\nabla\phi=-\gamma\frac{\delta E(\phi)}{\delta\phi} \quad \text{in } [0,T]\times Q \tag{1.7}$$

with the initial conditions

$$u(0,x)=u_0(x) \text{ with } \nabla\cdot u_0=0,\ \int_Q u_0dx=0 \text{ and } \ \phi(0,x)=\phi_0(x) \ \text{ in } x\in Q, \tag{1.8}$$

and the boundary conditions

$$u(t,x+e_i)=u(t,x), \quad \phi(t,x+e_i)=\phi(t,x) \text{ on } x\in[0,T]\times\partial Q, \tag{1.9}$$

where the set of vectors $\{e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)\}$ denotes an orthonormal basis of $\mathbb{R}^3$ and Q is the unit cube in $\mathbb{R}^3$. Here $u=(u_1,u_2,u_3)\in\mathbb{R}^3$ and $\pi\in\mathbb{R}$ denote, respectively, the velocity field and the pressure of the fluid, $\phi\in\mathbb{R}$ is the phase function of the vesicle membrane. $\frac{\delta E(\phi)}{\delta\phi}$ is the so-called chemical/physical potential that denotes the variational derivative of $E(\phi)$ in the variable ϕ. μ is the fluid viscosity which is assumed to be a positive constant throughout both fluid phases and the interface, and γ denotes the mobility coefficient which is assumed to be a small positive constant. It is easy to derive from (1.2)–(1.4) that if we denote

$$g(\phi)=-\Delta f(\phi)+\frac{1}{\varepsilon^2}(3\phi^2-1)f(\phi), \tag{1.10}$$

then

$$\frac{\delta E(\phi)}{\delta\phi}=kg(\phi)+M_1(A(\phi)-\alpha)+M_2(B(\phi)-\beta)f(\phi). \tag{1.11}$$

The system (1.5)–(1.7) describes the evolution of vesicle membranes immersed in an incompressible viscous fluid. Equations (1.5) and (1.6) are the momentum conservation equations and the mass conservation equations of a viscous fluid with unit density and with an external force caused by the phase field ϕ. Equation (1.6) is the condition of incompressibility. Equation (1.7) is a relaxed transport equation of ϕ with advection by the velocity field u. The right-hand side of (1.7) is a regularization term which ensures the consistent dissipation of energy. Roughly speaking, the system (1.5)–(1.7) is governed by the coupling of the hydrodynamic fluid flow and the bending elastic properties of the vesicle membrane. The resulting vesicle membrane configuration and the flow field reflect the competition and the coupling of the kinetic energy and membrane elastic energies.

Local and global well-posedness of the system (1.5)–(1.7) with the no-slip boundary condition for the velocity field u and the Dirichlet boundary condition for the phase field function ϕ have been studied in [4, 18]. In [4], by using the modified Galerkin argument, Du, Li and Liu proved global existence of weak solution, moreover, they also proved the weak solution is unique under an additionally regularity assumption $u\in L^8(0,T;L^4(Q))$. Similar results also hold for periodic boundary case, see Theorem 2.1 in [27]. However, as for the conventional Navier-Stokes equations, the question of regularity and uniqueness of weak solution of the system (1.5)–(1.9) in three dimensional space is still an outstanding open problem. For some regularity criteria of weak solutions, we refer the reader to see [27, 28]. In this paper, we are interested in finding sufficient conditions for weak solutions of the system (1.5)–(1.9) such that they become unique. Let us recall the definition of weak solution (for definitions of functional settings for periodic problems we refer the reader to see Section 2).

DEFINITION 1. Let $u_0 \in L^2_{per}(Q)$ and $\phi_0 \in H^2_{per}(Q)$ with $\nabla \cdot u_0 = 0$ and $\int_Q u_0 dx = 0$. A measurable pair of functions (u,ϕ) is called a weak solution of (1.5)–(1.9) on $(0,T)\times Q$ if it satisfies the following conditions:

(i) $u \in L^\infty(0,T;L^2_{per}(Q)) \cap L^2(0,T;H^1_{per}(Q))$ and $\phi \in L^\infty(0,T;H^2_{per}(Q)) \cap L^2(0,T;H^4_{per}(Q))$.

(ii) $\nabla \cdot u = 0$ in the sense of distribution.

(iii) For any $\eta \in C_0^{\infty(per)}([0,t]\times Q)$, $\nabla\cdot\eta = 0$, $\xi \in C_0^{\infty(per)}([0,t]\times Q)$ with $0 < t \leqslant T$, we have:

$$\int_0^t \int_Q \big(u\cdot\partial_t\eta - \mu\nabla u\nabla\eta + u\cdot\nabla\eta\cdot u\big)(\tau,x)dxd\tau$$
$$= -\int_0^t\int_Q \Big(\frac{\delta E(\phi)}{\delta\phi}\nabla\phi\cdot\eta\Big)(\tau,x)dxd\tau + \int_Q u(t,x)\eta(t,x)dx - \int_Q u_0(x)\eta(0,x)dx,$$

and

$$\int_0^t \int_Q \big(\phi\cdot\partial_t\xi + u\cdot\nabla\xi\cdot\phi\big)(\tau,x)dxd\tau$$
$$= \gamma\int_0^t\int_Q \Big(\frac{\delta E(\phi)}{\delta\phi}\cdot\xi\Big)(\tau,x)dxd\tau + \int_Q \phi(t,x)\xi(t,x)dx - \int_Q \phi_0(x)\xi(0,x)dx,$$

where $f \in C_0^{\infty(per)}([0,t]\times Q)$ means that $f \in C_0^\infty([0,t]\times Q)$ and $f(t,x+e_i) = f(t,x)$ for all $t\in[0,T]$.

(iv) $u(0,x) = u_0(x)$, $\phi(0,x) = \phi_0(x)$.

Since the Navier-Stokes equations is a subsystem of (1.5)–(1.9), one cannot expect better results than for the Navier-Stokes equations. For the three dimensional Navier-Stokes equations, Prodi [22] and Serrin [25] proved that uniqueness holds in the class

$$\mathscr{P} = L^q(0,T;L^p(\mathbb{R}^3)) \text{ with } \frac{3}{p}+\frac{2}{q} = 1,\ 3 < p \leqslant \infty.$$

Von Wahl [26] and Giga [14] improved this result in the class

$$\mathscr{P} = C([0,T],L^3(\mathbb{R}^3)).$$

Moreover, this last result was further extended in the limit case by Kozono and Sohr [15], and Escauriaza, Seregin and Šverák [10], who proved that uniqueness holds in the class

$$\mathscr{P} = L^\infty(0,T;L^3(\mathbb{R}^3)).$$

Some uniqueness criteria related to the Sobolev spaces we refer the reader to see [23]. Recently, many researches devoted to improving the above results. Kozono and Taniuchi [16] proved that uniqueness holds in the class

$$\mathscr{P} = L^2(0,T;BMO).$$

Gallagher and Planchon [12] proved that uniqueness holds in the class

$$\mathscr{P} = L^q(0,T;\dot{B}_{p,q}^{-1+3/p+2/q}(\mathbb{R}^3)) \text{ with } 2 \leqslant p < \infty, 2 < q < \infty \text{ and } \frac{3}{p} + \frac{2}{q} > 1.$$

Lemarié-Rieusset [17] and Germain [13] proved that uniqueness holds in the class

$$\mathscr{P} = C([0,T],X_1^{(0)}) \ \text{ or } \ \mathscr{P} = L^{2/(1-r)}(0,T;X_r) \ \text{ with } \ r \in [-1,1).$$

Finally, Chen, Miao and Zhang [2] improved the above results that uniqueness holds in the class $\mathscr{P} = L^q(0,T;B^s_{p,\infty}(\mathbb{R}^3))$ with

$$\frac{2}{q} + \frac{3}{p} = 1 + s, \frac{3}{1+s} < p \leqslant \infty, s \in (0,1] \text{ and } (p,s) \neq (\infty,1).$$

We refer the reader to see [13] and [17] for definitions of these function spaces.

Motivated by the above uniqueness criteria for weak solutions of the Navier-Stokes equations, the purpose of this paper is to consider uniqueness criteria of weak solutions for the system (1.5)–(1.9). The result indicates that the regularity of velocity field alone controls the uniqueness of weak solutions, and reveals that the velocity field u plays a more dominant role than that of the phase function ϕ in the uniqueness theory of weak solutions to the phase field Navier-Stokes vesicle-fluid interaction system (1.5)–(1.9).

Now we state the main result of this paper.

THEOREM 1. *Let* $(u_0, \phi_0) \in L^2_{per}(Q) \times H^2_{per}(Q)$ *with* $\nabla \cdot u_0 = 0$ *and* $\int_Q u_0 dx = 0$. *Let* (u_1, ϕ_1) *and* (u_2, ϕ_2) *be two weak solutions of the system* (1.5)–(1.9) *on* $(0,T)$ *with the same initial data* (u_0, ϕ_0). *If one of the following conditions holds for* $i = 1,2$*:*

$$(i) \ \ u_i \in L^q(0,T;L^p_{per}(Q)) \ \ \text{with } \frac{2}{q} + \frac{3}{p} = 1, \ 3 < p \leqslant \infty; \tag{1.12}$$

$$(ii) \ \ \nabla u_i \in L^q(0,T;L^p_{per}(Q)) \ \text{with } \frac{2}{q} + \frac{3}{p} = 2, \ \frac{3}{2} < p \leqslant \infty; \tag{1.13}$$

$$(iii) \ \ u_i \in C([0,T],L^3_{per}(Q)); \tag{1.14}$$

$$(iv) \ \ u_i \in L^2(0,T;BMO_{per}); \tag{1.15}$$

$$(v) \ \ u_i \in L^q(0,T;B^{s(per)}_{p,\infty}(Q)) \text{ with } \frac{2}{q} + \frac{3}{p} = 1 + s, \ \frac{3}{1+s} < p \leqslant \infty, \ s \in (0,1] \text{ and } (p,s) \neq (\infty,1), \tag{1.16}$$

then we have $u_1 = u_2$ *and* $\phi_1 = \phi_2$ *a.e. on* $(0,T) \times Q$.

The rest of this paper is organized as follows. In Section 2, we shall give the definitions of functional spaces for periodic problems used in this paper, then in Section 3, we shall present the proof of Theorem 1. Throughout the paper, we denote by C the generic constant which may depend on the coefficients of the system (1.5)–(1.9).

2. Preliminaries

In this section we shall recall some preliminaries on the Littlewood-Paley decomposition theory and the definitions of functional spaces for periodic problems. Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function and $\mathscr{S}'(\mathbb{R}^3)$ be its dual. Given $f \in \mathscr{S}(\mathbb{R}^3)$, we denote by $\mathscr{F}(f) = \widehat{f}$ the Fourier transform of f which is defined by

$$\mathscr{F}(f)(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{-ix\cdot\xi} dx.$$

Let

$$\mathscr{B} = \{\xi \in \mathbb{R}^3,\ |\xi| \leqslant \frac{4}{3}\} \quad \text{and} \quad \mathscr{C} = \{\xi \in \mathbb{R}^3,\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}.$$

Choose two nonnegative smooth radial functions $\chi, \varphi \in \mathscr{S}(\mathbb{R}^3)$ respectively supported on $\mathscr{B}$ and $\mathscr{C}$ which satisfy

$$\chi(\xi) + \sum_{j \geqslant 0} \varphi(2^{-j}\xi) = 1, \ \ \xi \in \mathbb{R}^3.$$

Let $h = \mathscr{F}^{-1}\varphi$ and $\tilde{h} = \mathscr{F}^{-1}\chi$, where $\mathscr{F}^{-1}$ is the inverse Fourier transform. Then we define the frequency localization operators Δ_j and S_j as follows:

$$\Delta_j f = \varphi(2^{-j}\mathrm{D})f = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x-y) dy \ \text{ for } \ j \geqslant 0,$$
$$S_j f = \chi(2^{-j}\mathrm{D})f = \sum_{-1 \leqslant k \leqslant j-1} \Delta_k f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) f(x-y) dy,$$

and

$$\Delta_{-1} f = S_0 f, \ \ \Delta_j f = 0 \ \text{ for } \ j \leqslant -2.$$

Here $\mathrm{D} = (\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3)$ and $\mathrm{D}_j = i^{-1}\partial_{x_j}$ $(i^2 = -1,\ j = 1,2,3)$. With the introduction of Δ_j and S_j, we have

$$f = S_0 f + \sum_{j \geqslant 0} \Delta_j f \ \text{ for } \ f \in \mathscr{S}'(\mathbb{R}^3),$$

which is called the Littlewood-Paley decomposition of f. Now let us recall the definition of the inhomogeneous Besov spaces.

DEFINITION 2. Let $s \in \mathbb{R}$, $1 \leqslant p, r \leqslant \infty$, inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^3)$ is defined by

$$B^s_{p,r}(\mathbb{R}^3) = \{f \in \mathscr{S}'(\mathbb{R}^3) : \ \ \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \left(\sum_{j=-1}^{\infty} 2^{jsr} \|\Delta_j f\|_{L^p}^r\right)^{1/r} & \text{for } \ 1 \leqslant r < \infty, \\ \sup_{j \geqslant -1} 2^{js} \|\Delta_j f\|_{L^p} & \text{for } \ r = \infty. \end{cases}$$

Let us point out that $B^s_{2,2}(\mathbb{R}^3)$ is the usual inhomogeneous Sobolev space $H^s(\mathbb{R}^3)$ which is endowed with the usual norm

$$\|f\|_{H^s} = \|(-\Delta)^{s/2} f\|_{L^2} + \|f\|_{L^2}.$$

Now we introduce some well-established functional settings for periodic problems: for $1 \leqslant p \leqslant \infty$, we denote

$$L^p_{per}(Q) = \{u \in L^p(\mathbb{R}^3) \mid u(x+e_i) = u(x)\}$$

endowed with the usual norm $\|\cdot\|_{L^p}$. For an integer $m > 0$, we denote

$$H^m_{per}(Q) = \{u \in H^m(\mathbb{R}^3) \mid u(x+e_i) = u(x)\}$$

endowed with the usual norm $\|\cdot\|_{H^m}$. For $s \in \mathbb{R}$ and $(p,r) \in [1,\infty] \times [1,\infty]$, we denote

$$B^{s(per)}_{p,r}(Q) = \{u \in B^s_{p,r}(\mathbb{R}^3) \mid u(x+e_i) = u(x)\}$$

associated with the usual norm $\|\cdot\|_{B^s_{p,r}}$. For the space of the Bounded Mean Oscillation BMO, which is defined as a set for locally $L^1(\mathbb{R}^3)$ function u such that

$$\|u\|_{BMO} = \sup_{R,x\in\mathbb{R}^3} \frac{1}{|B_R(x)|} \int_{B_R(x)} |u(y) - \bar{u}_{B_R}| dy < \infty,$$

where $\bar{u}_{B_R}$ stands for the average of u over the ball $B_R(x)$, i.e.,

$$\bar{u}_{B_R} = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y) dy.$$

We denote the corresponding space BMO_{per} for periodic problems by

$$BMO_{per} = \{u \in BMO \mid u(x+e_i) = u(x)\}$$

associated with the usual norm $\|\cdot\|_{BMO}$.

Before ending this section, we recall the following result from [27] which reveals that the average of the velocity field u is conserved.

LEMMA 1. (Lemma 2.1 in [27]) *Let* (u,ϕ) *be a weak solution of the system* (1.5)-(1.9) *on* $[0,T]$. *Then*

$$\int_Q u(t,x)dx = \int_Q u_0(x)dx = 0 \text{ for all } t \in [0,T]. \tag{2.1}$$

By using the well-known Poincaré-Wirtinger inequalities, we infer from Lemma 1 that

$$\|u\|_{H^1} \approx \|\nabla u\|_{L^2} \text{ and } \|u\|_{H^2} \approx \|\Delta u\|_{L^2}. \tag{2.2}$$

This combining the Sobolev embedding $H^1(Q) \hookrightarrow L^6(Q)$ implies that

$$\|u\|_{L^6} \leqslant C\|\nabla u\|_{L^2}. \tag{2.3}$$

This result will be used frequently in the proof of Theorem 1.

We emphasize here that, as the authors pointed out in [27], in the initial conditions (1.8), we have assumed that the average of the initial velocity field vanishes, i.e., $\int_Q u_0(x)dx = 0$. The advantage of this assumption is that one can apply the Poincaré-Wirtinger inequalities to the solution u such that the H^1-norm of u can be controlled by $\|\nabla u\|_{L^2}$. However, when a flow with non-vanishing average velocity field u is considered, we can introduce the new variable $\tilde{u} = u - \frac{1}{|Q|}\int_Q u(t,x)dx$ and transform the problem (1.5)–(1.9) into a new system in terms of $\tilde{u}$ and ϕ. Note that $\frac{1}{|Q|}\int_Q u(t,x)dx$ is a known constant determined by (2.1), it is not difficult to verify that our main result for the initial velocity with zero mean can be extended to this case with minor modifications.

3. The proof of Theorem 1

The idea of the proof of Theorem 1 comes from [4]. To simplify the proof, we introduce the following three notations:

$$G(\phi) = \frac{1}{2}\int_Q \left(k\varepsilon|\Delta\phi|^2 + \frac{k}{\varepsilon}|\nabla\phi|^2 + |\phi|^2\right)dx,$$
$$L(\phi) = \frac{\delta G(\phi)}{\delta\phi} = k\varepsilon\Delta^2\phi - \frac{k}{\varepsilon}\Delta\phi + \phi, \quad N(\phi) = \frac{\delta E(\phi)}{\delta\phi} - L(\phi).$$

It is clear that there exists a constant C depending only on k and ε such that

$$\frac{1}{C}\|\phi\|_{H^2}^2 \leqslant G(\phi) \leqslant C\|\phi\|_{H^2}^2.$$

Observe that $N(\phi)$ is the nonlinear term in $\frac{\delta E(\phi)}{\delta\phi}$ and is the main difficult term we shall deal with.

Assume that (u_1,ϕ_1) and (u_2,ϕ_2) are two weak solutions to the system (1.5)-(1.9) associated with the same initial data (u_0,ϕ_0), and assume that u_1 and u_2 satisfy one of the assumptions (1.12)–(1.16). Then there exist two functions π_1 and π_2 such that (u_1,π_1,ϕ_1) and (u_2,π_2,ϕ_2) satisfy the system (1.5)–(1.9). Set

$$\hat{u} = u_1 - u_2, \quad \hat{\pi} = \pi_1 - \pi_2, \quad \hat{\phi} = \phi_1 - \phi_2.$$

Due to (u_1,π_1,ϕ_1) and (u_2,π_2,ϕ_2) both are weak solutions, one obtains

$$\partial_t\hat{u} + \hat{u}\cdot\nabla u_1 + u_2\cdot\nabla\hat{u} - \mu\Delta\hat{u} + \nabla\hat{\pi} = (L(\phi_1) + N(\phi_1))\nabla\phi_1 - (L(\phi_2) + N(\phi_2))\nabla\phi_2, \tag{3.1}$$
$$\nabla\cdot\hat{u} = 0, \tag{3.2}$$
$$\partial_t\hat{\phi} + u_1\cdot\nabla\hat{\phi} + \hat{u}\cdot\nabla\phi_2 = -\gamma(L(\hat{\phi}) + N(\phi_1) - N(\phi_2)). \tag{3.3}$$

We first multiply (3.1) by $\hat{u}$, integrate over Q, after integration by parts, one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{u}\|_{L^2}^2+\mu\|\nabla\hat{u}\|_{L^2}^2\\ &=-\int_Q \hat{u}\cdot\nabla u_1\cdot\hat{u}dx+\int_Q(L(\phi_1)\nabla\phi_1-L(\phi_2)\nabla\phi_2)\cdot\hat{u}dx\\ &\qquad+\int_Q(N(\phi_1)\nabla\phi_1-N(\phi_2)\nabla\phi_2)\cdot\hat{u}dx, \end{aligned} \tag{3.4}$$

where we have used the fact $\int_Q u_2\cdot\nabla\hat{u}\cdot\hat{u}dx=0$ due to the periodicity of u_i $(i=1,2)$ and $\nabla\cdot u_2=0$.

Next, we multiply (3.3) by $L(\hat{\phi})$, integrate over Q, after integration by parts, we have

$$\begin{aligned}\frac{d}{dt}G(\hat{\phi})+\gamma\|L(\hat{\phi})\|_{L^2}^2\\ &=-\int_Q(u_1\cdot\nabla\hat{\phi}+\hat{u}\cdot\nabla\phi_2)L(\hat{\phi})dx-\gamma\int_Q(N(\phi_1)-N(\phi_2))L(\hat{\phi})dx. \end{aligned}\tag{3.5}$$

Adding (3.4) and (3.5) together, we conclude that

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\hat{u}\|_{L^2}^2+G(\hat{\phi})\right)+\mu\|\nabla\hat{u}\|_{L^2}^2+\gamma\|L(\hat{\phi})\|_{L^2}^2\\ &=-\int_Q\hat{u}\cdot\nabla u_1\cdot\hat{u}dx+\int_Q L(\phi_1)\nabla\hat{\phi}\cdot\hat{u}dx-\int_Q u_1\cdot\nabla\hat{\phi}L(\hat{\phi})dx\\ &\quad+\int_Q(N(\phi_1)\nabla\phi_1-N(\phi_2)\nabla\phi_2)\cdot\hat{u}dx-\gamma\int_Q(N(\phi_1)-N(\phi_2))L(\hat{\phi})dx. \end{aligned}\tag{3.6}$$

Let us derive the desired estimates for each terms appeared in the right-hand side of (3.6).

LEMMA 2. *Under the assumption* (1.12)*, we have*

$$\left|\int_Q\hat{u}\cdot\nabla u_1\cdot\hat{u}dx\right|\leqslant\frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2+C\|u_1\|_{L^p}^{2p/(p-3)}\|\hat{u}\|_{L^2}^2. \tag{3.7}$$

Proof. By using the divergence free condition $\nabla\cdot\hat{u}=0$, the periodicity of u_i $(i=1,2)$, the Hölder's inequality and Young's inequality, one obtains

$$\begin{aligned}\left|\int_Q\hat{u}\cdot\nabla u_1\cdot\hat{u}dx\right|&=\left|\int_Q\hat{u}\cdot\nabla\hat{u}\cdot u_1dx\right|\\ &\leqslant\|\hat{u}\|_{L^{2p/(p-2)}}\|\nabla\hat{u}\|_{L^2}\|u_1\|_{L^p}\\ &\leqslant C\|\hat{u}\|_{L^2}^{1-3/p}\|\nabla\hat{u}\|_{L^2}^{1+3/p}\|u_1\|_{L^p}\\ &\leqslant\frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2+C\|u_1\|_{L^p}^{2p/(p-3)}\|\hat{u}\|_{L^2}^2,\end{aligned}$$

where we have used the interpolation inequality

$$\|\hat{u}\|_{L^{2p/(p-2)}}\leqslant C\|\hat{u}\|_{L^2}^{1-3/p}\|\nabla\hat{u}\|_{L^2}^{3/p}.$$

LEMMA 3. *Under the assumption* (1.13), *we have*

$$\Big|\int_Q \hat{u}\cdot\nabla u_1\cdot\hat{u}dx\Big| \leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\|\nabla u_1\|_{L^p}^{2p/(2p-3)}\|\hat{u}\|_{L^2}^2. \tag{3.8}$$

Proof. By using the Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\Big|\int_Q \hat{u}\cdot\nabla u_1\cdot\hat{u}dx\Big| &\leqslant \|\nabla u_1\|_{L^p}\|\hat{u}\|_{L^{2p/(p-1)}}^2 \leqslant C\|\nabla u_1\|_{L^p}\|\hat{u}\|_{L^2}^{2-3/p}\|\nabla\hat{u}\|_{L^2}^{3/p}\\ &\leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\|\nabla u_1\|_{L^p}^{2p/(2p-3)}\|\hat{u}\|_{L^2}^2,\end{aligned}$$

where we have used the interpolation inequality

$$\|\hat{u}\|_{L^{2p/(p-1)}} \leqslant C\|\hat{u}\|_{L^2}^{1-3/(2p)}\|\nabla\hat{u}\|_{L^2}^{3/(2p)}.$$

LEMMA 4. *Under the assumption* (1.14), *we can split* u_1 *on* $[0,T]$ *as* $u_1 = u_{11} + u_{12}$ *such that* $u_{11} \in L^\infty(0,T;L^\infty_{per}(Q))$ *and* $\|u_{12}\|_{L^\infty(0,T;L^3)} < \frac{\mu}{12}$. *Moreover, we have*

$$\Big|\int_Q \hat{u}\cdot\nabla u_1\cdot\hat{u}dx\Big| \leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\|u_{11}\|_{L^\infty}^2\|\hat{u}\|_{L^2}^2. \tag{3.9}$$

Proof. The proof of this lemma is due to [26]. Since $u_1 \in C([0,T],L^3_{per}(Q))$, by the uniform continuity of u_1, we can choose N large enough such that

$$\Big\|u_1(t,x) - \sum_{k=0}^{N-1}\chi_{[\frac{k}{N}T,\frac{k+1}{N}T]}(t)u_1(\frac{k}{N}T,x)\Big\|_{L^\infty(0,T;L^3)} < \frac{\mu}{24},$$

where $\chi_{[a,b]}$ denotes the characteristic function on the interval $[a,b]$. Now we may approximate each $u_1(\frac{k}{N}T,\cdot)$ by a function $U_{k,N} \in L^\infty_{per}(Q)$ with an error controlled in L^3-norm by $\|u_1(\frac{k}{N}T,\cdot) - U_{k,N}(\cdot)\|_{L^3} < \frac{\mu}{24}$. Define

$$u_{11}(t,x) = \sum_{0\leqslant k\leqslant N-1}\chi_{[\frac{k}{N}T,\frac{k+1}{N}T]}(t)U_{k,N}(x)$$

and $u_{12} = u_1 - u_{11}$. Then it is clear that $u_{11} \in L^\infty(0,T;L^\infty_{per}(Q))$ and $\|u_{12}\|_{L^\infty(0,T;L^3)} < \frac{\mu}{12}$. Moreover, by using $\nabla\cdot\hat{u} = 0$, the periodicity of u_i $(i = 1,2)$ and (2.3), we have

$$\begin{aligned}\Big|\int_Q \hat{u}\cdot\nabla u_1\cdot\hat{u}dx\Big| &= \Big|\int_Q \hat{u}\cdot\nabla\hat{u}\cdot u_1 dx\Big|\\ &\leqslant \|u_{12}\|_{L^3}\|\hat{u}\|_{L^6}\|\nabla\hat{u}\|_{L^2} + \|u_{11}\|_{L^\infty}\|\nabla\hat{u}\|_{L^2}\|\hat{u}\|_{L^2}\\ &\leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\|u_{11}\|_{L^\infty}^2\|\hat{u}\|_{L^2}^2.\end{aligned}$$

This completes the proof of (3.9).

LEMMA 5. *Under the assumption* (1.15)*, we have*

$$\left| \int_Q \hat{u} \cdot \nabla u_1 \cdot \hat{u} dx \right| \leqslant \frac{\mu}{6} \|\nabla \hat{u}\|_{L^2}^2 + C \|u_1\|_{BMO}^2 \|\hat{u}\|_{L^2}^2. \tag{3.10}$$

Proof. The idea of the proof comes from [16]. Since $\nabla \cdot \hat{u} = 0$, it follows from [16] that

$$\hat{u} \cdot \nabla \hat{u} \in \mathscr{H}^1 \quad \text{and} \quad \|\hat{u} \cdot \nabla \hat{u}\|_{\mathscr{H}^1} \leqslant C \|\hat{u}\|_{L^2} \|\nabla \hat{u}\|_{L^2},$$

where $\mathscr{H}^1$ denotes the Hardy space. Recall that the dual space of Hardy space $\mathscr{H}^1$ is BMO, thus we have

$$\begin{aligned}
\left| \int_Q \hat{u} \cdot \nabla u_1 \cdot \hat{u} dx \right| &= \left| \int_Q \hat{u} \cdot \nabla \hat{u} \cdot u_1 dx \right| \\
&\leqslant C \|\hat{u} \cdot \nabla \hat{u}\|_{\mathscr{H}^1} \|u_1\|_{BMO} \\
&\leqslant C \|\hat{u}\|_{L^2} \|\nabla \hat{u}\|_{L^2} \|u_1\|_{BMO} \\
&\leqslant \frac{\mu}{6} \|\nabla \hat{u}\|_{L^2}^2 + C \|u_1\|_{BMO}^2 \|\hat{u}\|_{L^2}^2.
\end{aligned}$$

We complete the proof of Lemma 5.

LEMMA 6. *Under the assumption* (1.16)*, we can decompose u_1 as $u_1 = u_{13} + u_{14}$ such that $u_{13} \in L^1(0,T;Lip_{per})$ and $u_{14} \in L^{\tilde{q}}(0,T;L^{\tilde{p}}_{per}(Q))$ with some $\tilde{p}, \tilde{q}$ satisfying $\frac{2}{\tilde{q}} + \frac{3}{\tilde{p}} = 1$ and $\tilde{p} > 3$, where Lip_{per} denotes the periodic Lipschitz space which is a set of functions u such that $u \in L^\infty_{per}(Q)$ and $\nabla u \in L^\infty_{per}(Q)$. Moreover,*

$$\left| \int_Q \hat{u} \cdot \nabla u_1 \cdot \hat{u} dx \right| \leqslant \frac{\mu}{6} \|\nabla \hat{u}\|_{L^2}^2 + C(\|\nabla u_{13}\|_{L^\infty} + \|u_{14}\|_{L^{\tilde{p}}}^{\tilde{q}}) \|\hat{u}\|_{L^2}^2. \tag{3.11}$$

Proof. Inspired by [2], we set

$$u_{13} = S_N u_1, \quad u_{14} = u_1 - u_{13}.$$

Then by choosing

$$N = \left[\frac{q}{2} \log_2 (e + \|u\|_{B^s_{p,\infty}}) \right] + 1,$$

it is easy to prove that $u_{13} \in L^1(0,T;Lip_{per})$ and $u_{14} \in L^{\tilde{q}}(0,T;L^{\tilde{p}}_{per}(Q))$ for some $\tilde{p}, \tilde{q}$ satisfying $\frac{2}{\tilde{q}} + \frac{3}{\tilde{p}} = 1$ and $\tilde{p} > 3$, for details, see [2]. To prove (3.11), by using the facts $\nabla \cdot \hat{u} = 0$ and the periodicity of $\hat{u}$, and the Hölder's inequality, we obtain

$$\begin{aligned}
\left| \int_Q \hat{u} \cdot \nabla u_1 \cdot \hat{u} dx \right| &\leqslant \left| \int_Q \hat{u} \cdot \nabla u_{13} \cdot \hat{u} dx \right| + \left| \int_Q \hat{u} \cdot \nabla \hat{u} \cdot u_{14} dx \right| \\
&\leqslant \|\nabla u_{13}\|_{L^\infty} \|\hat{u}\|_{L^2}^2 + \|u_{14}\|_{L^{\tilde{p}}} \|\hat{u}\|_{L^{2\tilde{p}/(\tilde{p}-2)}} \|\nabla \hat{u}\|_{L^2} \\
&\leqslant \|\nabla u_{13}\|_{L^\infty} \|\hat{u}\|_{L^2}^2 + C \|u_{14}\|_{L^{\tilde{p}}} \|\hat{u}\|_{L^2}^{1-3/\tilde{p}} \|\nabla \hat{u}\|_{L^2}^{1+3/\tilde{p}} \\
&\leqslant \frac{\mu}{6} \|\nabla \hat{u}\|_{L^2}^2 + C(\|\nabla u_{13}\|_{L^\infty} + \|u_{14}\|_{L^{\tilde{p}}}^{\tilde{q}}) \|\hat{u}\|_{L^2}^2,
\end{aligned}$$

which completes the proof of (3.11).

The second and third terms on the right-hand side of (3.6) can be estimated as follows:

LEMMA 7. *Under the assumptions of Theorem 1, we have*

$$\Big|\int_Q L(\phi_1)\nabla\hat{\phi}\cdot\hat{u}dx\Big| \leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\|L(\phi_1)\|_{L^2}^2\|\hat{\phi}\|_{H^2}^2, \tag{3.12}$$

$$\Big|\int_Q u_1\cdot\nabla\hat{\phi}L(\hat{\phi})dx\Big| \leqslant \frac{\gamma}{4}\|L(\hat{\phi})\|_{L^2}^2 + C\|\nabla u_1\|_{L^2}^2\|\hat{\phi}\|_{H^2}^2. \tag{3.13}$$

Proof. By using the Hölder's inequality, Young's inequality and (2.3), it easy to derive that

$$\begin{aligned}\Big|\int_Q L(\phi_1)\nabla\hat{\phi}\cdot\hat{u}dx\Big| &\leqslant \|L(\phi_1)\|_{L^2}\|\nabla\hat{\phi}\|_{L^3}\|\hat{u}\|_{L^6} \leqslant C\|L(\phi_1)\|_{L^2}\|\hat{\phi}\|_{H^2}\|\nabla\hat{u}\|_{L^2}\\ &\leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\|L(\phi_1)\|_{L^2}^2\|\hat{\phi}\|_{H^2}^2\end{aligned}$$

and

$$\begin{aligned}\Big|\int_Q u_1\cdot\nabla\hat{\phi}L(\hat{\phi})dx\Big| &\leqslant \|u_1\|_{L^6}\|\nabla\hat{\phi}\|_{L^3}\|L(\hat{\phi})\|_{L^2}\\ &\leqslant \frac{\gamma}{4}\|L(\hat{\phi})\|_{L^2}^2 + C\|\nabla u_1\|_{L^2}^2\|\hat{\phi}\|_{H^2}^2.\end{aligned}$$

In order to estimate the last two terms on the right-hand side of (3.6), we need to establish the following estimate for the nonlinear term $N(\phi)$.

LEMMA 8. *Under the assumptions of Theorem 1, we have*

$$\|N(\phi_1) - N(\phi_2)\|_{L^2} \leqslant C\|\hat{\phi}\|_{H^2}, \tag{3.14}$$

where C is a constant depending only on the $\|\phi_i\|_{L^2(0,T;H^4(Q))}$, $\|\phi_i\|_{L^\infty(0,T;H^2(Q))}$ $(i = 1,2)$ and coefficients of the system.

Proof. By (1.10) and (1.11), it can be easily calculate that

$$\begin{aligned}N(\phi) = -\frac{k}{\varepsilon}\Delta(\phi^3) + \frac{2k}{\varepsilon}\Delta\phi - \phi + \frac{3k}{\varepsilon^2}\phi^2 f(\phi) - \frac{k}{\varepsilon^2}f(\phi)\\ + M_1(A(\phi)-\alpha) + M_2(B(\phi)-\beta)f(\phi).\end{aligned}$$

Hence, there exists a constant C such that

$$\begin{aligned}&\|N(\phi_1) - N(\phi_2)\|_{L^2}\\ &\quad\leqslant C\Big(\|\Delta(\phi_1^3) - \Delta(\phi_2^3)\|_{L^2} + \|\Delta\hat{\phi}\|_{L^2} + \|\hat{\phi}\|_{L^2} + \|f(\phi_1) - f(\phi_2)\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+\|\phi_1^2 f(\phi_1)-\phi_2^2 f(\phi_2)\|_{L^2}+\|A(\hat{\phi})\|_{L^2}+\|B(\phi_1)f(\phi_1)-B(\phi_2)f(\phi_2)\|_{L^2}\Big)\\
&\leqslant C\Big(\|\Delta(\phi_1^3)-\Delta(\phi_2^3)\|_{L^2}+\|f(\phi_1)-f(\phi_2)\|_{L^2}+\|\phi_1^2 f(\phi_1)-\phi_2^2 f(\phi_2)\|_{L^2}\\
&+\|A(\hat{\phi})\|_{L^2}+\|B(\phi_1)f(\phi_1)-B(\phi_2)f(\phi_2)\|_{L^2}\Big)+C\|\hat{\phi}\|_{H^2}\\
&=C\Big(\sum_{j=1}^{5}K_j\Big)+C\|\hat{\phi}\|_{H^2}.
\end{aligned}\tag{3.15}
$$

Since $\phi_i\in L^\infty(0,T;H^2(Q))$ $(i=1,2)$, $H^2(Q)\hookrightarrow L^\infty(Q)$ and $H^2(Q)$ is a Banach algebra, one can easily obtain that for $i,j=1,2$,

$$
\begin{aligned}
&\|\phi_i\|_{L^\infty(0,T;L^\infty(Q))}\leqslant C,\quad |\nabla\phi_i\|_{L^\infty(0,T;L^6(Q))}\leqslant C,\\
&\|\Delta\phi_i\|_{L^\infty(0,T;L^2(Q))}\leqslant C,\quad \|\phi_i\phi_j\|_{H^2}\leqslant C.
\end{aligned}
$$

Hence, we estimate K_1, K_2 and K_3 as follows:

$$
\begin{aligned}
K_1&=\|\Delta(\phi_1^3-\phi_2^3)\|_{L^2}=\|\Delta(\hat{\phi}(\phi_1^2+\phi_1\phi_2+\phi_2^2))\|_{L^2}\\
&\leqslant\|\Delta\hat{\phi}(\phi_1^2+\phi_1\phi_2+\phi_2^2)\|_{L^2}+\|\hat{\phi}\Delta(\phi_1^2+\phi_1\phi_2+\phi_2^2)\|_{L^2}\\
&\qquad+\|\nabla\hat{\phi}\nabla(\phi_1^2+\phi_1\phi_2+\phi_2^2)\|_{L^2}\\
&\leqslant\|\Delta\hat{\phi}\|_{L^2}\|\phi_1^2+\phi_1\phi_2+\phi_2^2\|_{L^\infty}+\|\hat{\phi}\|_{L^\infty}\|\Delta(\phi_1^2+\phi_1\phi_2+\phi_2^2)\|_{L^2}\\
&\qquad+\|\nabla\hat{\phi}\|_{L^6}\|\nabla(\phi_1^2+\phi_1\phi_2+\phi_2^2)\|_{L^6}\\
&\leqslant C\|\hat{\phi}\|_{H^2},\\
K_2&\leqslant C\big(\|\Delta\hat{\phi}\|_{L^2}+\|\hat{\phi}\|_{L^2}+\|\phi_1^3-\phi_2^3\|_{L^2}\big)\leqslant C\|\hat{\phi}\|_{H^2},\\
K_3&\leqslant\|(\phi_1^2-\phi_2^2)f(\phi_1)\|_{L^2}+\|\phi_2^2(f(\phi_1)-f(\phi_2))\|_{L^2}\\
&\leqslant\|\hat{\phi}\|_{L^\infty}\|\phi_1+\phi_2\|_{L^\infty}\|f(\phi_1)\|_{L^2}+\|\phi_2\|_{L^\infty}^2\|f(\phi_1)-f(\phi_2)\|_{L^2}\\
&\leqslant C\|\hat{\phi}\|_{H^2}.
\end{aligned}
$$

Since $A(\phi)$ and $B(\phi)$ are functions depending only on time, by (1.3), we can estimate K_4 and K_5 as follows:

$$
\begin{aligned}
K_4&\leqslant C\|\hat{\phi}\|_{L^1}\leqslant C\|\hat{\phi}\|_{H^2},\\
K_5&\leqslant\|B(\phi_1)(f(\phi_1)-f(\phi_2))\|_{L^2}+\|(B(\phi_1)-B(\phi_2))f(\phi_2)\|_{L^2}\\
&\leqslant|B(\phi_1)|\|(f(\phi_1)-f(\phi_2))\|_{L^2}+\|f(\phi_2)\|_{L^2}|B(\phi_1)-B(\phi_2)|\\
&\leqslant C(\|\nabla\phi_1\|_{L^2}^2+\|\phi_1^2-1\|_{L^2}^2)\|\hat{\phi}\|_{H^2}+C|B(\phi_1)-B(\phi_2)|\\
&\leqslant C\|\hat{\phi}\|_{H^2}.
\end{aligned}
$$

Combining the above estimates, we complete the proof of Lemma 8.

Now we can establish the estimates for the last two terms on the right-hand side of (3.6).

LEMMA 9. *Under the assumptions of Theorem 1, we have*

$$\Big|\int_Q (N(\phi_1)\nabla\phi_1 - N(\phi_2)\nabla\phi_2)\cdot\hat{u}dx\Big| \leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\big(\|N(\phi_1)\|_{L^2}^2 + \|\phi_2\|_{H^2}^2\big)\|\hat{\phi}\|_{H^2}^2, \tag{3.16}$$

and

$$\Big|\gamma\int_Q (N(\phi_1)-N(\phi_2))L(\hat{\phi})dx\Big| \leqslant \frac{\gamma}{4}\|L(\hat{\phi})\|_{L^2}^2 + C\|\hat{\phi}\|_{H^2}^2. \tag{3.17}$$

Proof. It follows from Lemma 8 and (2.3) that

$$\begin{aligned}
&\Big|\int_Q (N(\phi_1)\nabla\phi_1 - N(\phi_2)\nabla\phi_2)\cdot\hat{u}dx\Big| \\
&\leqslant \Big|\int_Q N(\phi_1)\nabla\hat{\phi}\cdot\hat{u}dx\Big| + \Big|\int_Q (N(\phi_1)-N(\phi_2))\nabla\phi_2\cdot\hat{u}dx\Big| \\
&\leqslant \|N(\phi_1)\|_{L^2}\|\nabla\hat{\phi}\|_{L^3}\|\hat{u}\|_{L^6} + \|N(\phi_1)-N(\phi_2)\|_{L^2}\|\nabla\phi_2\|_{L^3}\|\hat{u}\|_{L^6} \\
&\leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C\|N(\phi_1)\|_{L^2}^2\|\hat{\phi}\|_{H^2}^2 + C\|\phi_2\|_{H^2}^2\|N(\phi_1)-N(\phi_2)\|_{L^2}^2 \\
&\leqslant \frac{\mu}{6}\|\nabla\hat{u}\|_{L^2}^2 + C(\|N(\phi_1)\|_{L^2}^2 + \|\phi_2\|_{H^2}^2)\|\hat{\phi}\|_{H^2}^2.
\end{aligned}$$

This completes the proof of (3.16). The proof of (3.17) is obvious.

Finally, we complete the proof of Theorem 1 through the following five cases.

Case 1 Under the assumption (1.12), putting Lemmas 2, 7 and 9 together and noticing that $\|\hat{\phi}\|_{H^2}^2 \leqslant CG(\hat{\phi})$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\hat{u}\|_{L^2}^2 + 2G(\hat{\phi})\Big) + \mu\|\nabla\hat{u}\|_{L^2}^2 + \gamma\|L(\hat{\phi})\|_{L^2}^2 \\
&\leqslant C\Big(\|\hat{u}\|_{L^2}^2 + 2G(\hat{\phi})\Big) \\
&\qquad \times\Big(\|u_1\|_{L^p}^q + \|\nabla u_1\|_{L^2}^2 + \|L(\phi_1)\|_{L^2}^2 + \|N(\phi_1)\|_{L^2}^2 + 1\Big).
\end{aligned} \tag{3.18}$$

It is easy to verify that

$$\Big(\|u_1\|_{L^p}^q + \|\nabla u_1\|_{L^2}^2 + \|L(\phi_1)\|_{L^2}^2 + \|N(\phi_1)\|_{L^2}^2 + 1\Big)$$

is integrable in time. Moreover, due to $\hat{u}(0,x) = 0$ and $\hat{\phi}(0,x) = 0$, it follows from Gronwall's inequality that $\hat{u} = 0$ and $\hat{\phi} = 0$ for a.e. $t \in (0,T)$.

Case 2 Under the assumption (1.13), putting Lemmas 3, 7 and 9 together, we obtain

$$\frac{d}{dt}\Big(\|\hat{u}\|_{L^2}^2 + 2G(\hat{\phi})\Big) + \mu\|\nabla\hat{u}\|_{L^2}^2 + \gamma\|L(\hat{\phi})\|_{L^2}^2$$

$$\leqslant C\left(\|\hat{u}\|_{L^2}^2+2G(\hat{\phi})\right)\times\left(\|\nabla u_1\|_{L^p}^q+\|\nabla u_1\|_{L^2}^2+\|L(\phi_1)\|_{L^2}^2+\|N(\phi_1)\|_{L^2}^2+1\right). \quad (3.19)$$

Similarly, by applying Gronwall's inequality to (3.19), it follows from $\hat{u}(0,x)=0$ and $\hat{\phi}(0,x)=0$ that $\hat{u}=0$ and $\hat{\phi}=0$ for a.e. $t\in(0,T)$.

Case 3 Under the assumption (1.14), putting Lemmas 4, 7 and 9 together, we obtain

$$\frac{d}{dt}\left(\|\hat{u}\|_{L^2}^2+2G(\hat{\phi})\right)+\mu\|\nabla\hat{u}\|_{L^2}^2+\gamma\|L(\hat{\phi})\|_{L^2}^2 \leqslant C\left(\|\hat{u}\|_{L^2}^2+2G(\hat{\phi})\right)\times\left(\|u_{11}\|_{L^\infty}^2+\|\nabla u_1\|_{L^2}^2+\|L(\phi_1)\|_{L^2}^2+\|N(\phi_1)\|_{L^2}^2+1\right). \quad (3.20)$$

Proceeding the same proof as Case 1, we can show that $\hat{u}=0$ and $\hat{\phi}=0$ for a.e. $t\in(0,T)$.

Case 4 Under the assumption (1.15), putting Lemmas 5, 7 and 9 together, we obtain

$$\frac{d}{dt}\left(\|\hat{u}\|_{L^2}^2+2G(\hat{\phi})\right)+\mu\|\nabla\hat{u}\|_{L^2}^2+\gamma\|L(\hat{\phi})\|_{L^2}^2 \leqslant C\left(\|\hat{u}\|_{L^2}^2+2G(\hat{\phi})\right)\times\left(\|u_1\|_{BMO}^2+\|\nabla u_1\|_{L^2}^2+\|L(\phi_1)\|_{L^2}^2+\|N(\phi_1)\|_{L^2}^2+1\right). \quad (3.21)$$

Proceeding the same proof as Case 1, we can show that $\hat{u}=0$ and $\hat{\phi}=0$ for a.e. $t\in(0,T)$.

Case 5 Under the assumption (1.16), putting Lemmas 6, 7 and 9 together, we obtain

$$\frac{d}{dt}\left(\|\hat{u}\|_{L^2}^2+2G(\hat{\phi})\right)+\mu\|\nabla\hat{u}\|_{L^2}^2+\gamma\|L(\hat{\phi})\|_{L^2}^2 \leqslant C\left(\|\hat{u}\|_{L^2}^2+2G(\hat{\phi})\right)\cdot\left(\|\nabla u_{13}\|_{L^\infty}+\|u_{14}\|_{L^{\tilde{p}}}^{\tilde{q}}+\|\nabla u_1\|_{L^2}^2+\|L(\phi_1)\|_{L^2}^2+\|N(\phi_1)\|_{L^2}^2+1\right). \quad (3.22)$$

Proceeding the same proof as Case 1, we can show that $\hat{u}=0$ and $\hat{\phi}=0$ for a.e. $t\in(0,T)$.

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