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SECOND–ORDER DIFFERENTIAL EQUATIONS: SOME SIGNIFICANT RESULTS DUE TO JAMES S.W. WONG

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Abstract. Wong's contribution in the qualitative theory of second-order differential equations is well-known to a large mathematical auidence. Among a huge number of published Wong's papers, in this survey article, we analyze only a few Wong's theorems including their consequences, examples and many influences to other mathematicians dealing with oscillations of second-order differential and functional differential equations as well as of corresponding dynamic equations on time scales.

Contents

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1. Introduction

Among Wong's 151 scientific papers, we are focused here only to the following ones: Wong theorem [108, Theorem 1] - from 1999 (see Section 2), Sun and Wong theorems [92, Theorems 1 and 2] - from 2007 (see Section 3), and Wong theorems [109, Theorems 1 and 3] - from 2000 (see Section 4).

Each section, except this, contains: main result with examples, remarks, and comments, influences to related results by other authors and a review on several different results on these topics. Some examples are presented here in more general forms than their original ones.

Unless otherwise stated, we always assume that every solution $x = x(t)$ of any considered equation is smooth enough on $[t_0,\infty)$, that is, $x \in C([t_0,\infty),\mathbb{R}) \cap C^2((t_0,\infty),\mathbb{R})$, and as usual, a nontrivial solution $x = x(t)$ is said to be oscillatory (at $t = \infty$) if for any large enough $T > t_0$ there exists $t \geqslant T$ such that $x(t) = 0$.

2. A variational technique and Wong's interval oscillation criterion

In this section we consider the forced second-order linear differential equation:

$$\big(r(t)x'\big)' + q(t)x = e(t),\ t \geqslant t_0, \tag{2.1}$$

where $r(t)$ is continuously differentiable function, $q(t)$ and $e(t)$ are continuous functions. On the qualitative properties of solutions of equation (2.1) we refer the reader to some well-known books such as: Hartman [40, Chapters XI. and XIV.], Coppel [19, Chapter 1], Swanson [94, Chapters 1 and 2], Agarwal, Grace and O'Regan [2, Chapter 2], Amrein, Hinz and Pearson [5], etc. On the existence of a kind of continuable solutions of the forced equation (2.1), we refer reader, for instance, to Kelley and Peterson [48, Theorem 7.6] and references therein.

2.1. Main result - from 1999

James S.W. Wong in his paper [108] stated and proved the next interval oscillation criterion, that is called anywhere in this paper as *Wong's interval oscillation criterion*.

THEOREM 1. ([108, Theorem 1] - from 1999) *Assume $r(t) > 0$ and for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant a_1$, $b_1 \leqslant a_2$, such that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$. Let $D(a_i,b_i)$ be a set of functions defined by*

$$D(a_i,b_i) = \{u \in C^1([a_i,b_i]) : u(t) \not\equiv 0, u(a_i) = u(b_i) = 0\},\ i = 1,2. \tag{2.2}$$

If there exists a function $u \in D(a_i,b_i)$ such that

$$\int_{a_i}^{b_i} \big(q(t)u^2(t) - r(t)u'^2(t)\big)dt \geqslant 0,\ i = 1,2, \tag{2.3}$$

then equation (2.1) is oscillatory.

REMARK 1. Instead of "for any $T > 0$" we can always write "for any large enough $T > 0$" since both phrases give the same conclusion because of the asymptotic nature of oscillations at $t = \infty$. □

REMARK 2. The function $u \in D(a_i,b_i)$ can be replaced by two ones $u_i \in D(a_i,b_i)$, $i=1,2$ so that (2.3) is fulfilled with $u(t)=u_i(t)$, $i=1,2$. $\square$

In Section 2.2 below, some important results that preceded Wong's interval oscillation criterion will be discussed such as: Leighton and Hartman variational principles, and El-Sayed's and Nasr's interval oscillation criteria.

The variational technique that Wong developed in the proof of Theorem 1 is composed of three steps: the classic Riccati transformation of a nonoscillatory solution, a variational trick of testing of a differential inequality by a given function $u \in D(a_i,b_i)$, and finally, an integration by parts. In order to avoid any repetition from the original Wong's proof of Theorem 1, we recall this method in proving the next analogous theorem to Theorem 1 that is valid for the forced linear differential equation with damping term:

$$\big(r(t)x'\big)' + p(t)x' + q(t)x = e(t),\ t \geqslant t_0. \tag{2.4}$$

A rather complete review about the oscillation criteria for such a class of forced linear differential equations will be presented in the final subsection of this section.

THEOREM 2. *Let $r(t)>0$. Assume for any large enough $T>0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant a_1$, $b_1 \leqslant a_2$, such that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$. Let $D(a_i,b_i)$ be the set of functions define by (2.2). If there exists a function $u \in D(a_i,b_i)$, $i=1,2$, such that*

$$\int_{a_i}^{b_i} \left[q(t)u^2(t) - r(t)\left(u'(t) - \frac{p(t)}{2r(t)}u(t)\right)^2\right]dt > 0,\ i=1,2, \tag{2.5}$$

then equation (2.4) is oscillatory.

Proof. To the contrary, if equation (2.4) is not oscillatory, then there is a solution $x=x(t)$ and a $T \geqslant t_0$ such that $x(t) \neq 0$ for all $t \geqslant T$. We set $i=1$ if $x(t)>0$ and $i=2$ if $x(t)<0$. Since $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$, it is simple to show that the function

$$\omega(t) = -\frac{r(t)x'(t)}{x(t)} \quad \text{for } t > T \tag{2.6}$$

satisfies the Riccati-type differential inequality:

$$\omega' \geqslant \frac{1}{r(t)}\omega^2 - \frac{p(t)}{r(t)}\omega + q(t),\ t \in (a_i,b_i).$$

Multiplying this inequality by $u^2(t)$, where $u \in D(a_i,b_i)$, we obtain:

$$\int_{a_i}^{b_i} \omega'(t)u^2(t)dt \geqslant \int_{a_i}^{b_i} \frac{1}{r(t)}\omega^2(t)u^2(t)dt - \int_{a_i}^{b_i} \frac{p(t)}{r(t)}\omega(t)u^2(t)dt + \int_{a_i}^{b_i} q(t)u^2(t)dt.$$

Using the partial integration on the left hand side and $u(a_i)=u(b_i)=0$, we obtain:

$$\begin{aligned} 0 \geqslant \int_{a_i}^{b_i} &\left[\frac{1}{\sqrt{r(t)}}\omega(t)u(t) + \sqrt{r(t)}\left(u'(t) - \frac{p(t)}{2r(t)}u(t)\right)\right]^2 dt \\ &+ \int_{a_i}^{b_i} \left[q(t)u^2(t) - r(t)\left(u'(t) - \frac{p(t)}{2r(t)}u(t)\right)^2\right]dt, \end{aligned}$$

which contradicts assumption (2.5). Thus, equation (2.4) is oscillatory. □

Theorem 1 has been illustrated in [108, Example 1] by proving the oscillation of the equation

$$\big(\sqrt{t}x'\big)' + x = \sin(\sqrt{t}),\ t \geqslant 0. \tag{2.7}$$

In the next example, we show that Theorem 1 can be also applied in the study of oscillations of a one-parametric class of equations that contains the previous equation as a special case.

EXAMPLE 1. For any real parameter $\beta > 0$ we consider the equation

$$\big(t^{\frac{\beta-1}{\beta}}x'\big)' + t^{\frac{2-\beta}{\beta}}x = \sin(t^{\frac{1}{\beta}}),\ t > 0. \tag{2.8}$$

Obviously, this one-parametric class of equations contains equation (2.7) as particular case where $\beta = 2$. We claim that equation (2.8) is oscillatory. In fact, by using Theorem 1 for any large $T > 0$ and $e(t) = \sin(t^{\frac{1}{\beta}})$, it is enough to find numbers a_i, b_i and a function $u \in D(a_i, b_i)$ such that $e(t) \leqslant 0$ on $[a_1, b_1]$ and $e(t) \geqslant 0$ on $[a_2, b_2]$ and condition (2.3) is fulfilled. For any $T > 0$, there exists an $n \in \mathbb{N}$ such that $\big((2n-1)\pi\big)^{\beta} \geqslant T$. Let

$$a_1 = \big((2n-1)\pi\big)^{\beta},\ b_1 = a_2 = (2n\pi)^{\beta} \quad \text{and} \quad b_2 = \big((2n+1)\pi\big)^{\beta}.$$

Hence, for $u = \sin(t^{\frac{1}{\beta}})$, we have:

$$\int_{a_i}^{b_i} \big(q(t)u^2(t) - r(t)u'^2(t)\big)dt = \int_{a_i}^{b_i} \Big(t^{\frac{2-\beta}{\beta}} \sin^2(t^{\frac{1}{\beta}}) - \frac{1}{\beta^2} t^{\frac{\beta-1}{\beta}+\frac{2}{\beta}-2} \cos^2(t^{\frac{1}{\beta}})\Big)dt$$

$$= \frac{\beta\pi^2}{4}(4n-1) - \frac{\pi}{2\beta} > 0.$$

Thus, by Theorem 2 (or Theorem 1) equation (2.8) is oscillatory. □

2.2. Related results that preceded Wong's interval oscillation criterion - from 1962 to 1998

In this subsection, we present in the chronological order some results by other authors which preceded Wong's interval oscillation criterion: results due to Leighton - 1962, Hartman - 1964, Komkov - 1972, Butler, Erbe and Mingarelli - 1987, El-Sayed - 1993, and Nasr - 1998.

•1) In 1962 Walter Leighton in his paper [57] presented the following oscillation principle for unforced linear differential equations.

THEOREM 3. ([57, Theorem 1] - from 1962) *Let $e(t) \equiv 0$ and $D(a,b)$ be the set of functions define by* (2.2). *Assume there is a function $u \in D(a,b)$ such that assumption* (2.3) *holds with $a = a_i$, $b = b_i$. Then for any solution $x(t)$ of the unforced equation* (2.1) *such that $x(a) = 0$, there is a $c \in (a,b)$ such that $x(c) = 0$.*

The main limitations of this theorem are $e(t) \equiv 0$ and $x(a) = 0$.

•2) Philip Hartman in the original edition of his famous book in 1964 proved the following variational principle so that the unforced equation (2.1) is disconjugate on an interval J, that is, every nontrivial solution has at most one zero in J. In order to recall Hartman's result, let

$$A_1\big((a,b),\mathbb{R}^N\big) = \Big\{u : [a,b] \to \mathbb{R}^N \mid u(a) = u(b) = 0,\ u \in \mathrm{AC}\big([a,b],\mathbb{R}^N\big),\ u' \in L^2\big((a,b),\mathbb{R}^N\big)\Big\},$$

where $N \geqslant 1$, $\mathrm{AC}\big((a,b),\mathbb{R}^N\big)$ is the set of all absolutely continuous functions on $[a,b]$. For $N=1$, let

$$I(u;a,b) = \int_a^b \big(r(t)u'^2(t) - q(t)u^2(t)\big)dt \quad \text{for } u \in A_1((a,b),\mathbb{R}).$$

REMARK 3. If $x(t)$ is a solution of the unforced equation (2.1) such that $x(a)=x(b)=0$, then multiplaying (2.1) by $x(t)$ and integrating by parts, we obtain $I(x(t);a,b)=0$. □

THEOREM 4. ([40, Theorem 6.2] - from 1964) *Let $r(t)>0$, $q(t)$ be real-valued continuous functions on J and $e(t)\equiv 0$. Then equation (2.1) is disconjugate on J if and only if, for every closed bounded subinterval $a \leqslant t \leqslant b$ of J, the functional $I(u;a,b)$ is positive-definite on $A_1((a,b),\mathbb{R})$, that is, $I(u;a,b) \geqslant 0$ for $u \in A_1((a,b),\mathbb{R})$ and $I(u;a,b)=0$ for $u\equiv 0$.* □

COMMENT 1. Wong's interval oscillation criterion (2.3) can also be written in terms of the energy functional $I(u;a,b)$ in this way: "there exists a function $u \in D(a_i,b_i)$, $i=1,2$, such that $I(u;a_i,b_i) \leqslant 0$, $i=1,2$". From Theorem 4 and Sturm's separation theorem ([40, Corollary 3.1]), one can derive Wong's oscillation criterion in the unforced case. It is left to the reader. The main limitation of Theorem 4 is $e(t)\equiv 0$.

On the half-linear generalization of Hartman's Theorem 4, we refer the reader to Došlý and Rehak [22], Došlý and Fišnarová [20].

•3) In 1972 Komkov in [49] obtained the following generalization of the Leighton variational priniciple given in Theorem 3.

THEOREM 5. ([49, Theorem 2] - from 1972) *Let $r(t)>0$ and $e(t)\equiv 0$. Suppose there exist a function $u \in C^1([a,b],\mathbb{R})$ and a non-constant function $G(v)$, $v\in\mathbb{R}$, such that $G(u(a)) = G(u(b)) = 0$, $G'(v)$ is continuous, $[G'(v)]^2 \leqslant 4G(v)$ on $\mathbb{R}$ and*

$$\int_a^b \big(q(t)G(u(t)) - r(t)u'^2(t)\big)dt > 0.$$

The every solution of (2.1) must vanish on $[a,b]$.

It is clear that for $G(v)=v^2$ we have that $G(v)$ is not constant, $G'(v)=2v$ is continuous and $(G'(v))^2 = 4v^2 = 4G(v)$. Hence, Theorem 5 generalizes Theorem 3. About the application of Komkov's Theorem 5 in the nonoscillation of a class of the second-order nonlinear differential equation associated to (2.1), we refer the reader to the results due to Graef and Spikes in [31].

•4) In their paper [13] from 1987 Butler, Erbe and Mingarelli studied the oscillation of the second-order differential linear system $Y''+Q(t)Y=0$, where $Y(t),Q(t)$ are $N\times N$ real continuous matrix functions with $Q(t)$ symmetric. They derived and proved several oscillation criteria (see [13, Theorems 3.1, 3.2 and 3.3]) based on the Hartman's variational principle of Theorem 4 exploited in the next matrix form ([13, p. 267]):

Variational principle in matrix form. *System $Y''+Q(t)Y=0$ is oscillatory if and only if there is a sequence of intervals $[a_n,b_n]$, with $\lim_{n\to\infty} a_n = \infty$, and a sequence of functions $u_n \in A_1\big((a_n,b_n),\mathbb{R}^{\mathbb{N}}\big)$, such that*

$$\int_{a_n}^{b_n} \big(u_n^*(t)Q(t)u_n(t) - |u_n'(t)|^2\big)dt > 0.$$

It is not difficult to compare this inequality with assumption (2.3). The main limitation of this variational principle is $e(t) \equiv 0$.

•5) In his paper [24] from 1993, El-Sayed proved the following interval oscillation criterion:

THEOREM 6. ([24, Theorem 1] - from 1993) *Let there exist two positive increasing divergent sequences* $\{a_n^+\}$, $\{a_n^-\}$ *and two sequences of positive numbers* $\{c_n^+\}$, $\{c_n^-\}$ *such that*

$$\int_{a_n^\pm}^{a_n^\pm+\pi/\sqrt{c_n^\pm}} \left(c_n^\pm[1-r(t)]\cos^2\left(\sqrt{c_n^\pm}(t-a_n^\pm)\right)+[q(t)-c_n^\pm]\sin^2\left(\sqrt{c_n^\pm}(t-a_n^\pm)\right)\right)dt \geqslant 0 \quad (2.9)$$

for every $n \in \mathbb{N}$. *Assume that the function* $e(t)$ *satisfies*

$$e(t)\begin{cases} \geqslant 0, & t \in \left[a_n^+, a_n^+ + \pi/\sqrt{c_n^+}\right] \\ \leqslant 0, & t \in \left[a_n^-, a_n^- + \pi/\sqrt{c_n^-}\right], \end{cases}$$

for every $n \in \mathbb{R}$. *Then the linear forced equation* (2.1) *is oscillatory.*

With the help of this result, El-Sayed was able to give an answer to a question posed by Wong in [104] concerning the oscillation of the Mathieu's equation, for the details see Remark 4 below.

COMMENT 2. In contrast to Theorem 1, where we have a flexibility to choose a test function $u \in D(a_i, b_i)$ to satisfy condition (2.3), in Theorem 6 the test function is fixed. □

COMMENT 3. One can show that El-Sayed's criterion (2.9) is a special case of Wong's criterion (2.3). In fact, for $T \leqslant a_i < b_i$, $i = 1,2$, let c_i and $u(t)$ be defined by

$$c_i = \left(\frac{\pi}{b_i - a_i}\right)^2 \quad \text{and} \quad u(t) = \sin\left(\sqrt{c_i}(t-a_i)\right),\ t \in [a_i, b_i],\ i = 1,2.$$

Obviously, we have:

$$b_i = a_i + \frac{\pi}{\sqrt{c_i}},\ u(a_i) = u(b_i) = 0,\ u(t) \not\equiv 0 \text{ on } (a_i, b_i) \quad \text{and} \quad u \in C^1([a_i, b_i], \mathbb{R}),\ i = 1,2,$$

and moreover,

$$\int_{a_i}^{a_i+\frac{\pi}{\sqrt{c_i}}} \left[c_i\cos^2\left(\sqrt{c_i}(t-a_i)\right) - c_i\sin^2\left(\sqrt{c_i}(t-a_i)\right)\right]dt$$
$$= c_i\int_{a_i}^{a_i+\frac{\pi}{\sqrt{c_i}}} \cos\left(2\sqrt{c_i}(t-a_i)\right)dt = \frac{c_i}{2\sqrt{c_i}}\sin\left(2\sqrt{c_i}(t-a_i)\right)\Big|_{a_i}^{a_i+\frac{\pi}{\sqrt{c_i}}} = 0,\ i = 1,2.$$

Hence, El-Sayed's criterion (2.9) can be rewritten in the following equivalent form:

$$\int_{a_i}^{a_i+\frac{\pi}{\sqrt{c_i}}} \left[q(t)\sin^2\left(\sqrt{c_i}(t-a_i)\right) - c_i r(t)\cos^2\left(\sqrt{c_i}(t-a_i)\right)\right]dt \geqslant 0.$$

But, this inequality is contained in Wong's criterion (2.3) especially for $u(t) = \sin\left(\sqrt{c_i}(t-a_i)\right)$.
□

•6) For the forced Emden-Fowler equation:

$$x'' + q(t)|x|^{\gamma}\mathrm{sgn}(x) = e(t),\ \gamma > 0, \tag{2.10}$$

we recall the following Nasr's oscillation criterion.

THEOREM 7. ([68, Theorem] - from 1998) *Let $\gamma > 1$ and for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant a_1$, $b_1 = a_2$, such that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$, and $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(a_1,b_1)\cup(a_2,b_2)$. Let $D(a_i,b_i)$ be the set of functions defined by* (2.2)*. If there exists a function $u \in D(a_i,b_i)$, $i = 1,2$, such that*

$$\int_{a_i}^{b_i} \left(|e(t)|^{1-1/\gamma}[q(t)]^{1/\gamma}u^2(t) - u'^2(t) \right) dt \geqslant 0,\ i = 1,2, \tag{2.11}$$

then equation (2.10) *is oscillatory.*

This theorem answered a question raised by Wong in his paper [104].

COMMENT 4. Besides $\gamma > 1$, Nasr's Theorem 7 has another two essential limitations that do not appear in Wong's Theorem 1: the condition $b_1 = a_2$ and the condition $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(a_1,b_1)\cup(a_2,b_2)$. In some cases, it is not appropriate to require that these two conditions simultaneously hold. For instance, if $e(t) = -\cos(t)$ and $q(t) = \sin(2t)$, then only one possible choice exists for intervals $[a_1,b_1]$ and $[a_2,b_2]$ so that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$, and $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(a_1,b_1)\cup(a_2,b_2)$ is $[0,\pi/2]$ and $[\pi,3\pi/2]$, but obviously $[a_1,b_1]\cap[a_2,b_2] = \emptyset$. □

For the cases $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$, equation (2.10) is called respectively sub-linear, linear and super-linear. The oscillatory behaviour of the unforced Emden-Fowler equation (2.10) (with $e(t) \equiv 0$) have been studied by many authors in the past. Results on the nonoscillation of the unforced sub-linear Emden-Fowler equation (2.10) can be found in the books of Agarwal, Grace and O'Regan [2, Chapters 4 and 5] and Agarwal, Bohner and Li [1, Chapter 4]. One of the most important results on the nonoscillation of unforced equation (2.10) can be found in Kwong and Wong [56].

The case when both sub-linear and super-linear terms are appearing in the equation at the same time i.e., when the equation is of the so-called "mixed nonlinearities", will be considered in Section 3. The oscillation criteria with integral mean of the unforced sub- and super-linear Emden-Fowler equation (2.10) will be discussed in Section 4.

REMARK 4. (on the Mathieu's equation) We consider the well-known Mathieu's equation:

$$x'' + (a + b\cos(2t))x = 0, t \geqslant 0. \tag{2.12}$$

This equation was firstly introduced by M. Émile Mathieu in 1868 for the purpose of determining the vibration of a stretched membrane with an elliptic boundary; Heine in 1878 derived the solution of (2.12) by the cosine and sine series without evaluating the coefficients of the series; Floquet in 1883 published well-known theorem on the periodic solutions of linear differential equations with the periodic coefficients (for historical details see [69]). In the modern literature, the following facts about Mathieu's equation (2.12) are often emphasized: for each $a,b \in \mathbb{R}$, equation (2.12) has one odd and one even solution as well as at least one solution $x(t)$ such that $x(t+\pi) = \sigma x(t)$, where the constant σ is called the periodicity factor depending on parameters a

and b. About this facts and the application of Mathieu-type equations in some modern problems of applied sciences we refer the reader to [29].

El-Sayed in [24, Example 1] answered a question posed by Wong in [104] on the oscillation of Mathieu's equation (2.12). Precisely, Wong in [104] derived that equation (2.12) is oscillatory if

$$a+\frac{1}{2}|b|\geqslant 1,\ a,b\in\mathbb{R}.$$

Moreover, Wong posed an open question about the oscillations of the Mathieu's equation (2.12) for other values of $a,b\in\mathbb{R}$. In order to answer this question, in [24, Example 1] it has been proved that equation (2.12) is also oscillatory if

$$a,b\in\mathbb{R}\quad\text{and}\quad a>0.$$

Furthermore, in [88] the authors did speculate that for $a=-1$ and $b>\sqrt{2.5}$ equation (2.12) is still oscillatory. □

2.3. Influences to related results by other authors - from 2002 to 2013

Now, we give a brief review on some results written by other authors that have been inspired by Wong's Theorem 1. These results are arranged in the chronological order from 2002 to 2013:

- in 2002, Wan-Tong Li and Sui Sun Cheng [62];
- in 2003, Yuan Gong Sun [87], Qigui Yang [117];
- in 2004, Cakmak and Tiryaki [14];
- in 2006, Yuan Gong Sun and Fan Wei Meng [89], Wenying Shi [83];
- in 2007, Zhaowen Zheng and Fan Wei Meng [123], Qi-Ru Wang [98];
- in 2008, Douglas R. Anderson [6], Lynn H. Erbe, Allan C. Peterson and Samir H. Saker [27];
- in 2009, Douglas R. Anderson and Agacik Zafer [7], A. Feza Guvenilir [36];
- in 2012, Zhonghai Guo, Xiaoliang Zhou and Wu-Sheng Wang [35];
- in 2013, Yibing Sun, Zhenlai Han, Shurong Sun and Chao Zhang [93].

First of all, the analogous half-linear equation to (2.1) is:

$$\big(r(t)|x'|^{\alpha-1}x'\big)'+q(t)|x|^{\alpha-1}x=e(t),\ t\geqslant t_0, \tag{2.13}$$

where $\alpha>0$. On the qualitative properties of solutions of the second-order half-linear differential equation we refer the reader to the excellent book by Ondrej Dosly and Pavel Rehak [22]. Obviously, for $\alpha=1$ this equation becomes the linear equation (2.1).

In 2002 Wan-Tong Li and Sui Sun Cheng in their paper [62] studied the oscillations of (2.13) and they proved the following interval oscillation criterion.

THEOREM 8. ([62, Theorem 2] - from 2002) *Let for any $T>0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T\leqslant a_1$, $b_1\leqslant a_2$, such that $e(t)\leqslant 0$ on $[a_1,b_1]$ and $e(t)\geqslant 0$ on $[a_2,b_2]$. Let $D(a_i,b_i)$ be the set of functions defined by (2.2). If there exist a function $u\in D(a_i,b_i)$ and a positive nondecreasing function $\phi\in C([t_0,\infty),\mathbb{R})$ such that*

$$\int_{a_i}^{b_i}\phi(t)\left[q(t)u^2(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}|u(t)|^{\alpha-1}}\left(2|u'(t)|+|u(t)|\frac{\phi'(t)}{\phi(t)}\right)^{\alpha+1}\right]dt>0, \tag{2.14}$$

for $i=1,2$, then equation (2.13) is oscillatory.

For $\alpha = 1$ and $\phi(t) \equiv 1$ assumption (2.14) becomes (2.3), and therefore Theorem 8 generalizes Wong's interval oscillation criterion. The proof of Theorem 8 is a half-linear generalization of the proof of Wong's Theorem 1. It contains: the half-linear Riccati transformation of a nonoscillatory solution $x(t) \neq 0$, $t > T$,

$$\omega(t) = -\phi(t)\frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)} \text{ for } t > T, \tag{2.15}$$

that generalizes (2.6) in particular for $\phi(t) \equiv 1$ and $\alpha = 1$, then an appropriate Riccati differential inequality is derived, which is multiplied by u^2 and using partial integration together with some elementary inequalities, the proof of Theorem 8 follows. Theorem 8 was illustrated by the next half-linear equation

$$\left((2+\cos t)|x'|^{\alpha-1}x'\right)' + 5\left(\frac{3}{2}\right)^{4/3}|x|^{\alpha-1}x = \sin t,\ t \geqslant 1, \tag{2.16}$$

where $\alpha = 1/3$. Authors showed that equation (2.16) is oscillatory by using Theorem 8 in particular for $u(t) = \sin t$, $a_1 = 2n\pi$, $b_1 = (2n+1)\pi$ and $\phi(t) \equiv 1$.

Next, we are concerned with the functional-differential analogue of the forced linear differential equation (2.1):

$$x''(t) + q(t)x(\tau(t)) = e(t), \tag{2.17}$$

with delayed argument $\tau(t)$ satisfying that $\tau(t) \leqslant t$ and $\lim_{t\to\infty} \tau(t) = \infty$. On the qualitative properties of second-order linear differential equations with delay, we refer the reader to Erbe, Kong and Zhang [26, Chapter 4].

In 2003 Yuan Gong Sun studied the oscillation of equation (2.17) in the first part of his paper [87]. Later, we will also discuss the second result from [87] about the oscillations of more general Emden-Fowler type equations with delayed arguments.

THEOREM 9. ([87, Theorem 1] - from 2003) *Let for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant \tau(a_1)$, $b_1 \leqslant \tau(a_2)$, such that $e(t) \leqslant 0$ on $[\tau(a_1),b_1]$ and $e(t) \geqslant 0$ on $[\tau(a_2),b_2]$, and $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(\tau(a_1),b_1) \cup (\tau(a_2),b_2)$. Let $D(a_i,b_i)$ be the set of functions defined by* (2.2). *If there exists a function $u \in D(a_i,b_i)$ such that*

$$\int_{a_i}^{b_i} \left(q(t)\frac{\tau(t)-\tau(a_i)}{t-\tau(a_i)}u^2(t) - u'^2(t) \right) dt \geqslant 0,\ i = 1,2, \tag{2.18}$$

then equation (2.17) *is oscillatory.*

Obviously, for $\tau(t) = t$ equation (2.17) and condition (2.18) become respectively (2.1) and (2.3). Hence, Theorem 9 is a generalization of Wong's interval oscillation criterion under the restriction that $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(\tau(a_1),b_1) \cup (\tau(a_2),b_2)$. The proof of Theorem 9 follows the same steps as in the proof of Wong's Theorem 1. However, when the classic Riccati transformation (2.6) is applied to a nonoscillatory solution $x(t) \neq 0$, $t \geqslant T$, the corresponding Riccati differential equation derived involves the delay argument $\tau(t)$. Then, this term is eliminated by using some qualitative properties of concave functions on $(\tau(a_i),b_i))$, $i = 1,2$, and hence a Riccati differential inequality is obtained without delay argument $\tau(t)$.

Next, we consider the forced second-order nonlinear differential equation:

$$(r(t)x')' + q(t)f(x) = e(t),\ t \geqslant t_0, \tag{2.19}$$

where the function $f = f(x)$ satisfies the following lower growth condition:

$$f(x)/x \geqslant K|x|^{\gamma-1} \text{ for } x \neq 0, \text{ where } K > 0 \text{ and } \gamma \geqslant 1. \tag{2.20}$$

Specially for $K = \gamma = 1$, condition (2.20) is satisfied by the linear function $f(x) = x$ and therefore equation (2.19) is a generalization of forced linear equation (2.1). On the other hand, for $f(x) = |x|^\gamma \mathrm{sgn}(x)$ and $K = 1$, equation (2.19) generalizes the super-linear Emden-Fowler equation (2.10).

In 2003 Qigui Yang in his paper [117] derived and proved the next criterion for the oscillations of equation (2.19).

THEOREM 10. (the second part of [117, Theorem 1] - from 2003) *Let assumption (2.20) hold. Let for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant a_1$, $b_1 \leqslant a_2$, such that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$, and $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(a_1,b_1) \cup (a_2,b_2)$. Let $D(a_i,b_i)$ be the set of functions defined by (2.2). If there exists a function $u \in D(a_i,b_i)$ such that*

$$\int_{a_i}^{b_i} \left(|e(t)|^{1-1/\gamma} [Kq(t)]^{1/\gamma} u^2(t) - r(t) u'^2(t) \right) dt \geqslant 0, \; i = 1,2, \tag{2.21}$$

then equation (2.19) is oscillatory.

If we put $K = \gamma = 1$ into assumption (2.21), then we get assumption (2.3) and thus, Theorem 10 is a generalization of Wong's interval oscillation criterion under the restriction that $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(a_1,b_1) \cup (a_2,b_2)$. Also, if we put $f(x) = |x|^\gamma \mathrm{sgn}(x)$ and $K = 1$ into (2.19), then we conclude that Theorem 10 also generalizes Nasr's Theorem 7. In contrast to Theorems 1, 8 and 9, where the forced term $e(t)$ is zero, in the main condition of Theorem 10 the function $e(t)$ plays an active role. The proof of Theorem 10 is essentially a modification of the proof of Theorem 1. Using Theorem 10 in particular for $u(t) = -\sin 2t$, $a_1 = 2n\pi - \pi/2$, $b_1 = a_2 = 2n\pi$ and $b_2 = 2n\pi + \pi/2$, the author showed the oscillation of the equation

$$((1 + a\sin^2 t)x'(t))' + (\beta \cos t)\,|x(t)|^\gamma \Big[1 + \sum_{i=1}^{m} b_i x^{2i}(t)\Big] \mathrm{sgn}\, x(t) = \sin t, \; \gamma \geqslant 1, \; t \geqslant 0,$$

where $a \geqslant 0$, $b_i \geqslant 0$, $\gamma > 1$, provided

$$\beta^{1/\gamma} \geqslant \pi\Big(1 + \frac{a}{2}\Big) \Big/ \Big[2 \frac{\Gamma(3 + \frac{1}{\gamma})\Gamma(4 + \frac{1}{\gamma})}{\Gamma(7)}\Big].$$

Yuan Gong Sun studied in the second part of his paper [87] the oscillations of the Emden-Fowler type forced equation with a deviating argument:

$$x''(t) + q(t)|x(\tau(t))|^\gamma \mathrm{sgn}(x(\tau(t))) = e(t), \; \gamma > 1. \tag{2.22}$$

As a continuation of his Theorem 9, he proved the following criterion for (2.22).

THEOREM 11. ([87, Theorem 2] - from 2003) *Let for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant \tau(a_1)$, $b_1 \leqslant \tau(a_2)$, such that $e(t) \leqslant 0$ on $[\tau(a_1),b_1]$ and $e(t) \geqslant 0$ on $[\tau(a_2),b_2]$, and $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $(\tau(a_1),b_1) \cup (\tau(a_2),b_2)$. Let $D(a_i,b_i)$ be the set of functions defined by (2.2). If there exists a function $u \in D(a_i,b_i)$ such that*

$$\int_{a_i}^{b_i} \left(\theta |e(t)|^{1-1/\gamma} q^{1/\gamma}(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} u^2(t) - u'^2(t) \right) dt \geqslant 0, \; i = 1,2, \tag{2.23}$$

where $\theta = \gamma(\gamma - 1)^{-1+1/\gamma}$, then equation (2.22) is oscillatory.

Applying Theorem 9 in particular for $u(t) = \sin 2t \cos 2t$, $a_1 = 2n\pi + 3\pi/4$, $b_1 = 2n\pi + \pi$, $a_2 = 2n\pi + \pi/4$, $b_1 = 2n\pi + \pi/2$, the authors proved the oscillation of the following equation

$$x''(t) + m \sin t\, |x(t-\pi/4)|^{\gamma} \operatorname{sgn}\big(x(t-\pi/4)\big) = \cos t,\ \gamma \geqslant 1,\ t \geqslant 0,$$

is oscillatory, where $m \geqslant 0$ is a constant.

COMMENT 5. If $f(x) = |x|^{\gamma} \operatorname{sgn} x$, $\gamma > 1$, then equation (2.22) and Sun's Theorem 11 can be considered respectively as a generalization of equation (2.19) and Yang's Theorem 10. □

REMARK 5. In [14, Theorems 1 and 2] - from 2004 Devrim Cakmak and Aidyn Tiryaki gave a quasilinear generalization of Wong's Theorem 1 and Li-Cheng's Theorem 8 by studying the oscillation of the following quasilinear generalization of linear equation (2.1) and nonlinear equation (2.19):

$$(r(t)\Psi(x)|x'|^{\alpha-1}x')' + q(t)f(x) = e(t),\ t \geqslant t_0, \tag{2.24}$$

where $\Psi \in C(\mathbb{R},\mathbb{R})$. □

In 2006 Yuan Gong Sun and Fan Wei Meng in [89] studied the oscillation of the quasilinear forced differential equation:

$$\big(r(t)|x'|^{\alpha} \operatorname{sgn} x'\big)' + q(t)|x|^{\beta} \operatorname{sgn} x = e(t),\ t \geqslant t_0, \tag{2.25}$$

where $r(t) \equiv 1$, $\beta > \alpha > 0$ and $q(t), e(t)$ are continuous functions on $[t_0,\infty)$. In the case $\beta = \alpha > 0$, equation (2.25) becomes the half-linear equation (2.13).

THEOREM 12. ([89, Theorem 2.3] - from 2006) *Let $r(t) \equiv 1$. Assume for any $T > 0$ there exist intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T < a_1 < b_1 \leqslant a_2 < b_2$ such that $q(t) \geqslant 0$ on $[a_i,b_i]$, $i = 1,2$ and $e(t)$ has different signs on $[a_1,b_1]$ and $[a_2,b_2]$. If there exist two differentiable functions $u_1(t)$ and $u_2(t)$ such that $u_i(a_i) = u_i(b_i) = 0$, $|u_i'(t)| \in L^{1+\alpha}([a_i,b_i],\mathbb{R})$ and*

$$\int_{a_i}^{b_i} \Big[\theta q^{\alpha/\beta}(t)|e(t)|^{1-\alpha/\beta} u_i^{\alpha+1}(t) - |u_i'(t)|^{\alpha+1}\Big] dt \geqslant 0,\ i = 1,2,$$

where $\theta = \frac{\beta}{\alpha}\left(\frac{\beta}{\alpha} - 1\right)^{\frac{\beta}{\alpha}-1} > 1$, then equation (2.25) is oscillatory.

This theorem was illustrated to the quasilinear equation

$$\big(|x'|^{\alpha}\big)' + m \sin t |x|^{\beta} \operatorname{sgn} x = \cos t,\ t \geqslant 0, \tag{2.26}$$

where $\beta > \alpha > 0$, $\alpha = \text{odd}/\text{odd}$ and $m > 0$. By using Theorem 12 especially for $a_1 = 2n\pi$, $b_1 = a_2 = 2n\pi + \pi/2$, $b_2 = 2n\pi + \pi$ and $u_1(t) = u_2(t) = \sin t$, the author showed that equation (2.26) is oscillatory.

In 2006 Wenying Shi [83] studied the oscillation of a general class of nonlinear second-order differential equations with damping:

$$\big(r(t)k_1(x,x')\big)' + p(t)k_2(x,x')x' + q(t)f(x) = e(t),\ t \geqslant t_0, \tag{2.27}$$

where $r(t) > 0$ and $r'(t), p(t), q(t), e(t), k_1(u,v), k_2(u,v)$ and $f(u)$ are continuous functions on their domains which are respectively $[t_0,\infty)$, $\mathbb{R}^2$ and $\mathbb{R}$. The nonlinear functions $k_1(u,v)$ and $k_2(u,v)$ satisfy the following structural assumptions:

$$k_1(u,v)v \geqslant \alpha_1 k_1^2(u,v) \text{ for some } \alpha_1 > 0 \text{ and all } (u,v) \in \mathbb{R}^2, u \neq 0, \tag{2.28}$$

$$k_2(u,v)uv \geqslant \alpha_2 k_1^2(u,v) \text{ for some } \alpha_2 \geqslant 0 \text{ and all } (u,v) \in \mathbb{R}^2, u \neq 0. \tag{2.29}$$

COMMENT 6. The choice for the functions $k_1(u,v)$ and $k_2(u,v)$ satisfying (2.28) and (2.29) should be careful, because it is supposed that $k_1, k_2 \in C(\mathbb{R}^2,\mathbb{R})$. For instance, if we chose $k_1(u,v) = v$ and $k_2(u,v) = \phi_2(u)v$, then (2.27) becomes $\left(r(t)x'\right)' + p(t)\phi_2(x)x'^2 + q(t)f(x) = e(t)$ and $\phi_2(u)$ should satisfy $\phi_2 \in C(\mathbb{R},\mathbb{R})$ and $\phi_2(u)u \geqslant \alpha_2 > 0$ for all $u \neq 0$. But such a $\phi_2(u)$ does not exist. Thus, $k_1(u,v)$ should also depend on the variable u such as $k_1(u,v) = \phi_1(u)v$ for some function $\phi_1 \in C(\mathbb{R},\mathbb{R})$. □

THEOREM 13. ([83, Theorem 1] - from 2006) *Let for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant a_1$, $b_1 \leqslant a_2$, such that $e(t) \leqslant 0$ on $[a_1,b_1]$, $e(t) \geqslant 0$ on $[a_2,b_2]$ and $q(t) \geqslant 0$ on . Let $D(a_i,b_i)$ be the set of functions defined by* (2.2). *Let* (2.20), (2.28) *and* (2.29) *hold. If there exist a function $u \in D(a_i,b_i)$ and a positive function $\phi \in C([t_0,\infty),\mathbb{R})$ such that*

$$\int_{a_i}^{b_i} \phi(t)\left[\theta|e(t)|^{1-1/\gamma}[Kq(t)]^{1/\gamma}u^2(t) - \frac{r^2(t)}{\alpha_1 r(t) + \alpha_2 p(t)}\left(u'(t) + u(t)\frac{\phi'(t)}{2\phi(t)}\right)^2\right]dt > 0, \quad (2.30)$$

where $\theta = \gamma(\gamma-1)^{-1+1/\gamma}$ for $i = 1,2$, then equation (2.27) *is oscillatory.*

If $p(t) \equiv 0$, $f(x) = x$, then $K = 1$ and $\phi(t) \equiv 1$ and consequently, assumption (2.30) is reduced to the Wong's interval oscillation criterion (2.4) and hence, Theorem 13 extends Wong's Theorem 1 from linear to a general nonlinear case. Theorem 13 was illustrated by the nonlinear damped diffferential equation (2.27) especially for:

$$r(t) = \sqrt{t}(2+\sin\sqrt{t}),\ p(t) = \sqrt{t}(2-\sin\sqrt{t}),\ q(t) = \frac{1}{\sqrt{t}} \quad \text{and} \quad e(t) = \sin\sqrt{t},\ t \geqslant 1,$$
$$k_1(u,v) = \frac{u^2 v}{1+u^2},\ k_2(u,v) = \frac{u^3 v(1+u^2+v^2)}{(1+u^2)^2} \quad \text{and} \quad \alpha_1 = \alpha_2 = 1,$$
$$f(x) = x(2+\cos x),\ K = 1 \quad \text{and} \quad \gamma = 1.$$

The main limitations $r(t) \equiv 1$ and $\beta > \alpha > 0$ appearing in equation (2.25) and Theorem 12 were relaxed in Zhaowen Zheng and Fan Wei Meng [123] with $r(t) > 0$ and $\beta \geqslant \alpha > 0$ as follows.

THEOREM 14. ([123, Theorem 2.2] - from 2007) *Let $r(t) > 0$ and $\beta \geqslant \alpha > 0$. Assume for any $T > 0$ there exist intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T < a_1 < b_1 \leqslant a_2 < b_2$ such that $e(t)$ has different signs on $[a_1,b_1]$ and $[a_2,b_2]$. If there exist a function $u \in C^1([a_i,b_i],\mathbb{R})$, $u^{\alpha+1}(t) > 0$ on (a_i,b_i) and $u(a_i) = u(b_i) = 0$ and a positive nondecreasing function $\phi \in C([t_0,\infty),\mathbb{R})$ such that*

$$\int_{a_i}^{b_i} \phi(t)\left[\theta[q(t)]^{\alpha/\beta}|e(t)|^{1-\alpha/\beta}u^{\alpha+1}(t) - r(t)\left(|u'(t)| + u(t)\frac{\phi'(t)}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]dt > 0, \quad (2.31)$$

then equation (2.25) *is oscillatory, where $\theta = \frac{\beta}{\alpha}\left(\frac{\beta}{\alpha}-1\right)^{\frac{\beta}{\alpha}-1}$ and $0^0 = 1$.*

This theorem was illustrated by the quasilinear equation:

$$\left(at^{-b/3}x'\right)' + t^{-b}|x|^2 x = -\sin^3 t,\ t \geqslant 1, \quad (2.32)$$

where the constants $a,b>0$. Especially for $\alpha=1<3=\beta$, $u(t)=\sin t$ and $\phi(t)=t^{b/3}$, the authors showed that equation (2.32) is oscillatory provided

$$0<a<\frac{9\sqrt[3]{2}}{8(1+b/6)^2}.$$

Next, we present further generalizations of Wong's interval oscillation criterion to a general class of nonlinear second-order differential equations:

$$(r(t)x')'+F(t,x,x')=0,\ t\geqslant t_0, \tag{2.33}$$

due to Qi-Ru Wang [98] in 2007. The general nonlinear term $F(t,y,z)$ is supposed to contain in particular the linear case $F(t,y,z)=q(t)y-e(t)$ as well as the nonlinear case $F(t,y,z)=q(t)f(y)-e(t)$ and therefore, linear and nonlinear equations (2.1) and (2.19) are two special cases of equation (2.33).

THEOREM 15. ([98, Theorems 2.2 and 2.3] - from 2007) *Let for any $T>0$ there exist an $[a,b]$, $T\leqslant a<b$, and let $D(a,b)$ be the set of functions defined by (2.2). If there exists a function $u\in D(a,b)$ such that for any $y\in C^1([a,b],\mathbb{R})$ with $y(t)\neq 0$ on $[a,b]$ the following inequality holds:*

$$\int_a^b\left(\frac{F(t,y(t),y'(t))}{y(t)}u^2(t)-r(t)u'^2(t)\right)dt\geqslant 0, \tag{2.34}$$

then equation (2.33) is oscillatory.

THEOREM 16. ([98, Corollary 2.4] - from 2007) *Let $q,e\in C([t_0,\infty),\mathbb{R})$ and the nonlinear term $F(t,y,z)$ satisfy*

$$yF(t,y,z)\geqslant q(t)y^2-e(t)y\quad \textit{for all } (t,y,z)\in[t_0,\infty)\times\mathbb{R}^2. \tag{2.35}$$

Let for any $T>0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T\leqslant a_1$ and $b_1\leqslant a_2$, such that $e(t)\leqslant 0$ on $[a_1,b_1]$ and $e(t)\geqslant 0$ on $[a_2,b_2]$, and $q(t)\geqslant 0$, $q(t)\not\equiv 0$ on $(a_1,b_1)\cup(a_2,b_2)$. Let $D(a_i,b_i)$ be the set of functions defined by (2.2). If there exists a function $u\in D(a_i,b_i)$ such that (2.3) holds, then equation (2.33) is oscillatory.

Comparing Wong's Theorem 1 with Wang's Theorem 16 it is simple to deduce that the latter generalizes the former when $F(t,y,z)=q(t)y-e(t)$. The author used Theorem 16 in order to show the oscillation of the following class of equations (see [98, Example 3.1]):

$$\left(\sqrt{t+1}\,x'(t)\right)'+q(t)x(t)\left(1+\alpha x^2(t)+\beta x'^2(t)\right)=\sin\sqrt{t+1},\ t\geqslant 0,$$

where $\alpha,\beta\geqslant 0$, $q(t)\in C([0,\infty),(0,\infty))$ and $q(t)\geqslant 1/4\sqrt{t+1}$. See also [98, Example 3.1].

Next, we consider the time scale analogue equation to equation (2.33):

$$(r(t)x^\Delta(t))^\Delta+F(t,x^\sigma(t),x^\Delta(t))=0,\ t\in[t_0,\infty)_{\mathbb{T}}, \tag{2.36}$$

where $\mathbb{T}$ is a time scale, and the functions $r=r(t)$, $r:\mathbb{T}\to(0,\infty)$ and $F=F(t,y,z)$, $F:\mathbb{T}\times\mathbb{R}^2\to\mathbb{R}$ are right-dense in the time scale variable t. Instead of the set of functions $D(a,b)$ defined by (2.2) one uses its time scale analogue set:

$$J(a,b)=\{u\in C^\Delta_{\mathrm{rd}}[a,b]_{\mathbb{T}}:u(t)\not\equiv 0,u(a)=u(b)=0\}. \tag{2.37}$$

In 2008 Douglas R. Anderson in his paper [6] established the following generalizations of Wang's Theorems 15 and 16.

THEOREM 17. ([6, Theorem 3.2] - from 2008) *Let for any* $T \in [t_0,\infty)_{\mathbb{T}}$ *there exist an interval* $[a,b]_{\mathbb{T}} \subset [T,\infty)_{\mathbb{T}}$ *and a function* $u \in J(a,b)$ *such that for any* $y \in C^{\Delta}_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ *with* $yy^{\sigma} > 0$ *on* $[a,b]_{\mathbb{T}}$ *the following inequality holds:*

$$\int_a^b \left(\frac{F(t,y^{\sigma}(t),y^{\Delta}(t))}{y^{\sigma}(t)}(u^{\sigma}(t))^2 - r(t)(u^{\Delta}(t))^2\right)\Delta t \geqslant 0. \tag{2.38}$$

Then equation (2.36) *is oscillatory.*

THEOREM 18. ([6, Corollary 3.3] - from 2008) *Let* $q,e \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *and the nonlinear term* $F(t,y,z)$ *satisfy*

$$yF(t,y,z) \geqslant q(t)y^2 - e(t)y \quad \text{for all } (t,y,z) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^2. \tag{2.39}$$

Let for any $T > 0$ *there exist* a_i,b_i, $i = 1,2$, $T \leqslant a_1 < b_1 \leqslant a_2 < b_2$ *such that* $e(t)$ *has different signs on* $[a_1,b_1]_{\mathbb{T}}$ *and* $[a_2,b_2]_{\mathbb{T}}$. *If there exists a function* $u \in J(a_i,b_i)$ *such that*

$$\int_{a_i}^{b_i} \left(q(t)(u^{\sigma}(t))^2 - r(t)(u^{\Delta}(t))^2\right)\Delta t \geqslant 0,\ i = 1,2, \tag{2.40}$$

then equation (2.36) *is oscillatory.*

When $\mathbb{T} = \mathbb{R}$, Anderson's Theorems 17 and 18 become respectively Wang's Theorems 15 and 16.

Lynn H. Erbe, Allan C. Peterson and Samir H. Saker [27] in 2008 considered the time scale analogue of the second-order Emden-Fowler equation with forcing term:

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)|x^{\sigma}(t)|^{\gamma}\mathrm{sgn}x^{\sigma}(t) = e(t),\ t \in \mathbb{T}, \tag{2.41}$$

where $\mathbb{T}$ is a time scale and $\gamma \geqslant 1$. The main assumption is:

$$\int_{a_i}^{b_i} \left(\frac{\gamma}{(\gamma-1)^{1-1/\gamma}}[q(t)]^{1/\gamma}|e(t)|^{1-1/\gamma}\left[u^{\sigma}(t)\right]^2 - r(t)\left[u^{\Delta}(t)\right]^2\right)\Delta t \geqslant 0,\ i = 1,2, \tag{2.42}$$

where $0^0 = 1$.

THEOREM 19. ([27, Theorem 2.1] - from 2008) *Let* $\gamma \geqslant 1$, $r(t) > 0$, $q,e \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. *Asumme for any* $T \in [t_0,\infty)_{\mathbb{T}}$ *there exist* a_i,b_i, $i = 1,2$, $T \leqslant a_1 < b_1 \leqslant a_2 < b_2$ *such that* $e(t)$ *has different signs on* $[a_1,b_1]_{\mathbb{T}}$ *and* $[a_2,b_2]_{\mathbb{T}}$. *Assume there exists a function* $u \in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *such that* $u(t) \not\equiv 0$ *on* $[a_i,b_i]_{\mathbb{T}}$, $u(a_i) = u(b_i) = 0$ *and* (2.42) *holds. Then the dynamic equation* (2.41) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Especially for $\mathbb{T} = \mathbb{R}$, this theorem generalizes Wong's Theorem 1 or Nasr's Theorem 7 respectively for $\gamma = 1$ or $\gamma > 1$. Furthermore, if $\mathbb{T} = \mathbb{R}$, $f(x) = |x|^{\gamma}\mathrm{sgn}(x)$, $\gamma > 1$ and $K = 1$, then Yang's Theorem 10 becomes a special case of Theorem 19.

In 2009 Douglas R. Anderson and Agacik Zafer [7] studied the oscillation of the α-Laplacian analogoue of equation (2.36) i.e.,

$$\left(r(t)\Phi_{\alpha}(x^{\Delta}(t))\right)^{\Delta} + F(t,x^{\sigma}(t),x^{\Delta}(t)) = 0,\ t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.43}$$

where $\Phi_{\alpha}(x) = |x|^{\alpha-1}x$ and $\alpha > 0$. The author extended the case $\alpha = 1$ given in Theorem 17 to the following one with any $\alpha > 0$.

THEOREM 20. ([7, Theorem 3.1] - from 2009) *Let for any* $T \in [t_0,\infty)_{\mathbb{T}}$ *there exist an interval* $[a,b]_{\mathbb{T}} \subset [T,\infty)_{\mathbb{T}}$ *and a function* $u \in J(a,b)$ *such that for any* $y \in C^{\Delta}_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ *with* $yy^{\sigma} > 0$ *on* $[a,b]_{\mathbb{T}}$ *the following inequality holds:*

$$\int_a^b \left(\frac{F(t, y^{\sigma}(t), y^{\Delta}(t))}{\Phi_{\alpha}(y^{\sigma}(t))} |u^{\sigma}(t)|^{\alpha+1} - r(t)|u^{\Delta}(t)|^{\alpha+1} \right) \Delta t \geqslant 0. \tag{2.44}$$

Then equation (2.43) *is oscillatory.*

Also, the authors in [7] established the oscillation of a class of delay dynamic equations associated to (2.43), see [7, Theorem 4.1].

In 2009 A. Feza Guvenilir [36] considered the oscillation of a class of second-order functional differential equations:

$$\big(r(t)x'(t)\big)' + p(t)x(h(t)) + q(t)|x(h(t))|^{\gamma-1}x(h(t)) = e(t),\ t \geqslant 0, \tag{2.45}$$

where the functions r, p, q and e are continuous and the functional term $h(t) = \sigma(t)$ in advanced and $h(t) = \tau(t)$ in delayed case.

THEOREM 21. ([36, Theorems 2.1] - from 2009) *Let* $r(t) > 0$ *be nondecreasing,* $\gamma > 1$*,* $q, e \in C([t_0,\infty),\mathbb{R})$ *and the functional term* $\sigma(t)$ *be nondecreasing such that* $\lim_{t\to\infty}\sigma(t) = \infty$*,* $\sigma(t) \geqslant t$*. Let for any* $T > 0$ *there exist a pair of intervals* $[a_1,b_1]$ *and* $[a_2,b_2]$*,* $T \leqslant a_1 < b_1$*,* $\sigma(b_1) \leqslant a_2 < b_2$*, such that* $e(t) \leqslant 0$ *on* $[a_1,\sigma(b_1)]$ *and* $e(t) \geqslant 0$ *on* $[a_2,\sigma(b_2)]$*,* $p(t) \geqslant 0$ *and* $q(t) \geqslant 0$*,* $q(t) \not\equiv 0$ *on* $[a_1,\sigma(b_1)] \cup [a_2,\sigma(b_2)]$*. Let* $D(a_i,b_i)$ *be the set of functions defined by* (2.2)*. If there exists a function* $u \in D(a_i,b_i)$ *such that*

$$\int_{a_i}^{b_i} \left[\left(p(t) + \theta |e(t)|^{1-1/\gamma}[q(t)]^{1/\gamma} \right) \frac{\sigma(b_i) - \sigma(t)}{\sigma(b_i) - t} u^2(t) - r(t)u'^2(t) \right] dt \geqslant 0, \tag{2.46}$$

where $\theta = \gamma(\gamma-1)^{-1+1/\gamma}$*, then the advanced equation* (2.45) *is oscillatory.*

According to this theorem, the author showed that the advanced equation

$$x''(t) + m_1 \sin t\, x(t+\pi/6) + m_2 \cos t\, x^3(t+\pi/6) = \cos 2t,\ t \geqslant 0,$$

is oscillatory provided m_1 or m_2 is large enough.

THEOREM 22. ([36, Theorems 2.1] - from 2009) *Let* $r(t) > 0$ *be nondecreasing,* $\gamma > 1$*,* $q, e \in C([t_0,\infty),\mathbb{R})$ *and the functional term* $\tau(t)$ *be nondecreasing such that* $\lim_{t\to\infty}\tau(t) = \infty$*,* $\tau(t) \leqslant t$*. Let for any* $T > 0$ *there exist a pair of intervals* $[a_1,b_1]$ *and* $[a_2,b_2]$*,* $T \leqslant \tau(a_1) \leqslant a_1 < b_1$*,* $b_1 \leqslant \tau(a_2) \leqslant a_2 < b_2$*, such that* $e(t) \leqslant 0$ *on* $[\tau(a_1),b_1]$ *and* $e(t) \geqslant 0$ *on* $[\tau(a_2),b_2]$*,* $p(t) \geqslant 0$ *and* $q(t) \geqslant 0$*,* $q(t) \not\equiv 0$ *on* $[\tau(a_1),b_1] \cup [\tau(a_2),b_2]$*. Let* $D(a_i,b_i)$ *be the set of functions defined by* (2.2)*. If there exists a function* $u \in D(a_i,b_i)$ *such that*

$$\int_{a_i}^{b_i} \left[\left(p(t) + \theta |e(t)|^{1-1/\gamma}[q(t)]^{1/\gamma} \right) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} u^2(t) - r(t)u'^2(t) \right] dt \geqslant 0, \tag{2.47}$$

where $\theta = \gamma(\gamma-1)^{-1+1/\gamma}$*, then the delayed equation* (2.45) *is oscillatory.*

Consequently, the author showed that the delayed equation

$$x''(t) + m_1 \sin t\, x(t-\pi/12) + m_2 \cos t\, x^3(t-\pi/12) = \cos 2t,\ t \geqslant 0,$$

is oscillatory provided m_1 or m_2 is large enough.

REMARK 6. In 2012 Zhonghai Guo, Xiaoliang Zhou and Wu-Sheng Wang in [35, Theorem 2.4] established an interval oscillation criterion related to the Wong's oscillation criterion for the following class of super-half-linear impulsive differential equations with delay:

$$\big(r(t)\Phi_\alpha(x'(t))\big)' + p(t)\Phi_\alpha(x(t-\tau)) + q(t)f(x(t-\tau)) = e(t),\ t \neq \tau_k,$$
$$x(t^+) = a_k x(t),\ x'(t^+) = b_k x'(t),\ t = \tau_k,\ k = 1,2,\ldots,$$

where $\Phi_\alpha(u) = |u|^{\alpha-1}u$, τ is a nonnegative constant, τ_k denotes the impulsive moments sequence with $\tau_1 < \tau_2 < \ldots < \tau_k < \ldots..$, $\lim_{k\to\infty}\tau_k = \infty$ and $\tau_{k+1} - \tau_k > \tau$. □

REMARK 7. In 2013 Yibing Sun, Zhenlai Han, Shurong Sun and Chao Zhang gave in [93, Theorems 3, 5 and 7] an interval oscillation criterion related to the Wong's oscillation criterion for the following second-order nonlinear dynamic equation with forcing and damping term:

$$(r(t)x^\Delta(t))^\Delta + p(t)x^{\Delta\sigma}(t) + q(t)\big(x^\sigma(t)\big)^\alpha = F(t, x^\sigma(t)),\ t \in \mathbb{T},$$

where α is a quotient of odd positive integers. □

2.4. Other types of interval oscillation criteria - from 1997 to 2013

In previous subsections, we have studied some known interval oscillation criteria motivated by Wong's Theorem 1. However, simultaneously with these results, there are some other types of interval oscillation criteria different from Theorem 1 presented in the chronological order:

– in 1997, C. Huang [45];

– in 1998, A. Elbert [23];

– in 1999, Q. Kong [50, 51];

– in 2000, W.T. Li and R.P. Agarwal [60, 61];

– in 2001, W.T. Li and H.F. Huo [63];

– in 2003, Q. Yang [117];

– in 2004, J.S.W. Wong [112], Q. Yang, R.M. Mathesen [118], Y.G. Sun, C.H. Ou, J.S.W. Wong [88];

– in 2005, Z. Xu and S. Peng [114];

– in 2007, Q. Kong [52], Y.V. Rogovchenko and F. Tuncay [81];

– in 2008, T. Hassan [41];

– in 2009, A.K. Nandakumaran and S. Panigrahi [67];

– in 2011, J. Tyagi [97];

– in 2013, E. Tunc and H. Avci [96], M. Pašić [72].

In 1997 Chunchao Huang in his paper [45] studied the nonoscillation and oscillation of the second-order linear differential equation

$$x'' + q(t)x = 0,\ t \geqslant 0, \tag{2.48}$$

where $q(t) \geqslant 0$. It is the most simple form of the second-order linear differential equation (2.1) where $r(t) \equiv 1$ and $e(t) \equiv 0$.

THEOREM 23. ([45, Theorem 2] - from 1997) *Let* $c_0 = 3-2\sqrt{2}$ *and* $q(t) \geqslant 0$, $q \in C([0,\infty),\mathbb{R})$. *If there exist* $t_0 > 0$ *and* $c > c_0$ *such that for every* $n \in \mathbb{N}$,

$$\int_{2^n t_0}^{2^{n+1}t_0} q(t)dt \geqslant \frac{c}{2^n t_0}, \tag{2.49}$$

then equation (2.48) *is oscillatory.*

Since in this paper we only discuss the oscillation criteria, we omit a nonoscillation result presented in [45, Theorem 1]. Moreover, the author derived the following interesting consequence of Theorem 23.

THEOREM 24. ([45, Corollary 2] - from 1997) *Let* $c_0 = 3-2\sqrt{2}$ *and* $q(t) \geqslant 0$, $q \in C([0,\infty),\mathbb{R})$. *If*

$$\lim_{t\to\infty} t\int_t^{2t} q(s)ds = c > c_0,$$

then equation (2.48) *is oscillatory.*

This result was illustrated with equation (2.48), where $q(t) = \sum q_n(t)$ and $q_n(t)$ is a sequence of nonnegative functions satisfying:

$$\operatorname{supp} q_n(t) \subset \left[2^n - \frac{1}{2^n}, 2^n\right] \quad \text{and} \quad \int_{\mathbb{R}} q_n(t)dt = \frac{1}{2^{n+1}},\ n = 1,2,\ldots.$$

REMARK 8. Agarwal and Li [61] pointed out that Huang's Theorem 23 is not sharp enough. In fact, it is well-known that the Euler equation $x'' + \gamma t^{-2}x = 0$ is oscillatory for $\gamma > 1/4$; however, this is not revealed by Huang's result, especially for $\gamma \in (3-2\sqrt{2}, 6-4\sqrt{2})$. □

In 1998 A. Elbert in [23] improved preceding Huang's results giving sharper ones.

THEOREM 25. ([23, Theorem 2] - from 1998) *Let* $q(t) \geqslant 0$. *Assume there exist two sequences* t_n *and* c_n *such that* $0 < t_0 < t_1 \ldots < t_n < \ldots$, $t_n \to \infty$ *and* $c_n > 0$, $n \in \mathbb{N}$. *If* $q(t)$ *satisfies*

$$(t_{n+1} - t_n)\int_{t_n}^{t_{n+1}} q(t)dt \geqslant c_n,\ n \in \mathbb{N}, \tag{2.50}$$

where the recurrence relation

$$v_{n+1} = \frac{c_{n+1}}{c_n}\frac{t_{n+1}-t_n}{t_{n+2}-t_{n+1}}\left(\frac{v_n}{1-v_n} + c_n\right),\ n \in \mathbb{N} \quad \text{and} \quad v_0 = 0,$$

has no solution such that $0 < v_n < 1$, $n \in \mathbb{N}$, *then equation* (2.48) *is oscillatory.*

On a related nonoscillation criterion we refer reader to [23, Theorem 2].

COMMENT 7. Theorem 23 is a special case of Theorem 25. First of all, it is clear that Huang's criterion (2.49) can be written in the form of Elbert's criterion (2.50) in particular for

$$t_n = 2^n t_0 \quad \text{and} \quad c_n = c > 3-2\sqrt{2}.$$

Since

$$\frac{t_{n+1}-t_n}{t_{n+2}-t_{n+1}} = \frac{2^{n+1}t_0 - 2^n t_0}{2^{n+2}t_0 - 2^{n+1}t_0} = \frac{1}{2} \qquad \text{and} \qquad \frac{c_{n+1}}{c_n} = \frac{c}{c} = 1,$$

the desired recurrence relation becomes

$$v_{n+1} = \frac{1}{2}\Big(\frac{v_n}{1-v_n} + c\Big),\ n \in \mathbb{N} \quad \text{and} \quad v_0 = 0. \tag{2.51}$$

We show that assumption $c > 3-2\sqrt{2}$ implies that (2.51) has no solution L such that $0 < v_n < 1$. On the contrary, if exists such an L, then it must be $L \in [0,1]$. Passing to the limit in (2.51) we have that L satisfies the following algebraic equation with corresponding solution's form:

$$2L^2 - (1+c)L + c = 0 \quad \text{and} \quad L_{1,2} = \frac{1}{4}\big(1+c \pm \sqrt{c^2-6c+1}\big), \tag{2.52}$$

where $c^2-6c+1 = \big(c-(3-2\sqrt{2})\big)\big(c-(3+2\sqrt{2})\big)$. On the first hand, if $c \in \big(3-2\sqrt{2}, 3+2\sqrt{2}\big)$, then equation in (2.52) has no real solutions. On the other hand, if $c \geqslant 3+2\sqrt{2}$, then (2.52) has solutions and since $(c-3)^2 > c^2-6c+1$, from (2.52) we obtain $L_{1,2} > 1$, that contradicts the conclusion $L \in [0,1]$. Thus, assumption $c > 3-2\sqrt{2}$ ensures that (2.51) has no solution L such that $0 < v_n < 1$. □

As a consequence of Theorem 25, the author derived the following interesting criterion that can be tested on the Euler equation.

THEOREM 26. ([23, Corollary 6] - from 1999) *Under the same assumptions as in Theorem 25 suppose*

$$t_{n+2} - t_{n+1} = t_{n+1} - t_n \quad \textit{and} \quad c_n \geqslant \frac{c}{(n+1)^2},\ n \in \mathbb{N},$$

where $c > 1/4$. *Then equation* (2.48) *is oscillatory.*

For the oscillation of the Euler equation $x'' + \gamma t^{-2}x = 0$, Elbert's Theorem 26 is sharper than Huang's Theorem 23. Indeed, if $\gamma > 1/4$, $t_n = n+1$, $q(t) = \gamma/t^2$ and $c_n = \gamma/[(n+1)(n+2)]$, then $t_{n+2}-t_{n+1} = t_{n+1}-t_n = 1$ and

$$(t_{n+1}-t_n)\int_{t_n}^{t_{n+1}} q(t)dt = \int_{n+1}^{n+2} \frac{\gamma}{t^2}dt = c_n > \frac{c}{(n+1)^2},\ \forall n \geqslant n_0,$$

for some $c > 1/4$ and large enough $n_0 \in \mathbb{N}$. Now, by Theorem 26 the oscillation of the Euler equation $x'' + \gamma t^{-2}x = 0$ is established in this way too.

Independently from Wong's Theorem 1, in 1999, Qingkai Kong derived the following type of interval oscillation criteria for unforced linear differential equation (2.1): $(r(t)x')' + q(t)x = e(t)$. This theorem inspired some other authors to study the oscillation of several types of second-order differential equations, that will be briefly pointed out below. Denote $D = \{(t,s)\ :\ -\infty < s \leqslant t < \infty\}$ and let

$$\mathscr{H} = \Big\{H = H(t,s) \mid H \in C\big(D,\mathbb{R}_+\big),\ H(t,t) = 0,\ H(t,s) > 0 \text{ for } t > s, \\ \text{there exist } \frac{\partial H}{\partial t}, \frac{\partial H}{\partial s} \text{ such that } h_1, h_2 \in L_{\text{loc}}(D,\mathbb{R})\Big\}, \tag{2.53}$$

where

$$h_1(t,s) = \frac{1}{\sqrt{H(t,s)}}\frac{\partial H}{\partial t}(t,s) \quad \text{and} \quad h_2(t,s) = -\frac{1}{\sqrt{H(t,s)}}\frac{\partial H}{\partial s}(t,s). \tag{2.54}$$

THEOREM 27. ([50, Theorem 2.1] - from 1999) *Let* $e(t) \equiv 0$ *and* $\mathscr{H}$ *be the class of functions defined by* (2.53). *Equation* (2.1) *is oscillatory provided for any* $T \geqslant t_0$ *there exists* $H \in \mathscr{H}$ *and either*

(i) *there exists* $a,b,c \in \mathbb{R}$ *such that* $T \leqslant a < c < b$ *and*

$$\frac{1}{H(c,a)}\int_a^c \left(q(s)H(s,a) - \frac{1}{4}r(s)h_1^2(s,a)\right)ds + \frac{1}{H(b,c)}\int_c^b \left(q(s)H(b,s) - \frac{1}{4}r(s)h_2^2(b,s)\right)ds > 0, \tag{2.55}$$

or

(ii) *there exists* $a,b \in \mathbb{R}$ *such that* $T \leqslant a < b$ *and for any* $c \in [a,b]$ *at least one of the next two inequalities holds:*

$$\int_a^c \left(q(s)H(s,a) - \frac{1}{4}r(s)h_1^2(s,a)\right)ds > 0 \tag{2.56}$$

or

$$\int_c^b \left(q(s)H(b,s) - \frac{1}{4}r(s)h_2^2(b,s)\right)ds > 0. \tag{2.57}$$

EXAMPLE 2. Consider equation (2.1): $(r(t)x')' + q(t)x = e(t)$, where $e(t) \equiv 0$, $r(t) = t$ and

$$q(t) = \begin{cases} k/t, & e^{2n} \leqslant t \leqslant e^{2n+1} \\ q_n(t), & e^{2n+1} < t < e^{2n+2}, \end{cases} \qquad n \in \mathbb{N},$$

with $k > 12$ and $q_n \in C((e^{2n+1}, e^{2n+2}), \mathbb{R})$ such that $\int_{e^{2n+1}}^{e^{2n+2}} q_n(t)\,dt = -n$ for $n \in \mathbb{N}$. It is easy to see that $\int_0^\infty q(t)\,dt = -\infty$.

For any $T \geqslant 0$ there exists $n \in \mathbb{N}$ such that $e^{2n} \geqslant T$. Let $a = e^{2n}, b = e^{2n+1}$, and $H(t,s) = (\ln t - \ln s)^2$. Then $h_1(t,s) = 2/t$ and $h_2(t,s) = 2/s$. To show that Eq. (2.1) is oscillatory by Theorem 27, (ii), it suffices to show that for all $c \in [a,b]$

$$\begin{aligned} G(c) := &\int_a^c H(s,a)q(s)\,ds + \int_c^b H(b,s)q(s)\,ds \\ &-\frac{1}{4}\left(\int_a^c r(s)h_1^2(s,a)\,ds + \int_c^b r(s)h_2^2(b,s)\,ds\right) > 0, \end{aligned}$$

and then we see that for any $c \in [a,b]$, at least one of (2.56) and (2.57) holds. In fact, by a simple computation we have that

$$\begin{aligned} G(c) &= \int_a^c (\ln s - \ln a)^2 \frac{k}{s} ds + \int_c^b (\ln b - \ln s)^2 \frac{k}{s} ds - \frac{1}{4}\int_a^b \frac{4}{s} ds \\ &= \frac{k}{3}[(\ln c - \ln a)^3 + (\ln b - \ln c)^3] - (\ln b - \ln a) \\ &= k(\ln b - \ln a)[(\ln c - \frac{1}{2}(\ln b + \ln a))^2 + \frac{1}{12}(\ln b - \ln a)^2] - (\ln b - \ln a) \\ &\geqslant \frac{k}{12}(\ln b - \ln a)^3 - (\ln b - \ln a) \\ &= \frac{k}{12} - 1 > 0. \end{aligned}$$

This means that Eq. (2.1) is oscillatory. □

Next, let $\mathscr{H}_0$ be a subclass of $\mathscr{H}$ defined by

$$\mathscr{H}_0 = \{H = H(t,s), H \in \mathscr{H} \mid \exists H_0 = H_0(y),\ H_0 : \mathbb{R} \to \mathbb{R} \text{ such that } H(t,s) = H_0(t-s)\}. \quad (2.58)$$

It is elementary to check that for $H \in \mathscr{H}_0$:

$$\frac{\partial H}{\partial t}(t,s) = \frac{\partial H}{\partial s}(t,s) = H_0'(t-s).$$

THEOREM 28. ([50, Theorem 2.2] - from 1999) *Let* $e(t) \equiv 0$ *and* $\mathscr{H}_0$ *be a class of functions defined by* (2.58). *Let for any* $T \geqslant t_0$ *there exists* $H \in \mathscr{H}_0$ *and* $a, c \in \mathbb{R}$ *such that* $T \leqslant a < c$ *and*

$$\int_a^c \left(H_0(s-a)\left[q(s) + q(2c-s)\right] - \frac{1}{4}\left[r(s) + r(2c-s)\right] \frac{H_0'^2(s-a)}{H_0(s-a)} \right) ds > 0. \quad (2.59)$$

Then equation (2.1) *is oscillatory.*

EXAMPLE 3. Consider equation (2.1) with $r(t) \equiv 1$, $e(t) \equiv 0$ and

$$q(t) = k\left(\sin(\frac{2\pi}{3}t) - \frac{1}{2} \right)$$

with $k > 32$. For any $T \geqslant 0$ there exists $n \in \mathbb{N}_0$ such that $3n \geqslant T$. Let $a = 3n + 1/4$ and $c = 3n + 3/4$. Note that $q_1(t) := \sin(\frac{2\pi}{3}t) - \frac{1}{2}$ satisfies that $q_1(a) = 0$, $(q_1(c) - q_1(a))/(c-a) = 1$, and q_1 is concave down on (a,c). We have $q(t) \geqslant k(t-a)$. Let $H_0(t-s) = (t-s)^2$. Then $H_0'(t-s) = 2(t-s)$. It follows that

$$\begin{aligned} \int_a^c H_0(s-a)[q(s) + q(2c-s)]\,ds &= 2\int_a^c (s-a)^2 q(s)\,ds \\ &\geqslant 2k \int_a^c (s-a)^3\,ds = \frac{k}{32} \end{aligned}$$

and

$$\int_a^c \frac{1}{4}[r(s) + r(2c-s)] \frac{H_0'^2(s-a)}{H_0(s-a)} ds = 1.$$

Hence

$$\int_a^c \left(H_0(s-a)[q(s) + q(2c-s)] - \frac{1}{4}[r(s) + r(2c-s)] \frac{H_0'^2(s-a)}{H_0(s-a)} \right) \geqslant \frac{k}{32} - 1 > 0.$$

By Theorem 28, Eq. (2.1) is oscillatory. However, in this equation we have $\int_0^\infty q(t)\,dt = -\infty$. □

REMARK 9. In 1999 Qingkai Kong [51, Corollary 2.2] generalized his Theorem 27 to the half-linear equation

$$\left(r(t)x'^{\alpha}\right)' + q(t)x^{\alpha} = 0,\ t \geqslant t_0,$$

where $\alpha = \text{odd}/\text{odd}$ and $1/r, q \in L^1_{\text{loc}}([t_0,\infty),\mathbb{R})$. □

REMARK 10. In 2000 Wan-Tong Li and Ravi P. Agarwal in [60, Theorem 2.1] and [61, Theorem 2.1] extended Kong's Theorem 27 respectively to the nonlinear differential equations with damping:

$$\big(r(t)x'\big)' + p(t)x' + q(t)f(x) = 0,\ t \geqslant t_0, \tag{2.60}$$

and without damping:

$$x'' + q(t)f(x)g(x') = 0,\ t \geqslant t_0,$$

where $f(x)x > 0$ for $x \neq 0$. See also Wan-Tong Li and Hai-Feng Huo [63, Theorem 2.1] from 2001 concerning the equation (2.60) with $p(t) \equiv 0$. □

REMARK 11. Besides Theorem 10 mentioned in previous subsection, in 2003 Qigui Yang in [117] generalized both Kong's interval oscillation criteria to the nonlinear equation (2.19): $(r(t)x')' + q(t)f(x) = e(t)$, see [117, Theorem 2 and Corollaries 3,4 and 5]. □

In 2004 James S.W. Wong in his paper [112] proved the following extensions of Huang's results.

THEOREM 29. ([112, Theorem 2] - from 2004) *Let $q(t) \geqslant 0$, $q \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R})$ and $\lambda > 1$. Assume for some t_0 and every positive integer n,*

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(t)dt \geqslant \frac{c}{(\lambda-1)\lambda^n t_0},$$

where $c > k_0(\lambda) = (\sqrt{\lambda} - 1)^2$. Then equation (2.48) is oscillatory.

For a related nonoscillatory criterion we refer the reader to [112, Theorem 1]. Especially for $\lambda = 2$ Wong's Theorem 29 covers Huang's Theorem 23. Furthermore, the author showed that such a class of interval criterion is not a special case of known Hille's oscillation criterion. Moreover, Wong extended Theorem 29 to the case of delay linear differential equation:

$$x''(t) + q(t)x(\tau(t)) = 0,\ t \geqslant t_0 > 0, \tag{2.61}$$

where $\tau(t)$ is a continuous function such that $\tau(t) \leqslant t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

THEOREM 30. ([112, Theorem 4] - from 2004) *Let $q \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R})$ such that $q(t) \geqslant 0$ and $\lambda > 1$. Assume for some t_0 and every positive integer n,*

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(t)\tau(t)dt \geqslant \frac{\hat{c}}{\lambda^n t_0},$$

where $\hat{c} > k_0(\lambda) = (\sqrt{\lambda} - 1)^2$. Then equation (2.61) is oscillatory.

REMARK 12. In 2004 Qigui Yang and Ronald M. Mathesen [118, Theorems 2.1 and 2.2] extended Kong's Theorem 27 to the nonlinear delay differential equation

$$\big(r(t)\psi(x(t))x'(t)\big)' + F\big(t, x(t), x'(t), x(\tau(t)), x'(\tau(t))\big) = 0,\ t \geqslant t_0,$$

where F is a continuous function on $[t_0,\infty) \times \mathbb{R}^4$, $\tau(t) \leqslant t$, $\tau(t)$ is increasing and $\lim_{t\to\infty} \tau(t) = \infty$. □

Yuan Gong Sun, C.H. Ou and James S.W. Wong in [88] studied the oscillation of the second-order inhomogeneous linear differential equation (2.1): $\left(r(t)x'\right)' + q(t)x = e(t)$, where $q(t)$ and $e(t)$ may have different signs on given intervals $[a_n, b_n] \subseteq [0,\infty)$, $a_n < b_n \leqslant a_{n+1} < b_{n+1}$ and $\lim_{n\to\infty} a_n = \infty$.

THEOREM 31. ([88, Theorem 1] - from 2004) *Let there exist a sequence of functions* $\varphi_n(t)$ *such that:*

$$\left(r(t)\varphi_n'\right)' + q(t)\varphi_n \geqslant e(t),\ \left(r(t)\varphi_n'\right)' + q(t)\varphi_n \not\equiv 0 \quad \text{and} \quad \varphi_n^j(a_n) = \varphi_n^j(b_n) = 0,\ j = 1,2,$$

and

$$(-1)^n \int_{a_n}^{b_n} \varphi_n(t)e(t)dt \geqslant 0.$$

Then equation (2.1) *is oscillatory.*

Also, the author proved that the inequality "$\geqslant$" in all assumptions of Theorem 31 can be replaced by "$\leqslant$" so that equation (2.1) is still oscillatory, see [88, Theorems 2 and 3]. Theorem 31 was illustrated with the equation:

$$x'' + c(\sin t)x = t^\beta \cos t,\ \beta \geqslant 0. \tag{2.62}$$

Since for some $\alpha > 1$, $a_{2n-1} = 2n\pi$, $b_{2n-1} = a_{2n} = 2n\pi + \pi$ and $b_{2n} = 2n\pi + 2\pi$, the sequence of functions

$$\varphi_n(t) = \sin^\alpha\left(\frac{t - a_n}{b_n - a_n}\right),\ t \in [a_n, b_n],$$

satisfies all assumptions of Theorem 31, the author showed that equation (2.62) is oscillatory provided

$$c \geqslant \max_{0\leqslant t\leqslant \pi/2} \left\{-\frac{(\alpha-1)\alpha\cos^2 t - \alpha\sin^2 t}{\sin^3 t}\right\}.$$

REMARK 13. In 2005 Zhiting Xu and Shiguo Peng in [114, Theorem 2.1] extended Kong's Theorem 27 to the second-order half-linear damped differential equation:

$$\left(r(t)\Phi_\alpha(x')\right)' + p(t)\Phi_\alpha(x') + q(t)f(x) = 0,\ t \geqslant t_0,$$

where $\Phi_\alpha(u) = |u|^{\alpha-2}u$, $\alpha > 1$ and $f(u)u > 0$ for $u \neq 0$. □

In 2007 Qingkai Kong in [52] studied the oscillation of the second-order half-linear differential equation

$$\left(r(t)\Phi_\alpha(x')\right)' + q(t)\Phi_\alpha(x) = 0, \tag{2.63}$$

where $\Phi_\alpha(u) = |u|^{\alpha-1}u$, $\alpha > 0$, $r(t) > 0$, $q(t) \geqslant 0$ and $\int_0^\infty r^{-1/\alpha}(t)dt = \infty$. He extended Huang's Theorem 23 from linear to the half-linear case as follows.

THEOREM 32. ([52, Theorem 2.2] - from 2007) *Let* $r(t) \equiv 1$, $\lambda > 1$ *and* $c^* = c^*(\alpha)$. *Assume there exist* $t_0 \in (0,\infty)$ *and* $c > c^*$ *such that for each* $n \in \mathbb{N}_0$,

$$(t_{n+1} - t_n)\left(\int_{t_n}^{t_{n+1}} q(t)dt\right)^{1/\alpha} \geqslant c,$$

where $t_n = \lambda^n t_0$. *Then equation* (2.63) *is oscillatory.*

The proof of Theorem 32 was motivated by that of Wong's Theorem 29.

THEOREM 33. ([52, Theorem 2.4] - from 2007) *Let* $\lambda > 1$ *and* $c^* = c^*(\alpha)$. *Assume there exist* $t_0 \in (0,\infty)$ *and* $c > c^*$ *such that for each* $n \in \mathbb{N}_0$,

$$\int_{t_n}^{t_{n+1}} r^{-1/\alpha}(t)dt\left(\int_{t_n}^{t_{n+1}} q(t)dt\right)^{1/\alpha} \geqslant c,$$

where $t_n = g^{-1}\left(\lambda^n \int_0^{t_0} r^{-1/\alpha}(s)ds\right)$ *and* $g(t) = \int_a^t r^{-1/\alpha}(s)ds$. *Then equation* (2.63) *is oscillatory.*

Related nonoscillation criteria for equation (2.63) can be found in [52, Theorems 2.1 and 2.3]. The interval oscillation criteria given in Kong's Theorems 32 and 33 are quite different from those given in Wan-Tong Li and Sui Sun Cheng's Theorem 8, which deals with the forced half-linear differential equations.

REMARK 14. In 2007 Yuri V. Rogovchenko and Fatos Tuncay in [81, Theorems 1-5] extended Kong's Theorem 27 to the following class of second-order nonlinear differential equations with damping term:

$$\left(r(t)\psi(x)x'\right)' + p(t)x' + q(t)f(x) = 0,\ t \geqslant t_0,$$

where $f(u)u > 0$ for $u \neq 0$. □

REMARK 15. In 2008 T. Hassan in [41, Theorem 2.1] extended Kong's Theorem 27 to a class of second-order nonlinear differential equations with a nonlinear damping term

$$\left(r(t)x'\right)' + p(t)x' + q(t)|x|^{\alpha-1}x = 0,\ t \geqslant t_0,$$

where $\alpha \geqslant 1$ and the case $\int_{t_0}^{\infty} q(t)dt = -\infty$ is allowed such as in Theorem 27. □

REMARK 16. In 2009, A.K. Nandakumaran and S. Panigrahi in [67, Theorems 2.3, 2.7, 3.1, 3.2] extended Kong's Theorem 27 to the nonlinear differential equation:

$$\left(r(t)x'\right)' + p(t)x' + q(t)f(x)g(x') = 0,\ t \geqslant t_0,$$

where $f(x)x > 0$ for $x \neq 0$. □

In 2011, J. Tyagi [97] studied an interval oscillation criterion of the unforced linear differential equation (2.1): $\left(r(t)x'\right)' + q(t)x = e(t)$ improving El-Sayed's Theorem 6 which is specialized for the forced case.

THEOREM 34. ([97, Theorem 2.4] - from 2011) *Let* $e(t) \equiv 0$. *Let there exist a monotonic sequence* $a_n > 0$ *such that* $a_n \to \infty$ *and a sequence* $k_n > 0$, $n \in \mathbb{N}$. *Denote by* $b_n = \pi/\sqrt{k_n}$, $n \in \mathbb{N}$. *If* $0 < r(t) \leqslant 1$ *on* $[a_n, a_n + b_n]$ *and for all* $n \in \mathbb{N}$,

$$\int_t^{a_n+\frac{b_n}{2}} q(s)ds \geqslant k_n\left(a_n + \frac{b_n}{2} - t\right),\ \forall t \in \left[a_n, a_n + \frac{b_n}{2}\right],$$
$$\int_{a_n+\frac{b_n}{2}}^{t} q(s)ds \geqslant k_n\left(t - a_n - \frac{b_n}{2}\right),\ \forall t \in \left[a_n + \frac{b_n}{2}, a_n + b_n\right].$$

Then equation (2.1) *is oscillatory.*

The author illustrated this theorem with the equation

$$\left((1-\alpha\sin^2 t)x'\right)' + (1+2\cos t)x = 0,\ 0\leqslant\alpha<1,$$

by choosing $a_n = 2n\pi$ and $k_n = 1/16$. It is interesting that the mean value of $q(t) = 1+2\cos t$ is non-zero.

REMARK 17. In 2013 E. Tunc and H. Avci in [96, Theorem 1] extended Kong's Theorem 27 to the unforced equation (2.27): $\left(r(t)k_1(x,x')\right)' + p(t)k_2(x,x')x' + q(t)f(x) = 0$, $t\geqslant t_0$, where the functions $k_1(u,v)$, $k_2(u,v)$ and $f(u)$ satisfy the same assumptions as in Shi's Theorem 13. However, their oscillation criterion is different from the one given in Theorem 13. □

In Pašić [72], the author studied a kind of pointwise interval oscillation criteria for equation (2.27): $\left(r(t)k_1(x,x')\right)' + p(t)k_2(x,x')x' + q(t)f(x) = e(t)$, where the functions $k_1(u,v)$ and $k_2(u,v)$ satisfy the following conditions that are more general than (2.28): $k_1(u,v)v\geqslant\alpha_1 k_1^2(u,v)$ and (2.29): $k_2(u,v)uv\geqslant\alpha_2 k_1^2(u,v)$ as imposed in Shi's Theorem 13:

$$k_1(u,v)v\geqslant\alpha_1|k_1(u,v)|^\beta|u|^{2-\beta}\ \text{ for some } \alpha_1>0,\ \beta>1 \text{ and all } (u,v)\in\mathbb{R}^2,\ u\neq 0, \tag{2.64}$$

$$k_2(u,v)uv\geqslant 0\ \text{ for all } (u,v)\in\mathbb{R}^2,\ u\neq 0. \tag{2.65}$$

Especially for $\beta=2$, assumption (2.64) becomes (2.28). Also, unlike (2.29), assumption (2.65) does not depend on $k_1(u,v)$. Consequently, (2.64)-(2.65) are more general than (2.28)-(2.29). Moreover, assumption (2.64) allows two main classes of quasilinear second-order differential operators: α-Laplacian and the prescribed mean curvature, see for details [73, Remark 1] or Example 4 below. Even for $\beta=2$, assumptions (2.64)-(2.65) give more possibilities than (2.28)-(2.29). Precisely, by Comment 6 we know that (2.28)-(2.29) restrict the choice for the functions $k_1(u,v)$ and $k_2(u,v)$ so that both of them should depend on the first variable u, for instance, $k_1(u,v)=\phi_1(u)v$ and $k_2(u,v)=\phi_2(u)v$, where $\phi_1,\phi_2\in C(\mathbb{R},\mathbb{R})$ are non-constant functions. However, from assumptions (2.64) and (2.65), it is possible to have, for instance, $k_1(u,v)=v$ and $k_2(u,v)=uv$, where $k_1(u,v)$ only depends on v.

COMMENT 8. Shang and Qin [84] pointed out that some properties of $k_1(u,v)$ and $k_2(u,v)$ related to (2.28) and (2.29) can contradict the basic assumption on the continuity of $k_1(u,v)$ and $k_2(u,v)$ in both variables. □

THEOREM 35. ([72, Theorem 1.1] - from 2013) *Let $r(t)>0$, $r',p,q,e\in C([t_0,\infty),\mathbb{R})$ and $f(u)$ satisfy (2.20). Let for any $T>0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T\leqslant a_1$ and $b_1\leqslant a_2$, such that $e(t)\leqslant 0$ on $[a_1,b_1]$ and $e(t)\geqslant 0$ on $[a_2,b_2]$, and $q(t)\geqslant 0$, $q(t)\not\equiv 0$ on $(a_1,b_1)\cup(a_2,b_2)$. Assume (2.64) and (2.65). Equation (2.27) is oscillatory provided there are a real parameter $\lambda>0$ and a function $C=C(t)$, $C\in L^1((a_1,b_1)\cup(a_2,b_2),\mathbb{R}_+)$, such that*

$$\frac{1}{c_i}C(t)\leqslant\frac{\beta}{2\pi}\left(\sin\frac{\pi}{\beta}\right)\min\left\{\frac{\alpha_1}{(\lambda r(t))^{\beta-1}},\lambda Q(t)\right\},\ \ t\in[a_i,b_i],\ i\in\{1,2\},$$

where $Q(t)$ and c_i are defined by:

$$Q(t)=\begin{cases}Kq(t) & \text{if } \gamma=1,\\ \gamma(\gamma-1)^{\frac{1-\gamma}{\gamma}}[Kq(t)]^{\frac{1}{\gamma}}|e(t)|^{\frac{\gamma-1}{\gamma}} & \text{if } \gamma>1,\end{cases}\quad \text{and}\quad c_i:=\int_{a_i}^{b_i}C(\tau)d\tau>0,\ i\in\{1,2\},$$

where the constants K and γ are given in (2.20).

EXAMPLE 4. (about the hypotheses (2.64)-(2.65)) Let the functions $r(t)$, $p(t)$, $q(t)$, $f(u)$ and $e(t)$ satisfy conditions from Theorem 35. Then the following three classes of equations are oscillatory:

$$\left(r(t)x'\right)' + p(t)|x|^{\alpha}\operatorname{sgn}(x)x'^2 + q(t)|x|^{\gamma}\operatorname{sgn}(x) = e(t), \tag{2.66}$$

where $\alpha > 0$ and $\gamma \geqslant 1$;

$$\left(r(t)\phi_1(x)\frac{x'}{\sqrt{1+x'^2}}\right)' + p(t)\phi_2(x)x'^2 + q(t)f(x) = e(t), \tag{2.67}$$

where $0 < \phi_1(u) \leqslant 1$ and $u\phi_2(u) \geqslant 0$ for all $u \neq 0$;

$$\left(r(t)\phi(x)|x'|^{\alpha-1}x'\right)' + p(t)|x|^{\alpha-1}x\psi(x')x' + q(t)f(x) = e(t), \tag{2.68}$$

where $\alpha > 0$, $0 < \phi(u) \leqslant |u|^{1-\alpha}$ and $v\psi(v) \geqslant 0$ for all $u \neq 0$ and $v \in \mathbb{R}$.

Indeed: in equation (2.66) we have $k_1(u,v) = v$ and $k_2(u,v) = |u|^{\alpha}\operatorname{sgn}(u)v$ that satisfies (2.64)-(2.65) for $\beta = 2$; in equation (2.67) we have

$$k_1(u,v) = \phi_1(u)\frac{v}{\sqrt{1+v^2}} \quad \text{and} \quad k_2(u,v) = \phi_2(u)v$$

that satisfies (2.64)-(2.65) for $\beta = 2$ because $0 < \phi_1(u) \leqslant 1$ and $u\phi_2(u) \geqslant 0$ for all $u \neq 0$; and, in equation (2.68) we have $k_1(u,v) = \phi(u)|v|^{\alpha-1}v$ and $k_2(u,v) = |u|^{\alpha-1}u\psi(v)$ that satisfies (2.64)-(2.65) for $\beta = 1 + 1/\alpha$ because $\alpha > 0$, $0 < \phi(u) \leqslant |u|^{1-\alpha}$ and $v\psi(v) \geqslant 0$ for all $u \neq 0$ and $v \in \mathbb{R}$. $\square$

3. Equations with mixed nonlinearities

In this section, we are concerned with a class of second-order differential equations having the mixed nonlinearities. Recall that the forced Emden-Fowler equation (2.10) has a single term of sub-linear or super-linear nonlinearity. The following class of differential equations contains the sub-linear and super-linear terms simultaneously:

$$\left(r(t)x'\right)' + q(t)x + \sum_{j=1}^{n} q_j(t)|x|^{\alpha_j}\operatorname{sgn}(x) = e(t),\ t \geqslant 0, \tag{3.1}$$

where $n \in \mathbb{N}$, $\alpha_1 > \ldots > \alpha_m > 1 > \alpha_{m+1} > \ldots > \alpha_n > 0$. Since equation (3.1) contains the sub-linear and super-linear terms, it is originally called the equation with mixed nonlinearities. In a larger sense, if a differential equation contains at least two different super-linear terms, then it is also called the equation with mixed nonlinearities, see in Section 3.2 about the so-called super-half-linear equations.

3.1. Main results

The first result on the oscillations of equations with mixed nonlinearities was given in 2007 due to Yuan Gong Sun and James S.W. Wong in the following theorem.

THEOREM 36. ([92, Theorem 1] - from 2007) *Let for any $T>0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $b_1 \leqslant a_2$, such that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$, and $q_j(t) \geqslant 0$, $q_j(t) \not\equiv 0$ on $(a_1,b_1) \cup (a_2,b_2)$, $j=1,2,..,n$. Let $D(a_i,b_i)$, $i=1,2$ be the set of functions defined by (2.2). Let $\eta_0,\eta_1,...,\eta_n$ be positive constants satisfying*

$$\sum_{j=1}^{n} \alpha_j \eta_j = 1,\ 0<\eta_j<1,\ \sum_{j=1}^{n} \eta_j < 1 \quad \textit{and} \quad \eta_0 = 1 - \sum_{j=1}^{n} \eta_j. \tag{3.2}$$

If there exists a function $u \in D(a_i,b_i)$ such that

$$\int_{a_i}^{b_i} \big(Q(t)u^2(t) - r(t)u'^2(t)\big)\,dt \geqslant 0,\ i=1,2, \tag{3.3}$$

where

$$Q(t) = q(t) + |e(t)|^{\eta_0} \prod_{j=0}^{n} \left(\frac{q_j(t)}{\eta_j}\right)^{\eta_j}, \tag{3.4}$$

then equation (3.1) is oscillatory.

To the best of our knowledge, it seems that this type of equation has not been considered in the oscillation theory before Sun and Wong have published Theorem 36. This theorem generalizes Wong's Theorem 1 and Yang's Theorem 10 in the method as well as in the result. In fact, if we put $q_j(t) \equiv 0$ into equation (3.1) and conditions (3.3)-(3.4), then we obtain equation (2.1) and condition (2.3) respectively. Thus, Theorem 1 is a special case of Theorem 36. The main results of [92, Theorem 1] was illustrated by the following forced Emden-Fowler equation of the mixed type (see [92, Example 1]):

$$x'' + c_0 \sin 2t\, x + 2c_1 \sin t\, |x|^{5/2} \operatorname{sgn} x + 2c_2 \cos t\, |x|^{1/2} \operatorname{sgn} x = -\cos 2t,\ t \geqslant 0,$$

where c_0, c_1, c_2 are three arbitrary positive constants.

In the case $e(t) \equiv 0$, the author established the following oscillation criterion.

THEOREM 37. ([92, Theorem 2] - from 2007) *Let for any $T>0$ there exists a subinterval $[a_1,b_1]$ of $[T,\infty)$ such that $q_j(t) \geqslant 0$, $q_j(t) \not\equiv 0$ on (a_1,b_1), $j=1,2,..,n$. Let $D(a,b)$ be the set of functions defined by (2.2). Let $r(t)>0$ and $q \in C([t_0,\infty),\mathbb{R})$. Let $\eta_1,...,\eta_n$ be positive constants satisfying*

$$\sum_{j=1}^{n} \alpha_j \eta_j = 1,\ \ 0<\eta_j<1 \quad \textit{and} \quad \sum_{j=1}^{n} \eta_j = 1.$$

If there exists a function $u \in C^1([a_1,b_1],\mathbb{R})$ such that $u(t) \not\equiv 0$ on $[a_1,b_1]$, $u(a_1)=u(b_1)=0$ and

$$\int_{a_1}^{b_1} \Big(Q(t)u^2(t) - r(t)u'^2\Big)\,dt \geqslant 0,$$

where

$$Q(t) = q(t) + \prod_{j=1}^{n} \left(\frac{q_j(t)}{\eta_j}\right)^{\eta_j}, \tag{3.5}$$

then equation (3.1) with $e(t) \equiv 0$ is oscillatory.

For the case when $(-1)^i e(t) > 0$ we refer the reader to [92, Theorem 3]. In [92, Section 3], the author presented several possibilities for choosing the test function $u(t)$ such that $u \in C^1([a_i,b_i],\mathbb{R})$, $u(a_i)=u(b_i)=0$, $u \not\equiv 0$ on $[a_i,b_i]$ and $u(t)$ satisfies (3.3).

3.2. Extensions by other authors

In this section we present several oscillation criteria for the second-order differential equations with mixed-nonlinearities whose publication were motivated by Sun and Wong's Theorems 36 and 37. All results are rearranged in the chronological order: – in 2008, Yuan Gong Sun and Fan Wei Meng [90]; – in 2009, Zhaowen Zheng, Xiao Wang and Hongmei Han [124], Ravi P. Agarwal and Agacik Zafer [3]; – in 2010, Ravi P. Agarwal, Douglas R. Anderson and Agacik Zafer [4], Yuzhen Bai and Lihua Liu [8], Sowdaiyan Murugadass, Ethiraju Thandapani and Sandra Pinelas [64]; – in 2011, Taher S. Hassan and Qingkai Kong [43], Taher S. Hassan, Lynn Erbe and Allan Peterson [42]; – in 2012, Jing Shao, Fanwei Meng and Xinqin Pang [82]; – in 2013, Mervan Pašić [71].

We first present the oscillation criteria for equation (3.1) due to Yuan Gong Sun and Fan Wei Meng in [90]. For this purpose, we introduce a class of functions:

$$\mathscr{D}_{a,b} = \Big\{ H = H(t,s) : H \in C^1\big([a,b]\times[a,b],\mathbb{R}\big),\ H(b,t) > 0,\ H(s,a) > 0,\ \forall s,t \in [a,b] \Big\}.$$

Also, one can use the notation

$$h_1(t,s) = \frac{1}{2\sqrt{H(t,s)}}\frac{\partial H}{\partial t}(t,s) \quad \text{and} \quad h_2(t,s) = -\frac{1}{2\sqrt{H(t,s)}}\frac{\partial H}{\partial s}(t,s). \tag{3.6}$$

THEOREM 38. ([90, Theorem 1] - from 2008) *Let for any $T>0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $b_1 \leqslant a_2$, and two intermediate points $c_i \in [a_i,b_i]$, $i=1,2$ such that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$, and $q_j(t) \geqslant 0$, $q_j(t) \not\equiv 0$ on $(a_1,b_1)\cup(a_2,b_2)$, $j=1,2,..,n$. Let $\eta_0,\eta_1,\dots,\eta_n$ be positive constants satisfying (3.2). If there exist two functions $H_i = H_i(t,s)$, $H_i \in \mathscr{D}_{a_i,b_i}$, $i=1,2$ with $h_{i1}(s,t), h_{i2}(s,t)$ defined in (3.6) such that*

$$\frac{1}{H_i(c_i,a_i)}\int_{a_i}^{c_i}\big(Q(t)H_i(t,a_i) - r(t)h_{i1}^2(t,a_i)\big)dt + \frac{1}{H_i(b_i,c_i)}\int_{c_i}^{b_i}\big(Q(t)H_i(b_i,t) - r(t)h_{i2}^2(b_i,t)\big)dt > 0,\ i=1,2, \tag{3.7}$$

where $Q(t)$ is defined in (3.4), then equation (3.1) is oscillatory.

By this theorem, the authors showed the oscillation of the following equation with mixed nonlinearities (see [90, Example 1]):

$$x'' + k\sin t\,|x|^{\alpha_1}\operatorname{sgn} x + l\cos t\,|x|^{\alpha_2}\operatorname{sgn} x = -m\cos 2t,\ t \geqslant 0,$$

where k,l,m are three arbitrary positive constants and $\alpha_1 > 1$, $0 < \alpha_2 < 1$. See also [90, Example 2].

COMMENT 9. It is interesting to compare Sun-Wong's Theorem 38 with $q_j(t) \equiv 0$, $j = 1,2,..,n$ and Kong's Theorem 27. □

In 2009, Zhaowen Zheng, Xiao Wang and Hongmei Han [124] studied the oscillation of the second-order forced half-linear differential equation with mixed nonlinearities:

$$\big(r(t)\Phi_\alpha(x')\big)' + q(t)\Phi_\alpha(x) + \sum_{j=1}^{n} q_j(t)\Phi_{\alpha_j}(x) = e(t),\ t \geqslant t_0, \tag{3.8}$$

where $\Phi_\alpha(x) = |x|^{\alpha-1}x$, $n \in \mathbb{N}$ and $\alpha_n > \ldots > \alpha_2 > \alpha_1 > \alpha > 0$. These exponents are not exactly mixed in the sense of Wong's Theorem since they are all super-half-linear. If $n = 1$, then equation (3.8) becomes super-half-linear equation (3.8) considered in Zheng and Meng's Theorem 14.

THEOREM 39. ([124, Theorem 2.2] - from 2009) *Let $r(t) > 0$, $e \in C([t_0,\infty),\mathbb{R})$. Assume for any $T > 0$ there exist intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T < a_1 < b_1 \leqslant a_2 < b_2$ such that $e(t)$ has different signs on $[a_1,b_1]$ and $[a_2,b_2]$. If there exist a function $u \in C^1([a_i,b_i],\mathbb{R})$, $u^{\alpha+1}(t) > 0$ on (a_i,b_i) and $u(a_i) = u(b_i) = 0$ and a positive function $\phi \in C([t_0,\infty),\mathbb{R})$ such that*

$$\int_{a_i}^{b_i} \phi(t)\left[\left(q(t)+\sum_{j=1}^{m} Q_j(t)\right)u^{\alpha+1}(t) - r(t)\left(|u'(t)|+|u(t)|\frac{|\phi'(t)|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]dt > 0,$$

where

$$Q_j(t) = \alpha^{-\frac{\alpha}{\alpha_j}}\alpha_j[n(\alpha_j-\alpha)]^{\frac{\alpha-\alpha_j}{\alpha_j}}[q(t)]^{\frac{\alpha}{\alpha_j}}|e(t)|^{\frac{\alpha_j-\alpha}{\alpha_j}},\ 1 \leqslant j \leqslant n, \tag{3.9}$$

then (3.8) is oscillatory.

The author tested this theorem by the forced Duffing equation

$$x'' + x + \varepsilon x^3 = \varepsilon \sin^3 t,\ t \geqslant 0, \tag{3.10}$$

which allows an explicit solution $x(t) = \sin t$. Even in the forced linear second-order differential equations, oscillation and nonoscillation can occur simultaneously (since the Sturm's theorem generally does not hold in the forced case). However, according to Theorem 39, the authors showed that equation (3.10) is oscillatory, by choosing for $u(t) = \sin t$, $\phi(t) \equiv 1$, $a_1 = 2k\pi$, $b_1 = (2k+1)\pi = a_2$ and $b_2 = (2k+2)\pi$. About the oscillation of a large class of forced Duffing equations, we refer the reader to M. Pašić [74] and references therein.

In the same year, Ravi P. Agarwal and Agacik Zafer [3] studied the oscillation of the second-order forced half-linear dynamic equation on time scales with mixed nonlinearities:

$$\left(r(t)\Phi_\alpha(x^\Delta)\right)^\Delta + q(t)\Phi_\alpha(x^\sigma) + \sum_{j=1}^{n} q_j(t)\Phi_{\alpha_j}(x^\sigma) = e(t),\ t \in [t_0,\infty)_\mathbb{T}, \tag{3.11}$$

where $\Phi_\alpha(x) = |x|^{\alpha-1}x$, $n \in \mathbb{N}$ and $\alpha_1 > \ldots > \alpha_m > \alpha > \alpha_{m+1} > \ldots > \alpha_n > 0$. This equation is a time scale generalization of equation (3.8).

THEOREM 40. ([3, Theorem 3.1] - from 2009) *Let $r(t) > 0$, $q,e \in C_{\rm rd}([t_0,\infty)_\mathbb{T},\mathbb{R})$. Assume for any $T \in [t_0,\infty)_\mathbb{T}$ there exist a_i,b_i, $i = 1,2$, $T \leqslant a_1 < b_1 \leqslant a_2 < b_2$ such that $q_j(t) \geqslant 0$ on $[a_1,b_1]_\mathbb{T} \cup [a_2,b_2]_\mathbb{T}$ for $j = 1,2,\ldots,n$ and $(-1)^i e(t) \geqslant 0 (\not\equiv 0)$ on $[a_i,b_i]_\mathbb{T}$, $i = 1,2$. Let $\eta_0,\eta_1,\ldots,\eta_n$ be positive constants satisfying*

$$\sum_{j=1}^{n} \alpha_j\eta_j = \alpha,\ 0 < \eta_j < 1,\ \sum_{j=1}^{n} \eta_j < 1 \quad \text{and} \quad \eta_0 = 1 - \sum_{j=1}^{n} \eta_j. \tag{3.12}$$

If there exists a function $u \in C^1_{\rm rd}(\mathbb{T},\mathbb{R})$ such that $u(t) \not\equiv 0$ on $[a_i,b_i]_\mathbb{T}$, $u(a_i) = u(b_i) = 0$ and

$$\int_{a_i}^{b_i}\left(Q(t)|u^\sigma(t)|^{\alpha+1} - r(t)|u^\Delta(t)|^{\alpha+1}\right)\Delta t \geqslant 0,\ i = 1,2,$$

where

$$Q(t)=q(t)+|e(t)|^{\eta_0}\prod_{j=1}^{n}\left(\frac{q_j(t)}{\eta_j}\right)^{\eta_j},$$

then equation (3.11) *is oscillatory.*

In the case $e(t)\equiv 0$, the author established the following criterion.

THEOREM 41. ([3, Theorem 3.2] - from 2009) *Let* $r(t)>0$, $q,e\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. *Assume for any* $T\in[t_0,\infty)_{\mathbb{T}}$ *there exists a subinterval* $[a_1,b_1]_{\mathbb{T}}$ *of* $[T,\infty)_{\mathbb{T}}$ *such that* $q_j(t)\geqslant 0$ *on* $[a_1,b_1]_{\mathbb{T}}$. *Let* $\eta_1,\dots,\eta_n$ *be positive constants satisfying*

$$\sum_{j=1}^{n}\alpha_j\eta_j=\alpha,\;\; 0<\eta_j<1 \quad \textit{and} \quad \sum_{j=1}^{n}\eta_j=1.$$

If there exists a function $u\in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *such that* $u(t)\not\equiv 0$ *on* $[a_1,b_1]_{\mathbb{T}}$, $u(a_1)=u(b_1)=0$ *and*

$$\int_{a_1}^{b_1}\left(Q(t)|u^{\sigma}(t)|^{\alpha+1}-r(t)|u^{\Delta}(t)|^{\alpha+1}\right)\Delta t\geqslant 0,$$

where $Q(t)$ *is from* (3.5), *then equation* (3.11) *with* $e(t)\equiv 0$ *is oscillatory.*

Especially for $\alpha=1$ and $\mathbb{T}=\mathbb{R}$, Theorems 40 and 41 become Sun and Wong's Theorems 36 and 37. About the case $(-1)^i e(t)>0$ on $[a_i,b_i]_{\mathbb{T}}$, $i=1,2$, we refer the reader to [3, Theorem 3.3].

Theorem 41 has been illustrated by the following half-linear differential equation with positive constant coefficients:

$$\left(|x'|^{\alpha-1}x'\right)'+a|x|^{\alpha-1}x+b|x|^{\alpha_1-1}x+b|x|^{\alpha_2-1}x=0,\; t\geqslant 0, \tag{3.13}$$

where $0<\alpha_2<\alpha<\alpha_1$. The author showed that equation (3.13) is oscillatory provided

$$a+\left(\frac{b}{\eta_1}\right)^{\eta_1}\left(\frac{c}{\eta_2}\right)^{\eta_2}\geqslant 1,$$

where

$$\eta_1=\frac{\alpha-\alpha_2}{\alpha_1-\alpha_2}\quad\text{and}\quad \eta_2=\frac{\alpha_1-\alpha}{\alpha_1-\alpha_2}.$$

REMARK 18. In 2010 Ravi P. Agarwal, Douglas R. Anderson and Agacik Zafer in [4] studied the oscillation of the equation (3.11) with delay arguments:

$$\left(r(t)\Phi_{\alpha}(x^{\Delta}(t))\right)^{\Delta}+q(t)\Phi_{\alpha}(x(\tau_0(t)))+\sum_{j=1}^{n}q_j(t)\Phi_{\alpha_j}(x(\tau_j(t)))=e(t),\; t\in[t_0,\infty)_{\mathbb{T}},$$

where $\tau_j(t):\mathbb{T}\to\mathbb{T}$ are nondecreasing right-dense continuous functions with $\tau_j(t)\leqslant t$ and $\lim_{t\to\infty}\tau_j(t)=\infty$. The author derived the delay version of Theorems 40 and 41, see [4, Theorems 3.1 and 3.2]. □

In 2010, Yuzhen Bai and Lihua Liu in [8] studied the oscillation of second-order delay differential equation:

$$\left(r(t)x'(t)\right)'+\sum_{j=1}^{n}r_j(t)x(t-\tau_j)+\sum_{j=1}^{n}q_j(t)|x(t-\tau_j)|^{\alpha_j}\operatorname{sgn}x(t-\tau_j)=e(t),\; t\geqslant t_0, \tag{3.14}$$

where $\tau_j \geqslant 0$ and the exponents α_j are given as before. Clearly, equation (3.14) with $\tau_j = 0$, $j = 1,2,...,n$, becomes equation (3.1). The following result is a differential-functional generalization of the preceding Theorem 38 with the same notations for the set $\mathscr{D}_{a_i,b_i}$, $i = 1,2$ and the function $h_1(t,s)$, except that

$$h_2(t,s) = -\frac{1}{2\sqrt{H(t,s)}}\frac{\partial H}{\partial s}(t,s). \tag{3.15}$$

THEOREM 42. ([8, Theorem 2.2] - from 2010) *Let for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $b_1 \leqslant a_2$, and two intermediate points $c_i \in [a_i,b_i]$, $i = 1,2$ such that $e(t) \leqslant 0$ on $[a_1 - \tau_j, b_1]$ and $e(t) \geqslant 0$ on $[a_2 - \tau_j, b_2]$, $r_j(t) \geqslant 0$ and $q_j(t) \geqslant 0$, $q_j(t) \not\equiv 0$ on $(a_1 - \tau_j, b_1) \cup (a_2 - \tau_j, b_2)$, $j = 1,2,..,n$. Let $\eta_0, \eta_1, ..., \eta_n$ be positive constants satisfying* (3.2). *Let there exist two functions $H_i = H_i(t,s)$, $H_i \in \mathscr{D}_{a_i,b_i}$, $i = 1,2$ with $h_{i1}(s,t)$ and $h_{i2}(s,t)$ defined respectively in* (3.6) *and* (3.15) *such that*

$$\frac{1}{H_i(c_i,a_i)}\int_{a_i}^{c_i}\big(Q_i(t)H_i(t,a_i) - r(t)h_{i1}^2(t,a_i)\big)dt \\ + \frac{1}{H_i(b_i,c_i)}\int_{c_i}^{b_i}\big(Q_i(t)H_i(b_i,t) - r(t)h_{i2}^2(b_i,t)\big)dt > 0,\ i = 1,2, \tag{3.16}$$

where $Q_i(t)$, $i = 1,2$ are two functions defined by

$$Q_i(t) = \sum_{j=1}^{n} r_j(t)\Big(\frac{t-a_i}{t-a_i+\tau_j}\Big) + |e(t)|^{\eta_0}\prod_{j=0}^{n}\eta_j^{-\eta_j}\prod_{j=1}^{n} q_j^{\eta_j}(t)\Big(\frac{t-a_i}{t-a_i+\tau_j}\Big)^{\alpha_j\eta_j}. \tag{3.17}$$

Then equation (3.1) *is oscillatory.*

Comparing the main conditions (3.7) and (3.16) of Theorems 38 and 42 respectively, we see that there is only one difference between them because instead of function $Q(t)$ the functions $Q_i(t)$, $i = 1,2$ are appearing in (3.16). Moreover, if we put $\tau_j = 0$, $j = 1,2,...,n$, and $r_1(t) \equiv q(t)$, $r_j(t) \equiv 0$, $j = 2,..,n$, then $Q(t) \equiv Q_i(t)$, $i = 1,2$ and hence, Theorem 42 completely generalizes Theorem 38. According to Theorem 42, the oscillation of the following equation with mixed nonlinearities was shown (see [8, Section 3]):

$$x'' + k\sin t\left|x\Big(t-\frac{\pi}{8}\Big)\right|^{\alpha_1}\operatorname{sgn} x\Big(t-\frac{\pi}{8}\Big) + l\cos t\left|x\Big(t-\frac{\pi}{4}\Big)\right|^{\alpha_2}\operatorname{sgn} x\Big(t-\frac{\pi}{4}\Big) = -m\cos 2t,\ t \geqslant 0,$$

where k,l,m are positive constants, $\alpha_1 > 1$ and $0 < \alpha_2 < 1$.

In 2010, Sowdaiyan Murugadass, Ethiraju Thandapani and Sandra Pinelas in [64] have studied the oscillation of the second-order quasilinear delay differential equation:

$$\big(r(t)(x'(t))^{\alpha}\big)' + q(t)x^{\alpha}(t-\tau) + \sum_{j=1}^{n} q_j(t)x^{\alpha_j}(t-\tau) = e(t),\ t \geqslant t_0, \tag{3.18}$$

where $n \in \mathbb{N}$, $\alpha_1 > ... > \alpha_m > \alpha > \alpha_{m+1} > ... > \alpha_n > 0$, and α, $\alpha_j > 0$ are ratio of odd positive integers. Analogously to (3.2), we need the positive constants $\eta_0, \eta_1, ..., \eta_n$ satisfying (3.12).

Two kinds of oscillation criteria have been obtained as follows.

THEOREM 43. ([64, Theorem 2.6] - from 2010) *Let for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T \leqslant a_1$, $b_1 \leqslant a_2$, such that $e(t) \leqslant 0$ on $[a_1 - \tau, b_1]$, $e(t) \geqslant 0$ on $[a_2 - \tau, b_2]$, $q(t) \geqslant 0$ and $q_j(t) \geqslant 0$ on $[a_1 - \tau, b_1] \cup [a_2 - \tau, b_2]$, $j = 1,2,...,n$. Let $D(a_i,b_i)$*

be the set of functions define by (2.2) *and* $\eta_0, \eta_1, \ldots, \eta_n$ *be positive constants satisfying* (3.12).*If there exist two functions* $u_i \in D(a_i, b_i)$, $u_i^{\alpha+1} > 0$ *on* (a_i, b_i) *and a positive, nondecreasing, function* $\phi \in C([t_0, \infty), \mathbb{R})$ *such that*

$$\int_{a_i}^{b_i} \phi(t)\left(Q_i(t)u_i^{\alpha+1}(t) - r(t)\left(|u_i'(t)| + u_i(t)\frac{\phi'(t)}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right)dt > 0, \tag{3.19}$$

for $i = 1,2$, *where*

$$Q_i(t) = \left(|e(t)|^{\eta_0} \prod_{j=0}^{n} \eta_j^{-\eta_j} \prod_{j=1}^{n} q_j^{\eta_j}(t) + q(t)\right)\left(\frac{t-a_i}{t-a_i+\tau}\right)^{\alpha}. \tag{3.20}$$

Then equation (3.18) *is oscillatory.*

One can use the similar notations to (3.6) and (3.15):

$$h_1(t,s) = \frac{1}{(\alpha+1)\sqrt{H(t,s)}}\frac{\partial H}{\partial t}(t,s) \quad \text{and} \quad h_2(t,s) = -\frac{1}{(\alpha+1)\sqrt{H(t,s)}}\frac{\partial H}{\partial s}(t,s). \tag{3.21}$$

Now we state the second oscillation criterion from [64].

THEOREM 44. ([64, Theorem 2.3] - from 2010) *Let for any* $T > 0$ *there exist a pair of intervals* $[a_1, b_1]$ *and* $[a_2, b_2]$, $b_1 \leqslant a_2$, *and two intermediate points* $c_i \in (a_i, b_i)$, $i = 1,2$ *such that* $e(t) \leqslant 0$ *on* $[a_1 - \tau, b_1]$ *and* $e(t) \geqslant 0$ *on* $[a_2 - \tau, b_2]$, $q(t) \geqslant 0$ *and* $q_j(t) \geqslant 0$ *on* $(a_1 - \tau, b_1) \cup (a_2 - \tau, b_2)$, $j = 1,2,..,n$. *Let* $\eta_0, \eta_1, \ldots, \eta_n$ *be positive constants satisfying* (3.12). *Let there exist two functions* $H_i = H_i(t,s)$, $H_i \in \mathscr{D}_{a_i,b_i}$, $i = 1,2$ *with* $h_{i1}(s,t)$ *and* $h_{i2}(s,t)$ *defined in* (3.21) *such that*

$$\begin{aligned}\frac{1}{H_i(c_i,a_i)}&\int_{a_i}^{c_i} H_i(t,a_i)\left(Q_i(t) - \frac{r(t)}{\alpha^\alpha}\left(\frac{h_{i1}(t,a_i)}{\sqrt{H_i(t,a_i)}}\right)^{\alpha+1}\right)dt \\ &+ \frac{1}{H_i(b_i,c_i)}\int_{c_i}^{b_i} H_i(b_i,t)\left(Q_i(t) - \frac{r(t)}{\alpha^\alpha}\left(\frac{h_{i2}(b_i,t)}{\sqrt{H_i(b_i,t)}}\right)^{\alpha+1}\right)dt > 0,\ i = 1,2,\end{aligned} \tag{3.22}$$

where $Q_i(t)$, $i = 1,2$ *are two functions defined by* (3.29). *Then equation* (3.18) *is oscillatory.*

The main results of [64] were illustrated by the following equation (see [64, Example 3.1]):

$$\left(t(x'(t))^3\right)' + l_1 \cos t\left(x\left(t - \frac{\pi}{8}\right)\right)^3 + l_2(\sin t)^{\frac{20}{11}}\left(x\left(t-\frac{\pi}{8}\right)\right)^5 + l_3\cos^4 t\, x\left(t - \frac{\pi}{8}\right) = -m\cos^5 2t,$$

where $t \geqslant 0$, and l_1, l_2, l_3, m are positive constants. See also [64, Examples 3.1 and 3.2].

In 2011, Taher S. Hassan and Qingkai Kong in [43] studied the oscillation of the half-linear differential equation (3.8) with damped term:

$$\left(r(t)\Phi_\alpha(x')\right)' + p(t)\Phi_\alpha(x') + q(t)\Phi_\alpha(x) + \sum_{j=1}^{n} q_j(t)\Phi_{\alpha_j}(x) = e(t),\ t \geqslant t_0, \tag{3.23}$$

where $\Phi_\alpha(u) = |u|^\alpha \operatorname{sgn}(u) = |u|^{\alpha-1}u$ and $n \in \mathbb{N}$. Unlike equation (3.8), where α_j are all super-half-linear, in equation (3.23) exponents α_j are of mixed-type i.e., $\alpha_j > \alpha > 0$ for $j = 1,2,\ldots,l$ and $\alpha > \alpha_j > 0$ for $j = l+1,\ldots,n$.

The authors proved in [43, Lemma 2.1] that if

$$m = \frac{\alpha}{n-l} \sum_{j=l+1}^{n} \alpha_j^{-1} \quad \text{and} \quad n = \frac{\alpha}{l} \sum_{j=1}^{l} \alpha_j^{-1}$$

then for any $\delta \in (m,n)$, there exists an n-tuple $(\eta_1, \eta_2, ..., n)$ with $\eta_j > 0$ satisfying:

$$\sum_{j=1} \alpha_j \eta_j = \alpha \quad \text{and} \quad \sum_{j=1} \eta_j = \delta. \tag{3.24}$$

This result improves the well-known lemma due to Sun and Wong [92, Lemma 1].

THEOREM 45. ([43, Theorem 2.2] - from 2011) *Let for any $T > 0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $b_1 \leqslant a_2$, such that $e(t) \leqslant 0$ on $[a_1,b_1]$ and $e(t) \geqslant 0$ on $[a_2,b_2]$, and $q_j(t) \geqslant 0$, $q_j(t) \not\equiv 0$ on $(a_1,b_1) \cup (a_2,b_2)$, $j = 1,2,..,n$. Let $D(a_i,b_i)$, $i = 1,2$ be the set of functions defined by (2.2). Let $\eta_1,...,\eta_n$ be positive constants satisfying (3.24). Let there exist two functions $u_i \in D(a_i,b_i)$, $i = 1,2$ such that*

$$\sup_{\delta \in (m,1]} \int_{a_i}^{b_i} \big(Q(t)|u_i(t)|^{\alpha+1} - \rho(t) r(t)|u_i'(t)|^{\alpha+1}\big) dt \geqslant 0, \; i = 1,2, \tag{3.25}$$

where

$$\rho(t) = \exp \int_0^t \frac{p(s)}{r(s)} ds \quad \textit{and} \quad Q(t) = \rho(t) \left(q(t) + \left(\frac{|e(t)|}{1-\delta} \right)^{1-\delta} \prod_{j=1}^{n} \left(\frac{q_j(t)}{\eta_j} \right)^{\eta_j} \right), \tag{3.26}$$

with the notations $0^{1-\delta} = 1$ and $(1-\delta)^{1-\delta} = 1$ for $\delta = 1$. Then equation (3.23) is oscillatory.

This theorem was illustrated with the following forced second-order differential equation with mixed nonlinearities and damping

$$\begin{aligned} &\big(r(t)\Phi_\alpha(x')\big)' - r^2(t)|\cos 4t|^{\alpha+1}\Phi_\alpha(x') + c_0 \cos 4t\, \Phi_\alpha(x) \\ &\qquad + c_1 \sin 2t\, \Phi_{\frac{1}{2}\alpha}(x) + c_2 \sin 2t\, \Phi_{\frac{3}{2}\alpha}(x) = -e(t)\cos 2t, \; t \geqslant 0, \end{aligned}$$

where $\alpha > 0$, $c_j > 0$ for $j = 0,1,2$, $r(t) > 0$ on $[0,\infty)$ and $e(t) \in C([0,\infty),[0,\infty))$.

In 2011, Taher S. Hassan, Lynn Erbe and Allan Peterson in [42] studied the oscillation of the half-linear functional differential equation:

$$\big(r(t)(x'(t))^\alpha\big)' + q_0(t)x^\alpha(h_0(t)) + \sum_{j=1}^{n} q_j(t)|x(h_j(t))|^{\alpha_j} \operatorname{sgn} x(h_j(t)) = e(t), \; t \geqslant t_0, \tag{3.27}$$

where the function $h_j = h_j(t)$ are positive continuous functions with $\lim_{t\to\infty} h_j(t) = \infty$, and $n \in \mathbb{N}$, $\alpha_1 > ... > \alpha_m > \alpha > \alpha_{m+1} > ... > \alpha_n > 0$. Here we use the next notation:

$$h_{\min}(t) = \min\{t, h_0(t), ..., h_n(t)\} \quad \text{and} \quad h_{\max}(t) = \max\{t, h_0(t), ..., h_n(t)\},$$

$$\phi_{i,j}(t) = \begin{cases} \delta_{i,j}(t), & h_j(t) < t, \\ 1, & h_j(t) = t, \\ \varsigma_{i,j}(t), & h_j(t) > t, \end{cases}$$

where

$$\delta_{i,j}(t)=\int_{h_j(a_i)}^{h_j(t)}\frac{ds}{r^{\frac{1}{\alpha}}(s)}\left(\int_{h_j(a_i)}^{t}\frac{ds}{r^{\frac{1}{\alpha}}(s)}\right)^{-1}\quad\text{and}\quad \varsigma_{i,j}(t)=\int_{h_j(t)}^{h_j(b_i)}\frac{ds}{r^{\frac{1}{\alpha}}(s)}\left(\int_{t}^{h_j(b_i)}\frac{ds}{r^{\frac{1}{\alpha}}(s)}\right)^{-1}.$$

In this way, the authors considered simultaneously three cases: retarded, nondeviating and advanced.

THEOREM 46. ([42, Theorem 2.5] - from 2011) *Let for any $T>0$ there exist a pair of intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T\leqslant a_1$, $b_1\leqslant a_2$, such that $e(t)\leqslant 0$ on $[h_{\min}(a_1),h_{\max}(b_1)]$, $e(t)\geqslant 0$ on $[h_{\min}(a_2),h_{\max}(b_2)]$, and $q_j(t)\geqslant 0$ on $[h_{\min}(a_1),h_{\max}(b_1)]\cup[h_{\min}(a_2),h_{\max}(b_2)]$, $j=0,2,\dots,n$. Let $D(a_i,b_i)$ be the set of functions define by (2.2) and $\eta_0,\eta_1,\dots,\eta_n$ be positive constants satisfying (3.12). If there exist a function $u\in D(a_i,b_i)$ and a positive differentiable function $\rho(t)$ such that*

$$\int_{a_i}^{b_i}\rho(t)\left(Q_i(t)u^{\alpha+1}(t)-r(t)\left(u'(t)+u(t)\frac{\rho'(t)}{(\alpha+1)\rho(t)}\right)^{\alpha+1}\right)dt>0, \tag{3.28}$$

for $i=1,2$, where

$$Q_i(t)=\left(|e(t)|^{\eta_0}\prod_{j=0}^{n}\eta_j^{-\eta_j}\prod_{j=1}^{n}q_j^{\eta_j}(t)\phi_{i,j}^{\alpha_j\eta_j}(t)+q_0(t)\phi_{i,0}^{\alpha}(t)\right). \tag{3.29}$$

Then equation (3.27) is oscillatory.

See also [42, Theorems 2.6 and 2.7]. Theorem 46 was illustrated with the following forced quasilinear differential equation with delay and advanced arguments:

$$\begin{aligned}((x'(t))^{\alpha})'+c_0\sin 2t\,x^{\alpha}\left(t-\frac{\pi}{16}\right)+c_1\sin t\left|x\left(t-\frac{\pi}{32}\right)\right|^{\frac{3\alpha}{2}}\operatorname{sgn}x\left(t-\frac{\pi}{32}\right)\\ +c_2\cos t\left|x\left(t+\frac{\pi}{32}\right)\right|^{\frac{\alpha}{3}}\operatorname{sgn}x\left(t+\frac{\pi}{32}\right)=-\cos 2t,\ t\geqslant 0,\end{aligned}$$

where the constants $c_i>0$, $i=1,2,3$ and α is a quotient of odd positive integers.

In 2012 Jing Shao, Fanwei Meng and Xinqin Pang in [82] studied the oscillation of equation (3.8) (the undamped case of equation (3.23) with super-half-linear exponents $\alpha_j>\alpha>0$ for all $j=1,2,..,n$. The author showed the following interval oscillation criterion different from related ones presented in Theorems 39 and 45, because it uses the Komkov's type function $G(v)$ introduced in Theorem 5.

THEOREM 47. ([82, Theorem 2.2] - from 2012) *Let $r(t)>0$, $e\in C([t_0,\infty),\mathbb{R})$. Assume for any $T>0$ there exist intervals $[a_1,b_1]$ and $[a_2,b_2]$, $T<a_1<b_1\leqslant a_2<b_2$ such that $e(t)$ has different signs on $[a_1,b_1]$ and $[a_2,b_2]$. Let G_1,G_2 be two non-negative functions such that $G_i(v)=0$, $G_i'(v)$ are continuous and $[G_i'(v)]^{\alpha+1}\leqslant(\alpha+1)^{\alpha+1}G_i^{\alpha+1}(v)$, $\forall v\in\mathbb{R}$. If there exist a function $u\in C^1([a_i,b_i],\mathbb{R})$, $u^{\alpha+1}(t)>0$ on (a_i,b_i) and $u(a_i)=u(b_i)=0$ and a positive function $\phi\in C([t_0,\infty),\mathbb{R})$ such that*

$$\int_{a_i}^{b_i}\phi(t)\left[\left(q(t)+\sum_{j=1}^{n}Q_j(t)\right)G_i(u(t))-r(t)\left(|u'(t)|+\frac{G_i^{1/(\alpha+1)}(u(t))|\phi'(t)|}{(\alpha+1)\phi(t)}\right)^{\alpha+1}\right]dt>0,$$

where $Q_j(t)$ is given in (3.9), then equation (3.8) is oscillatory.

Obviously, the functions $G_i(v) = v^2$, $i = 1,2$, satisfy all required assumptions of Theorem 47 and hence, Theorem 47 generalizes Theorems 39 in the sense of Komkov's Theorem 5. Theorem 47 was illustrated with the following forced nonlinear differential equation:

$$\left(r_0 t^{-r_1/3} x'\right)' + q(t)x + q_1(t)|x|^2 x = -\sin^3(t),\ t \geqslant 2\pi, \tag{3.30}$$

where r_0 and r_1 are positive constants, $q(t) = t^{-r_1/3}\exp(\sin t)$ and $q_1(t) = t^{-r_1}\exp(3\sin t)$ on $[a_1, b_1] = [2k\pi, (2k+1)\pi]$ and $q(t) = t^{-r_1/3}\exp(-\sin t)$ and $q_1(t) = t^{-r_1}\exp(-3\sin t)$ on $[a_2, b_2] = [(2k+1)\pi, (2k+2)\pi]$. The author showed that equation (3.30) is oscillatory by using Theorem 47 especially for $\phi(t) = t^{r_1/3}$ and $u(t) = \sin t \geqslant 0$, $G_1(u) = u^2\exp(-u)$ on $[a_1, b_1]$, and $u(t) = \sin t \leqslant 0$, $G_2(u) = u^2\exp(u)$ on $[a_2, b_2]$.

In 2013, Pašić in [71] considered the oscillation of the following class of functional differential equations of second-order:

$$\left(r(t)\Phi(x'(t))\right)' + \sum_{i=1}^{n} r_i(t) f(x(h_i(t))) + \sum_{i=1}^{n} q_i(t)|x(h_i(t))|^{\alpha_i}\operatorname{sgn} x(h_i(t)) = e(t), \tag{3.31}$$

where the deviating arguments $h_i(t)$ may be of delay, advanced or delay-advanced types. Equation (3.31) generalizes equation (3.27) in particular for $\Phi(v) = |v|^{\alpha-1}v$, since it satisfies the general assumptions:

$$\Phi \in C^1(\mathbb{R},\mathbb{R}),\ \Phi \text{ is odd and increasing function on } \mathbb{R}, \tag{3.32}$$

$$\Phi(v)v \geqslant |\Phi(v)|^{\frac{\alpha+1}{\alpha}} \quad \text{for all } v \in \mathbb{R} \text{ and some } \alpha > 0. \tag{3.33}$$

Besides the half-linear function $\Phi(v) = |v|^{\alpha-1}v$, the function of prescribed mean curvature $\Phi(v) = v(1+v^2)^{-1/2}$ also satisfies the required assumptions (3.32) and (3.33). The exponents α_i are as usual of mixed type and satisfy

$$\begin{cases} \alpha_1 \geqslant \ldots \geqslant \alpha_m > \alpha > \alpha_{m+1} \geqslant \ldots \geqslant \alpha_n > 0,\ m \in \mathbb{N},\ \alpha_i > \alpha_{i+1},\ i \in \{1,\ldots,n-1\}, \\ \text{there exists } (n+1)\text{-tuple } (\eta_0, \eta_1, \ldots\ldots, \eta_n) \text{ such that} \\ 0 < \eta_i < 1,\ \sum_{i=1}^n \eta_i < 1,\ \eta_0 = 1 - \sum_{i=1}^n \eta_i \quad \text{and} \quad \sum_{i=1}^n \alpha_i \eta_i = \alpha, \end{cases} \tag{3.34}$$

where α is from assumption (3.33). The function $f = f(y)$ satisfies two assumptions:

$$\begin{cases} f(y) \text{ is odd function on } \mathbb{R}, \\ f(y)/y^p \geqslant K > 0 \text{ for all } y > 0 \text{ and some } K \in \mathbb{R}, \end{cases} \tag{3.35}$$

where $\alpha > 0$ is from assumption (3.33). Unlike the previous oscillation criteria, the following criterion is based on an elementary integral inequality:

$$\frac{1}{2\pi_\alpha}\int_{a_j}^{b_j} \min\left\{\frac{\alpha}{(\lambda_j r(t))^{1/\alpha}}, \lambda_j\left(R_j(t) + Q_j(t)\right)\right\} dt \geqslant 1,\quad j \in \{1,2\}, \tag{3.36}$$

where $a_1 < b_1 \leqslant a_2 < b_2$, α is from (3.33), $\pi_\alpha = \frac{\alpha}{\alpha+1}\pi / \sin\frac{\alpha\pi}{\alpha+1}$, $\lambda_j > 0$, and functions $Q_j(t)$, $R_j(t)$ are explicitly expressed by the coefficients of equation (3.31). In the delay case, we have:

THEOREM 48. ([71, Theorem 1] - from 2013) *Let* (3.32), (3.33), (3.34) *and* (3.35) *hold. Let $r(t)$ be a nondecreasing positive function on $[t_0, \infty)$. Let $h_i(t) = \tau_i(t) \leqslant t$ on $[t_0, \infty)$ and*

$\lim_{t\to\infty}\tau_i(t)=\infty$, $i\in\{1,2,\dots,n\}$. *Let for every* $T\geqslant t_0$ *there exist* a_1,b_1,a_2,b_2, $T\leqslant a_1<b_1\leqslant \tau_{\min}(a_2)\leqslant a_2<b_2$ *such that:*

$$r_i(t)\geqslant 0 \quad \text{and} \quad q_i(t)\geqslant 0 \text{ on } [\tau_{\min}(a_1),b_1]\cup[\tau_{\min}(a_2),b_2], \tag{3.37}$$

$$e(t)\leqslant 0 \text{ on } [\tau_{\min}(a_1),b_1] \quad \text{and} \quad e(t)\geqslant 0 \text{ on } [\tau_{\min}(a_2),b_2], \tag{3.38}$$

where $\tau_{\min}(t)=\min\{\tau_1(t),\tau_2(t),\dots,\tau_n(t)\}$. *Then equation* (3.31) *is oscillatory provided there are two real parameters* $\lambda_1,\lambda_2>0$ *such that* (3.36) *is fulfilled, where:*

$$R_j(t)=K\sum_{i=1}^{n} r_i(t)\Big(\frac{\tau_i(t)-\tau_i(a_j)}{t-\tau_i(a_j)}\Big)^p, \tag{3.39}$$

$$Q_j(t)=\big(\eta_0^{-1}|e(t)|\big)^{\eta_0}\prod_{i=1}^{n}\big(\eta_i^{-1}q_i(t)\big)^{\eta_i}\prod_{i=1}^{n}\Big(\frac{\tau_i(t)-\tau_i(a_j)}{t-\tau_i(a_j)}\Big)^{\alpha_i\eta_i}, \tag{3.40}$$

for $t\in[a_j,b_j]$, $j\in\{1,2\}$ *and positive constants* p, K, η_i *appearing respectively in* (3.33), (3.35) *and* (3.34).

This theorem was illustrated with the following class of second-order nonlinear differential equations of multiple delay arguments and with Emden-Fowler type nonlinearities:

$$x''(t)+\sum_{i=1}^{n} q_i(t)|x(\tau_i(t))|^{\alpha_i}\operatorname{sgn} x(\tau_i(t))=e(t), \tag{3.41}$$

where $q_i(t)$, $\tau_i(t)$ and $e(t)$ are as in Theorem 48. The author showed that equation (3.41) is oscillatory provided

$$\frac{1}{\pi}\int_{a_j}^{b_j} Q_j(t)dt\geqslant\sqrt{\max_{t\in[a_j,b_j]} Q_j(t)}>0,\ j=1,2,$$

where $Q_j(t)$ is defined in (3.40). On related oscillation criteria in the advanced and delay-advanced cases, we refer reader to [71, Theorems 10 and 12] and [71, Corollaries 11 and 13].

Recently in the oscillation theory, the following type of differential equations with nonlinearities given by Riemann-Stieltjes integrals is considered, which in some extent, generalizes equations with mixed nonlinearities:

$$\big(r(t)|x'(t)|^p\operatorname{sgn}(x'(t))\big)'+q_0(t)|x(t)|^p\operatorname{sgn}(x(t))+\int_0^b q(t,s)|x(t)|^{\alpha(s)}\operatorname{sgn}(x(t))d\zeta(s)=e(t), \tag{3.42}$$

where $b\in(0,\infty)$, $\alpha\in C((0,b],\mathbb{R})$ is strictly increasing and $\int_0^b f(s)d\zeta(s)$ denotes the Riemann-Stieltjes integral of the function $f(s)$ on $[0,b)$ with respect to ζ. On the oscillation criteria for equation (3.42), we refer the reader to Sun and Kong [91], Hassan and Kong [44], Zeng [122] and references therein.

4. Wong's oscillation criteria involving a general mean

In 2000, James S.W. Wong in his paper [109] developed a general integral averaging method and related oscillation criteria for the following general class of second-order differential equations:

$$x''+q(t)f(x)=0,\ t\in[0,\infty), \tag{4.1}$$

where the continuous function $q(t)$ may change sign and

$$f \in C(\mathbb{R},\mathbb{R}) \cap C^1(\mathbb{R}/\{0\},\mathbb{R}),\ f'(y) \geqslant 0 \quad \text{and} \quad yf(y) > 0,\ \forall y \neq 0. \tag{4.2}$$

The unforced Emden-Fowler equation is equation (2.10) with $e(t) \equiv 0$ i.e.,

$$x'' + q(t)|x|^{\gamma}\mathrm{sgn}(x) = 0,\ \gamma > 0. \tag{4.3}$$

Equation (4.3) is a prototype of general equation (4.1), since the function $f(y) = |y|^{\gamma}\mathrm{sgn}(y)$, $\gamma > 0$, satisfies all properties given in (4.2), see (i)-Comment 10 below. As in equation (4.3), where $0 < \gamma < 1$ and $\gamma > 1$, two cases are studied:

– the sub-linear

$$0 < \int_{0\pm}^{y} \frac{dv}{f(v)} < \infty,\ \forall y \neq 0, \tag{4.4}$$

with the additional the so-called strictly sub-linear condition (introduced by Manabu Naito [65])

$$f'(y)\int_{0\pm}^{y} \frac{dv}{f(v)} \geqslant c^{-1} > 0,\ \forall y \neq 0, \tag{4.5}$$

where $0\pm = 0+$ if $y > 0$ and $0\pm = 0-$ if $y < 0$, and c is a positive constant only depending on f, and

– the super-linear

$$0 < \int_{y}^{\infty} \frac{dv}{f(v)} < \infty \quad \text{and} \quad 0 < \int_{-y}^{-\infty} \frac{dv}{f(v)} < \infty,\ \forall y > 0, \tag{4.6}$$

with the additional the so-called strictly super-linear condition (see [65])

$$\min\left\{ f'(y)\int_{y}^{\infty} \frac{dv}{f(v)}, f'(-y)\int_{-y}^{-\infty} \frac{dv}{f(v)} \right\} \geqslant d > 1,\ \forall y > 0, \tag{4.7}$$

where the constant d only depends on f. It is simple to check that if $f(y) = |y|^{\gamma}\mathrm{sgn}(y)$, $\gamma > 0$, then (4.4)-(4.5) and (4.6)-(4.7) are satisfied provided $0 < \gamma < 1$ and $\gamma > 1$ respectively.

COMMENT 10. About hypotheses (4.2) and (4.4)-(4.7) one can say the following:

(i) Although the function $\mathrm{sgn}(y)$ is not continuous at $y = 0$, the equality $f(y) = |y|^{\gamma}\mathrm{sgn}(y) = |y|^{\gamma-1}y$ ensures that $f(y)$ is continuous on $\mathbb{R}$ and $f'(y) = \gamma|y|^{\gamma-1}$. This implies $f \in C^1(\mathbb{R},\mathbb{R})$ for $\gamma \geqslant 1$ and $f \in C^1(\mathbb{R}/\{0\},\mathbb{R})$ for $0 < \gamma < 1$.

(ii) In assumptions (4.2) and (4.4)-(4.7), the case $y < 0$ is equivalently treated with the case $y > 0$. As a consequence, in the proofs of the main results one can work only with the positive nonoscillatory solution of (4.1) without loss of generality. However, it should be very careful with this trick in general, since for certain classes of equations the proposed assumptions are not enough to work only with positive nonoscillatory solutions without loss of generality. In such a case, the nonoscillatory solutions must be considered without assuming their signs.

(iii) Assumption (4.2) implies that $f(y)$ is positive (resp. negative) for y positive (resp. negative). Hence, the inequalities

$$0 < \int_{0\pm}^{y} \frac{dv}{f(v)},\ \forall y \neq 0, \quad \text{and} \quad 0 < \int_{y}^{\infty} \frac{dv}{f(v)},\ 0 < -\int_{-\infty}^{-y} \frac{dv}{f(v)},\ \forall y > 0,$$

can be removed from (4.4) and (4.6).

(iv) From (4.2) it follows $f(0) = 0$ and thus, the function $1/f(v)$ has a singular point at 0. Hence, the super-linear condition (4.7) can not be written only with one integral

$$f'(y)\int_{y}^{\infty}\frac{dv}{f(v)} \geqslant d > 1,\ \forall y \in \mathbb{R},$$

and so, it has to be divided into two ones over $(-\infty,-y]$ and $[y,\infty)$ where $y > 0$. Consequently, unlike sub-linear condition (4.5), the super-linear condition (4.6) (and (4.7)) should involve two integrals. □

4.1. Main results

Before we discuss the key point of an integral averaging technique and some kinds of integral means (classic, of convolution type and general), we consider the nonnegative kernel function $h(t,s)$ introduced for the first time by Philos [76] satisfying the following properties:

$$h(t,t) \equiv 0 \quad \text{and} \quad \frac{\partial h}{\partial s}(t,s)\Big|_{s=t} \equiv 0 \quad \text{for } t \geqslant t_0, \tag{4.8}$$

$$\frac{\partial h}{\partial s}(t,s) \leqslant 0 \quad \text{and} \quad \frac{\partial^2 h}{\partial s^2}(t,s) \geqslant 0 \quad \text{for } t \geqslant s \geqslant t_0, \tag{4.9}$$

$$-\frac{\frac{\partial h}{\partial s}(t,s)\Big|_{s=t_0}}{h(t,t_0)} \leqslant M_0 \quad \text{for } t \geqslant t_0, \tag{4.10}$$

where in (4.10) the indefinite limit form $\frac{0}{0}$ appears at $t = t_0$ and M_0 is a constant depending only on t_0. The first main result of this section is the following Wong's oscillation criterion for the sub-linear equation (4.1).

THEOREM 49. ([109, Theorem 1] - from 2000) *Let $q \in C([t_0,\infty),\mathbb{R})$ and $f(y)$ satisfy (4.2) and (4.4). Suppose that there exists a non-negative kernel function $h(t,s)$ on $\{(t,s) : t \geqslant s \geqslant t_0\}$ satisfying (4.8)-(4.10). If $q(t)$ satisfies*

$$\limsup_{t\to\infty}\frac{1}{h(t,t_0)}\int_{t_0}^{t} h(t,s)q(s)ds = \infty, \tag{4.11}$$

then the sub-linear equation (4.1) is oscillatory.

A basic example for the functions $h(t,s)$ satisfying hypotheses (4.8)-(4.10) is $h(t,s) = (t-s)^{\alpha}$, $\alpha > 1$. Regarding to the proof of Theorem 49, the additional "strictly sub-linear condition" (4.5) is used only in [109, Theorem 2]. A summary on the oscillation criteria concerning the sub-linear equation (4.1) is made in Section 4.3 including Theorem 49 as well as other results from Section 4.2.

Next, two additional conditions on the kernel function $h(t,s)$ are added: for two constants b_0 and B_0 depending only on t_0,

$$0 < b_0 \leqslant \lim_{t\to\infty}\frac{h(t,s)}{h(t,t_0)} \leqslant B_0 < \infty \quad \text{for } s \geqslant t_0, \tag{4.12}$$

$$-\frac{\frac{\partial h}{\partial s}(t,s)\Big|_{s=\tau}}{h(t,\tau)} = o(1) \quad \text{for all } \tau \geqslant t_0 \text{ as } t \to \infty. \tag{4.13}$$

The second main result of this section is the following Wong's oscillation criterion for the super-linear equation (4.1).

THEOREM 50. ([109, Theorem 3] - from 2000) *Let* $q \in C([t_0,\infty),\mathbb{R})$ *and* $f(y)$ *satisfy* (4.2), (4.6) *and* (4.7). *Suppose that there exist two non-negative kernel functions* $h_1(t,s)$ *and* $h_2(t,s)$ *on* $\{(t,s) : t \geqslant s \geqslant t_0\}$ *satisfying* (4.8)-(4.10) *and* (4.12)-(4.13), *and in addition,*

$$\left|\frac{\partial h_2}{\partial s}(t,s)\right|^2 \leqslant d_1 \frac{\partial^2 h_2}{\partial s^2}(t,s) h_2(t,s), \tag{4.14}$$

where the positive constant $d_1 < d$ *and* d *is from* (4.7). *If* $q(t)$ *satisfies*

$$\limsup_{t\to\infty} \frac{1}{h_1(t,t_0)} \int_{t_0}^t h_1(t,s)q(s)ds = \infty, \tag{4.15}$$

and

$$-\infty < \liminf_{t\to\infty} \frac{1}{h_2(t,t_0)} \int_{t_0}^t h_2(t,s)q(s)ds < 0, \tag{4.16}$$

then super-linear equation (4.1) *is oscillatory.*

Two basic kernel functions satisfying (4.8)-(4.10) and (4.12)-(4.14) are $h_1(t,s) = (t-s)^\alpha$, $\alpha > 1$ and $h_2(t,s) = (t-s)^\beta$, $\beta \geqslant 0$, see [109, Corollary 3]. By the main results of [109] we see that the Emden-Folwer equation:

$$x'' + t^\sigma \sin(t)|x|^\gamma \mathrm{sgn}(x) = 0,\ t > 0,\ \gamma > 0, \tag{4.17}$$

is oscillatory provided $\sigma > 1$. In the super-linear case $\gamma > 1$, it was known by Butler [12] that equation (4.17) is oscillatory if and only if $\sigma \geqslant -1$. In the sub-linear case $0 < \gamma < 1$, by Kura's Theorem 64 below, (4.17) is oscillatory if $\sigma > -\gamma$. For $\gamma = 1$, (4.17) becomes the well known Willet-Wong linear differential equation presented with details in Remark 21 in the next subsection.

Another example considered in [109] with a non-Emden-Fowler nonlinear term $f(y)$ and the same coefficient $q(t)$ as in (4.17):

$$x'' + t^\sigma \sin(t)|x|^{1/2}(1+|x|)\mathrm{sgn}(x) = 0,\ t > 0,$$

shows that this equation is oscillatory provided $\sigma > 1$ such as equation (4.17).

On some known important properties of the integral mean or integral average of a continuous function $Q(t)$ over the interval $[t_0,t]$ such as the Lebesgue-Besicovitch theorem, Poincaré and Jensen inequalities, etc. we refer the reader to [11, 17, 28].

REMARK 19. (i) Using integration by part, it is easy to show that the integral mean of the primitive function of $q(t)$ is equal to the integral average (of convolution-type) of $q(t)$, because of the integral identity:

$$\int_{t_0}^t \int_{t_0}^s q(\tau)d\tau ds = \int_{t_0}^t (t-s)q(s)ds. \tag{4.18}$$

(ii) In all observation in this section, the integral mean in various limit forms will apear as $\frac{1}{t}\int_{t_0}^t Q(t)$ instead of $\frac{1}{t-t_0}\int_{t_0}^t Q(t)$ for some $Q(t)$, because

$$\lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^t Q(\tau)d\tau = \lim_{t\to\infty} \frac{t-t_0}{t} \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^t Q(\tau)d\tau = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^t Q(\tau)d\tau. \tag{4.19}$$

(iii) From (4.18) and (4.19), it follows

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^t \int_{t_0}^s q(\tau)d\tau ds = \infty \quad \Rightarrow \quad \lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^t (t-s)q(s)ds = \infty, \tag{4.20}$$

and by Wong's survey paper [111], dedicated to Paul Waltman, for any $\alpha > 1$, it holds

$$\lim_{t\to\infty}\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)q(s)ds=\infty \quad\Rightarrow\quad \lim_{t\to\infty}\frac{1}{(t-t_0)^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}q(s)ds=\infty. \tag{4.21}$$

Thus, the integral mean of convolution type appearing on the right-hand side of (4.21) is more general in the limit sense than the classic integral mean appearing on the left-hand side of (4.20). Therefore, replacing the term $(t-s)^{\alpha}$ with the general kernel function of Philos-type $h(t,s)$, Wong called the general mean of $q(t)$ the next integral

$$\frac{1}{h(t,t_0)}\int_{t_0}^{t}h(t,s)q(s)ds. \ \Box$$

4.2. This type of oscillation criteria by other authors from 1949 to 1999

In this subsection, we present in the chronological order the oscillation criteria involving the integral mean by other authors published in the period from 1949 to 1999 and consequently, preceded Wong's Theorems 49 and 50 from 2000.

• The first oscillation criteria including integral mean were devoted to the unforced second-order linear differential equation:

$$(r(t)x')'+q(t)x=0,\ t\geqslant t_0. \tag{4.22}$$

In 1949, Aurel Wintner in his paper [100] gave the first such a criterion.

THEOREM 51. ([100] - from 1949) *Let* $r(t)\equiv 1$ *and* $q\in C([t_0,\infty),\mathbb{R})$. *If*

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(\tau)d\tau ds=\infty, \tag{4.23}$$

then the linear equation (4.22) *is oscillatory.*

We illustrate this criterion in the next example and remark.

EXAMPLE 5. For any constant $a>0$, the equation

$$x''+\frac{a}{t\ln(t)}x=0,\ t>1, \tag{4.24}$$

is oscillatory. In fact, by an elementary calculation we have:

$$\int\frac{d\tau}{\tau\ln(\tau)}=\ln(\ln(\tau)),\ \lim_{s\to\infty}[\ln(\ln(s))]=\infty \quad\text{and}\quad \lim_{t\to\infty}\frac{\int_{t_0}^{t}\ln(\ln(s))ds}{t}=\infty,$$

and therefore, the function $q(t)=a/[t\ln(t)]$ satisfies (4.23). Thus, by Theorem 51 one can conclude that equation (4.24) is oscillatory for any $a>0$. $\Box$

REMARK 20. If $a>1/4$, then the oscillation of equation (4.24) also follows from the Sturm comparison principle, since for $a>1/4$ and some $\lambda_0\in(1/4,a)$, it holds $a/(t\ln(t))\geqslant \lambda_0/t^2$ for all $t>1$ and using that Euler equation $x''+(\lambda_0/t^2)x=0$ is oscillatory. $\Box$

• In 1952 Philip Hartman in [39] showed that if "lim" in (4.23) is replaced by "lim sup", then the corresponding condition is not enough for the oscillation of equation (4.22).

THEOREM 52. ([39] - from 1952) *Let* $r(t) \equiv 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *If* $q(t)$ *satisfies*

$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds \leqslant \infty, \tag{4.25}$$

then linear equation (4.22) *is oscillatory.*

The key point of this criterion is that "lim inf" in (4.25) must be finite and not equal to "lim sup". For instance, if $q(t) = \sin t$, then (4.25) does not hold since:

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} \sin(\tau)d\tau ds = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} \sin(\tau)d\tau ds = 0. \tag{4.26}$$

Theorem 52 was illustrated with the equation $x'' + t\sin(t)x = 0$ showing that all its solutions oscillate, see also the next example. On this equation, Wintner's Theorem 51 fails.

EXAMPLE 6. For any amplitude $a \neq 0$ and frequency $\omega \neq 0$, the equation

$$x'' + at\sin(\omega t)x = 0,\ t > 0, \tag{4.27}$$

is oscillatory. For a given $q(t)$ and t_0, let

$$W_q(t;t_0) = \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds.$$

By an elementary calculation, it is easy to verify that for the function $q(t) = at\sin(\omega t)$,

$$\liminf_{t\to\infty} W_q(t;t_0) = c_0 - \frac{|a|}{\omega^2} < c_0 + \frac{|a|}{\omega^2} = \limsup_{t\to\infty} W_q(t;t_0), \tag{4.28}$$

where the constant $c_0 = a[t_0\omega\cos(\omega t_0) - \sin(\omega t_0)]/\omega^2$. Thus, condition (4.25) is satisfied in this case and by Theorem 52, equation (4.27) is oscillatory. For instance, if $q(t) = t\sin(t)$ and $t_0 = \pi$, then $W_q(t;\pi) = -\sin(t) - \pi - (2\cos(t) + 2 - \pi^2)/t$ and its graph is given in the next figure:

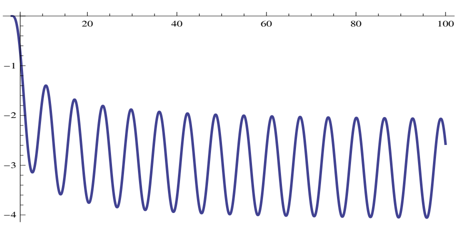


Figure 1: the graph of function $W_q(t;\pi)$;
$\liminf_{t\to\infty} W_q(t;\pi) = -\pi - 1 < -\pi + 1 = \limsup_{t\to\infty} W_q(t;\pi)$. □

REMARK 21. (on the Willet-Wong equation) The linear differential equation (4.27) is a special case of the following class of linear differential equations, the so-called Willet-Wong equation,

$$x'' + at^{\sigma}\sin(\omega t)x = 0,\ t > 0,\ a, \sigma, \omega \in \mathbb{R}/\{0\}. \tag{4.29}$$

For the first time, this class of linear differential equation was considered in Willet [99] and Wong [101] in particular for $\sigma = -1$ and $\omega = 1$. It was shown that equation (4.29) is oscillatory if and only if $a > 1/\sqrt{2}$. Moreover, Wong in [101] showed that for $\sigma = -1$ and $\omega \neq 0$ equation (4.29) is oscillatory if $|a/\omega| > 1/\sqrt{2}$. $\square$

- In 1954 Emilio Gagliardo in [30] proved the following criterion involving a classic integral mean.

THEOREM 53. ([30, Theorem] - from 1954) *Let* $r \in C^1([t_0,\infty),\mathbb{R})$ *and* $r(t) > 0$, $q \in C([t_0,\infty),\mathbb{R})$. *If there exists a function* $\eta \in C^2([t_0,\infty),\mathbb{R})$, $\eta > 0$, *such that*

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)\eta^2(s)}ds = \infty \quad \textit{and} \quad \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\Big[\big(r(\tau)\eta'(\tau)\big)' + q(\tau)\eta(\tau)\Big]\eta(\tau)d\tau ds = \infty, \tag{4.30}$$

then the linear equation (4.22) *is oscillatory.*

It is interesting that especially for $q(t) \geqslant 0$ and $\eta(t) \equiv 1$, from (4.30) we obtain the well-known Fite-Wintner-Leighton oscillation criterion: $\int^{\infty} 1/r(s)ds = \int^{\infty} q(s)ds = \infty$. In fact, since $q(t) \geqslant 0$ and $(t-s)/t \in [0,1]$ for all $s \in [t_0,t]$, from (4.18) and the second statement in (4.30), we obtain

$$\lim_{t\to\infty}\int_{t_0}^{t}q(s)ds \geqslant \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}(t-s)q(s)ds = \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(\tau)d\tau ds = \infty.$$

Thus, in this case, criterion (4.30) is a special case of the Fite-Wintner-Leighton oscillation criterion.

- In 1955 Calvin R. Putnam in [78] gave a variant of Hartman's Theorem 52.

THEOREM 54. ([78] - from 1955) *Let* $r(t) \equiv 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *If there exists* $c > 0$ *such that* $q(t)$ *satisfies*

$$\int_{t_0}^{t}q(s)ds > e^{-ct},\ t \geqslant t_0, \tag{4.31}$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(\tau)d\tau ds = \infty, \tag{4.32}$$

then linear equation (4.22) *is oscillatory.*

It seems that condition (4.31) is not necessarily a special case of Hartman's condition (4.25). Indeed, from (4.31) we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(\tau)d\tau ds \geqslant \lim_{t\to\infty}\frac{1}{ct}\left(e^{-ct_0} - e^{-ct}\right) = 0,$$

and thus, "lim inf" may be equal to ∞, which is not possible in (4.25). Hence, Theorem 54 is different from Hartman's Theorem 52.

• In 1968 William J. Coles in [18] introduced the so-called weighted average

$$\frac{1}{\int_{t_0}^{t} w(s)ds}\int_{t_0}^{t} w(s)Q(s)ds$$

of a function $Q(t)$ with respect to a nonnegative locally integrable function $w(t)$ such that $\int_{t_0}^{t} w(t)dt \not\equiv 0$, in the next called as "weighted function", see also Willett [99].

THEOREM 55. ([18, Theorem 1] - from 1968) *Let* $r(t)\equiv 1$ *and* $q\in C([t_0,\infty),\mathbb{R})$. *If there exists a weighted function* $w(t)$ *such that for some* k, $0\leqslant k<1$,

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{w(s)\left(\int_{t_0}^{s} w(\tau)d\tau\right)^k}{\int_{t_0}^{s} w^2(\tau)d\tau}ds=\infty \quad and \quad \lim_{t\to\infty}\frac{1}{\int_{t_0}^{t} w(s)ds}\int_{t_0}^{t} w(s)\int_{t_0}^{s} q(\tau)d\tau=\infty, \tag{4.33}$$

then linear equation (4.22) *is oscillatory.*

According to Theorem 55, one can show that equation $x''+t^2\sin(t)x=0$ is oscillatory. But, it can not be obtained by Wintner's Theorem 51 and Hartman's Theorem 52.

• In 1971, Coppel in his book [19] studied the oscillation of the one-parameter linear differential equation

$$x''+\lambda q(t)x=0, \tag{4.34}$$

where the coefficient $q(t)$ has the integral mean value equal to zero, i.e.

$$\lim_{t\to\infty}\frac{1}{t}\int_{a}^{a+t} q(\tau)d\tau=0, \tag{4.35}$$

uniformly for $a\in\mathbb{R}$.

THEOREM 56. ([19, Theorem 14] - from 1971) *Let* $q(t)\not\equiv 0$ *be an almost periodic function. If* (4.35) *holds, then equation* (4.34) *is oscillatory for every* $\lambda\neq 0$.

On the definition and properties of the almost periodic functions, we refer reader to the book by Besicovitch [10]. Since a periodic function is also almost periodic and $\int_0^{2\pi}\sin t dt=0$, with the help of Theorem 56 one can show that the equation $x''+\sin t\, x=0$ is oscillatory. The periodic case in Theorem 56 can be characterized by the properties (4.35), see Stanek [85]. It is also true in the almost periodic case which will be presented in Halvorsen and Mingarelli's Theorem 72 below.

• In 1971 Kamenev in [46] studied the unforced Emden-Fowler equation (4.3): $x''+q(t)|x|^{\gamma}\mathrm{sgn}(x)=0$, $\gamma>0$. He showed that in sub-linear case of (4.3), that is, $0<\gamma<1$, the Wintner oscillation criterion (4.23) still holds if "lim" is replaced by "lim sup". We mentioned before that Hartman proved that it is not true in the linear case.

THEOREM 57. ([46] - from 1971) *Let* $0<\gamma<1$ *and* $q\in C([t_0,\infty),\mathbb{R})$. *If* $q(t)$ *satisfies* (4.32)*, then sub-linear Emden-Fowler equation* (4.3) *is oscillatory.*

This Kamenev's result drew much attention from many authors. The function $q(t)=t^2\sin t$ satisfies the required condition (4.32) which is not the case with $q(t)=\sin t$ and $q(t)=t\sin t$ because of (4.26) and (4.28) respectively.

• In 1973 Wong in [102] studied the super-linear case of the Emden-Fowler equation (4.3), that is $\gamma>1$, and deduced the first oscillation criterion including integral mean for such a class of equations.

THEOREM 58. ([102] - from 1973) *Let* $\gamma > 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *If* $q(t)$ *satisfies* (4.32) *and*

$$-\infty < \liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds < 0, \tag{4.36}$$

then the super-linear Emden-Fowler equation (4.3) *is oscillatory.*

COMMENT 11. In Wong's condition (4.36) the term under "lim inf" is different from related term in Hartman's condition (4.25). Hence, one can pose question about the relation between them, that is, between

$$-\infty < \liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds < 0 \quad \text{and} \quad -\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds < \infty. \tag{4.37}$$

The answer to this question is left to the reader. □

We demonstrate Wong's Theorem 58 in the next example.

EXAMPLE 7. The super-linear Emden-Fowler equation

$$x'' + \big(t\cos(t) + \sin(t) + 1\big)|x|^{\gamma}\mathrm{sgn}(x) = 0,\ t \geqslant t_0,\ \gamma > 1, \tag{4.38}$$

is oscillatory. Choose $T_0 \geqslant t_0$ such that $T_0 \neq \frac{3}{2}\pi + 2n\pi$, $n \in \mathbb{N}$. Now, for $q(t) = t\cos(t) + \sin(t) + 1$, we have that the functions:

$$Q(t;T_0) = \int_{T_0}^{t} q(\tau)d\tau \quad \text{and} \quad W(t;T_0) = \frac{1}{t}\int_{T_0}^{t}\int_{T_0}^{s} q(\tau)d\tau ds,$$

satisfy:

$$Q(t;t_0) = t\sin(t) + t + c_0,$$

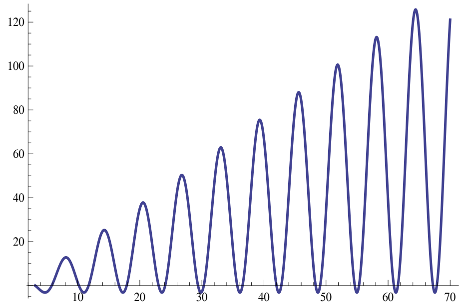
$$W(t;T_0) = \frac{1}{t}\Big(-t\cos(t) + \sin(t) + \frac{1}{2}t^2 + c_0 t + c_1\Big),$$

where $c_0 = -T_0\sin(T_0) - T_0 < 0$ for $T_0 \neq \frac{3}{2}\pi + 2n\pi$ and $c_1 \in \mathbb{R}$ depending only on T_0. Hence,

$$\liminf_{t\to\infty} Q(t;T_0) = \liminf_{t\to\infty} \big(t(\sin(t)+1)\big) + c_0 = 0 + c_0 \in (-\infty, 0),$$

$$\limsup_{t\to\infty} W(t;T_0) = 1 + c_0 + \frac{1}{2}\limsup_{t\to\infty} t = \infty.$$

Consequently, conditions (4.32) and (4.36) are satisfied. By Wong's Theorem 58, we conclude that equation (4.38) is oscillatory. The graphs of both functions $Q(t;T_0)$ and $W(t;T_0)$ for $q(t) = t\cos(t) + \sin(t) + 1$ and $T_0 = \pi$ are given in the next figure:



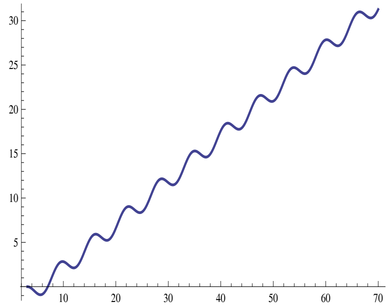


Figure 2 - left: the graph of function $Q(t;\pi)$; Figure 2 - right: the graph of function $W(t;\pi)$

$$\liminf_{t\to\infty} Q(t;\pi) = -\pi < 0; \qquad \limsup_{t\to\infty} W(t;\pi) = \infty. \ \Box$$

In the same way as in the previous example, we can show the next one.

EXAMPLE 8. For any amplitude $a > 0$ and frequency $\omega \neq 0$, the super-linear Emden-Fowler equation

$$x'' + a\big(\omega t\cos(\omega t) + \sin(\omega t) + 1\big)|x|^{\gamma}\mathrm{sgn}(x) = 0,\ t \geqslant t_0,\ \gamma > 1,$$

is oscillatory. $\Box$

• In 1975 Hiroshi Onose studied in [70] more general class of second-order differential equations containing the Emden-Fowler equation (4.3) as a particular case, it is equation (4.1): $x'' + q(t)f(x) = 0$ studied by Wong in the main results of this section, see Theorems 49 and 50 above. It seems that the following theorem is the first oscillation criterion involving integral mean for equation (4.1) in the super-linear case which extends Wong's Theorem 58 from the Emden-Fowler equation (4.3) to general equation (4.1).

THEOREM 59. ([70, Theorem 3] - from 1975) *Let $q \in C([t_0,\infty),\mathbb{R})$. Assume $f(y)$ satisfy the super-linear condition* (4.6) *and*

$$f \in C^1([0,\infty),\mathbb{R}),\ f'(y) \geqslant k > 0 \ \textit{for all}\ y \neq 0. \tag{4.39}$$

If $q(t)$ satisfies (4.32) *and* (4.36)*, then super-linear equation* (4.1) *is oscillatory.*

By Theorem 59, the author showed that the nonlinear equation $x'' + q(t)(x^3 + x) = 0$ is oscillatory.

COMMENT 12. The Emden-Fowler nonlinearity $f(y) = |y|^{\gamma}\mathrm{sgn}\, y$, $\gamma > 1$, satisfies the general super-linear condition (4.6) but not (4.39) since for $y > 0$ we have $f'(y) = \gamma y^{\gamma-1} \to 0$ as $y \to 0$. Hence, Theorem 59 can not be applied on the super-linear Emden-Fowler equation (4.3). $\Box$

EXAMPLE 9. A class of the nonlinear function $f(y)$ that satisfies both conditions (4.6) and (4.39) is the following linear perturbation of the Emden-Fowler super-nonlinearity:

$$f(y) = |y|^{\gamma}\mathrm{sgn}\, y + ky,\ \gamma > 1,\ k > 0. \ \Box$$

• In 1978 Yung-Ming Chen in his paper [15] extended Wong's Theorem 58 from the Emden-Fowler equation (4.3) to general equation (4.1).

THEOREM 60. ([15, Theorem 1] - from 1978) *Let $q \in L_{\mathrm{loc}}([t_0,\infty),\mathbb{R})$. Assume $f(y) > 0$ for $y > 0$ and for some α,β, $1 < \alpha < \beta < \infty$,*

$$y^{-\alpha} f(y)\ \textit{is nondecreasing} \quad \textit{and} \quad y^{-\beta} f(y)\ \textit{is nonincreasing in}\ (0,\infty). \tag{4.40}$$

If $q(t)$ satisfies (4.32) *and* (4.36)*, then general equation* (4.1) *is oscillatory.*

Condition (4.40) was illustrated with the function $f(y) = y^2(1 + \log^+ y)^3$, where $\eta^+ = \max\{\eta, 0\}$ for all $\eta \in \mathbb{R}$.

COMMENT 13. For the Emden-Fowler type of nonlinearity $f(y) = |y|^{\gamma}\mathrm{sgn}(y)$, $\gamma > 0$, Chen's condition (4.40) holds provided $y^{-\alpha}y^{\gamma}$ is nondecreasing and $y^{-\beta}y^{\gamma}$ is nonincreasing on $(0,\infty)$. It is for $\gamma - \alpha \geqslant 0$ and $\gamma - \beta \leqslant 0$, which implies $\beta \geqslant \gamma \geqslant \alpha > 1$. Thus, Theorem 60 can be applied on the super-linear Emden-Fowler equation since one can always chose α and β such that $1 < \alpha \leqslant \gamma \leqslant \beta$. □

• The problem whether or not "lim" in Winter's criterion (4.23) can be replaced by "lim sup" is solved by Kamenev in his Theorem 57 for the sub-linear case of Emden-Fowler equation (4.3). Although by Hartman's result this is not possible in the linear case, fortunately, in 1978 Kamenev in [47] gave an alternative solution of this problem in the linear case, as follows.

THEOREM 61. ([47] - from 1978) *Let* $r(t) \equiv 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *If for some* $n > 2$,

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds = \infty, \tag{4.41}$$

then linear equation (4.22) *is oscillatory.*

It was the first result where the weighted mean of $q(t)$ is involved. The Wintner condition (4.23) is weaker than (4.41) because of the statements (4.20) and (4.21). Thus, Kamenev's Theorem 61 is stronger than Wintner's Theorem 51.

• In 1980 Butler in his paper [12] proved that the Wintner oscillation criterion (4.23) still holds for the Emden-Fowler equation (4.3) with $\gamma > 1$.

THEOREM 62. ([12] - from 1980) *Let* $\gamma > 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *If* $q(t)$ *satisfies* (4.23), *then super-linear Emden-Fowler equation* (4.3) *is oscillatory.*

The main limitation of Theorem 62, in the contrast to Wong's Theorem 58, is the term "lim" appearing originally in Wintner condition (4.23).

• In 1989 Cheh-Chih Yeh in his paper [119] considered a general class of equations with a functional term:

$$x''(t) + q(t)F(x(t), x(h(t))) = 0,\ t \geqslant t_0, \tag{4.42}$$

where the nonlinear and functional terms $F(u,v)$ and $h(t)$ satisfy $h_1(t) \leqslant h(t)$, $0 < k \leqslant h_1'(t) \leqslant 1$ and:

$$\exists M > 0,\ \liminf_{v\to\infty} \left|\frac{F(u,v)}{v}\right| \geqslant c > 0 \text{ for all } v \geqslant M. \tag{4.43}$$

Especially for the Emden-Fowler nonlinearity $f(y) = |y|^{\gamma}\mathrm{sgn}(y)$, $\gamma > 0$, if we put $F(u,v) = |v|^{\gamma}\mathrm{sgn}(v)$, $h_1(t) = h(t) = t$, $k \equiv 1$, then (4.43) is fulfilled provided $\gamma \geqslant 1$.

THEOREM 63. ([119, Theorem 1] - from 1980) *If* (4.43) *holds,* $q(t) \geqslant 0$ *and for some* $n > 2$,

$$\limsup_{t\to\infty} \frac{1}{n!t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)ds = \infty, \tag{4.44}$$

then the nonlinear equation (4.42) *is oscillatory.*

On the one hand, Theorem 63 is a variant of Travis oscillation criterion of integral type not involving the integral mean, see [95]. On the other hand, Kamenev's oscillation criterion (4.41) is comparable in some sense with (4.44).

REMARK 22. Hamedani and Krenz in their paper [38] studied the second-order functional differential equation $x''(t)+q(t)f(x(t),x(h(t))g(x'(t))=0$ that especially for $g(y)=1$ contains the functional equation (4.42) considered in Trevis [95] and Yeh [119]. They established an integral oscillation criterion for this equation but without using integral mean and hence, it is not presented here. □

• In 1982 Takeshi Kura in [54] considered the sub-linear Emden-Fowler equation (4.3) and he generalized Kamenev's Theorem 57 in the following sense.

THEOREM 64. ([54, Theorem 1] - from 1982) *Let* $0<\gamma<1$ *and* $q\in C([t_0,\infty),\mathbb{R})$. *If for some* $\beta\in[0,\gamma]$,

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\tau^{\beta}q(\tau)d\tau ds=\infty, \tag{4.45}$$

then sub-linear Emden-Folwer equation (4.3) *is oscillatory.*

For $\beta=0$ condition (4.45) becomes (4.32) and thus, Theorem 64 is a generalization of Kamenev's Theorem 57. The author illustrated Theorem 64 by (4.17): $x''+t^{\sigma}\sin(t)|x|^{\gamma}\mathrm{sgn}(x)=0,\ 0<\gamma<1$, and showed that all its solutions oscillate provided $\sigma>-\gamma$. Let us remark that Butler in [12] conjectured that this equation is oscillatory if and only if $\sigma>-\gamma$.

• In 1982 Yeh in [120] studied the second-order nonlinear differential equations with damping term:

$$x''+p(t)x'+q(t)f(x)=0, \tag{4.46}$$

where $p(t),q(t),f(y)$ are continuous on their domains and $q(t)$ may change its sign.

THEOREM 65. ([120, Theorem 1] - from 1982) *Let* $p,q\in C([t_0,\infty),\mathbb{R})$. *Assume* $f(y)$ *satisfy* (4.2) *and* (4.39). *If for some integer* $n>2$,

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}sq(s)ds=\infty, \tag{4.47}$$

and

$$\lim_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}s\Big[(t-s)\Big(p(s)-\frac{1}{s}\Big)+n-1\Big]^2(t-s)^{n-3}ds<\infty, \tag{4.48}$$

then the damped equation (4.46) *is oscillatory.*

COMMENT 14. If $p(t)\equiv 0$, then (4.48) is not satisfied, and hence Theorem 65 does not contain the undamped case. In order to show that, let chose a large enough t such that $\frac{t_0}{t}\leqslant\frac{1}{2n}$ for a fixed $n\geqslant 3$. Then for all $\tau\leqslant\frac{1}{2n}$, we have $1-n\tau\geqslant\frac{1}{2}$ and $1-\tau\geqslant\frac{1}{2n}$. Hence, by substitution $\tau=\frac{s}{t}$ we obtain

$$\begin{aligned}
&\frac{1}{t^{n-1}}\int_{t_0}^{t}s\Big[(t-s)\Big(0-\frac{1}{s}\Big)+n-1\Big]^2(t-s)^{n-3}ds\\
&\qquad=\int_{\frac{t_0}{t}}^{1}\frac{1}{\tau}(1-n\tau)^2(1-\tau)^{n-3}d\tau\geqslant\int_{\frac{t_0}{t}}^{\frac{1}{2n}}\frac{1}{\tau}(1-n\tau)^2(1-\tau)^{n-3}d\tau\\
&\qquad\geqslant\frac{1}{4}\Big(1-\frac{1}{2n}\Big)^{n-3}\int_{\frac{t_0}{t}}^{\frac{1}{2n}}\frac{1}{\tau}d\tau=\frac{1}{4}\Big(1-\frac{1}{2n}\Big)^{n-3}\ln\frac{t}{2nt_0}\to\infty
\end{aligned}$$

as $t\to\infty$. Note that condition (4.47) is a little bit different from (4.41) because, instead of $q(s)$, the function $sq(s)$ is appearing. □

Theorem 65 was applied to the Euler differential equation

$$x'' + \frac{1}{t}x' + \frac{1}{t^2}x = 0,\ t > 0, \tag{4.49}$$

to show that all its solutions are oscillatory. Another way to verify the oscillation of equation (4.49) is to check that $x_1(t) = \cos(\ln t)$ and $x_2(t) = \sin(\ln t)$ are two its solutions and using Sturm's separation theorem.

COMMENT 15. By Comment 12, the Emden-Fowler nonlinearity $f(y) = |y|^\gamma \operatorname{sgn} y$, $\gamma > 1$, does not satisfy the required condition (4.39) and hence, Yeh's Theorem 65 can not be applied to the super-linear Emden-Fowler equation (4.3). □

• In 1983 Christos G. Philos in [75] studied oscillation of equation (4.1): $x'' + q(t)f(x) = 0$ in the sub-linear case.

THEOREM 66. ([75] - from 1983) *Let $q \in C([t_0,\infty),\mathbb{R})$. Assume $f(y)$ satisfy the basic condition* (4.2)*, the sub-linear condition* (4.4) *and there exist a nonnegative constant*

$$I_f = \min\left\{ \frac{\inf_{y>0}\left[f'(y)F(y)\right]}{1+\inf_{y>0}\left[f'(y)F(y)\right]}, \frac{\inf_{y<0}\left[f'(y)F(y)\right]}{1+\inf_{y<0}\left[f'(y)F(y)\right]} \right\}, \tag{4.50}$$

where $F(y) = \int_0^y 1/f(v)dv$. If for some $\beta \in [0, I_f]$ condition (4.45) *holds, then sub-linear equation* (4.1) *is oscillatory.*

It is clear that Theorem 66 extends Kura's Theorem 64 from the sub-linear Emden-Fowler equation to general sub-linear equation (4.1), where the Emden-Fowler exponent γ is replaced by I_f.

• In 1984 Jurang Yan in his paper [115] studied the second-order linear differential equation with damping:

$$(r(t)x')' + p(t)x' + q(t)x = 0,\ t \geqslant t_0, \tag{4.51}$$

where $p(t), q(t)$ may change sign and obtained the Kamenev-type oscillation criterion for (4.51).

THEOREM 67. ([115, Theorem 1] - from 1984) *Let $r(t) \equiv 1$ and $p, q \in C([t_0,\infty),\mathbb{R})$. If for some $\alpha \in (1,\infty)$ and $\beta \in [0,1)$ the functions $p(t)$ and $q(t)$ satisfy*

$$\limsup_{t\to\infty} \frac{1}{t^\alpha}\int_{t_0}^t (t-s)^\alpha s^\beta q(s)ds = \infty, \tag{4.52}$$

$$\limsup_{t\to\infty} \frac{1}{t^\alpha}\int_{t_0}^t \left[s(t-s)p(s) + \alpha s - \beta(t-s)\right]^2 (t-s)^{\alpha-2} s^{\beta-2} ds < \infty, \tag{4.53}$$

then damped linear equation (4.51) *is oscillatory.*

COMMENT 16. For $p(t) \equiv 0$ equation (4.51) becomes undamped linear equation (4.22). Moreover, if we chose $\beta = 0$ and $\alpha = n-1$, then conditions (4.52) and (4.53) reduce respectively to Kamenev's condition (4.41) and

$$\limsup_{t\to\infty} \frac{1}{t^\alpha}\int_{t_0}^t (\alpha s)^2 (t-s)^{\alpha-2} s^{-2} ds = \frac{\alpha^2}{\alpha-1}\lim_{t\to\infty}\frac{(t-t_0)^{\alpha-1}}{t^\alpha} = 0 < \infty.$$

Thus, Yan's Theorem 67 extends Kamenev's Theorem 61 from undamped to damped linear equation. □

The author showed in [115, Remark 2] that the two-parametric equation

$$x'' + \frac{\sin(t)}{t^{\mu}}x' + \frac{\cos(t)}{t^{\nu}}x = 0,\ t > 0, \tag{4.54}$$

is oscillatory if $1 \leqslant \mu < \infty$ and $0 \leqslant \nu < 1$.

REMARK 23. It is known that for $\mu = \nu = 0$ all solutions of (4.54) are nonoscillatory, because of Sturm's separation theorem and the fact that $x(t) = e^{\cos(t)}$ is an its nonoscillatory solution, see also Section 5. □

In the same paper, the author considered the second-order nonlinear differential equation with damping:

$$(r(t)x')' + p(t)x' + q(t)f(x) = 0,\ t \geqslant t_0, \tag{4.55}$$

and generalized Theorem 67 from linear to the related nonlinear case.

THEOREM 68. ([115, Corollary 2] - from 1984) *Let $r(t) \equiv 1$ and $p,q \in C([t_0,\infty),\mathbb{R})$. Assume $f(y)$ satisfy (4.39). If $p(t)$ and $q(t)$ satisfy (4.52) and (4.53), then the damped nonlinear equation (4.55) is oscillatory.*

With the help of Comment 16, one can conclude that especially for $p(t) \equiv 0$ and $\beta = 0$, condition (4.53) is fulfilled. Therefore, from Theorem 68 we can derive the following consequence.

THEOREM 69. *Let $r(t) \equiv 1$, $p(t) \equiv 0$ and $q \in C([t_0,\infty),\mathbb{R})$. Assume $f(y)$ satisfy (4.39). If $q(t)$ satisfy (4.52), then the nonlinear equation (4.55) is oscillatory.*

• In 1986 Wong in his paper [103] continued his preceding results from [102] on the Emden-Folwer equation (4.3): $x'' + q(t)|x|^{\gamma}\mathrm{sgn}(x) = 0$ and gave the following oscillation criterion based on Wong condition (4.36) and a variation of the classic Kamenev-type condition (4.41) in the general case $\gamma > 0$.

THEOREM 70. ([103] - from 1986) *Let $\gamma > 0$ and $q \in C([t_0,\infty),\mathbb{R})$. If $q(t)$ satisfies (4.36) and for some $\alpha > 1$,*

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-s)^{\alpha} q(s)ds = \infty, \tag{4.56}$$

then general Emden-Fowler equation (4.3) is oscillatory.

In Theorem 70 all three cases in the Emden-Fowler equation (4.3) are simultaneously studied: sub-linear $0 < \gamma < 1$, linear $\gamma = 1$ and super-linear $\gamma > 1$. In Kamenev's Theorem 61, condition (4.32) was replaced by (4.41) in the linear case of (4.3). However, in Wong's Theorem 70, it was made together in all three cases provided (4.36) holds.

• In 1986 Jurang Yan in his paper [116] extended his oscillation criterion given in Theorem 67 on the second-order linear differential equation with damping (4.51) from $r(t) \equiv 1$ to arbitrary $r(t) > 0$.

THEOREM 71. ([116, Theorem 1] - from 1986) *Let $r(t) > 0$ and $r,p,q \in C([t_0,\infty),\mathbb{R})$. Assume there exist a function $\varphi \in C^1([t_0,\infty),\mathbb{R})$, $\varphi > 0$ on $[t_0,\infty)$, and a constant $\alpha \in (0,\infty)$ such that the functions $r(t)$, $p(t)$ and $q(t)$ satisfy*

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha}}\int_{t_0}^{t}\Big[(t-s)^{\alpha}\varphi(s)q(s) \\ -\frac{1}{4}\Big[(t-s)\frac{p(s)\varphi(s)}{r(s)}+\alpha\varphi(s)-(t-s)\varphi'(s)\Big]^2(t-s)^{\alpha-2}\frac{r(s)}{\varphi(s)}\Big]ds=\infty. \quad (4.57)$$

Then damped linear equation (4.51) *is oscillatory.*

As a consequence, the author showed that all solutions of the equation $\left(t^{-1}x'\right)'+\sin(t)x'+t^2\cos(t)x=0$ is oscillatory. Moreover, according to [116, Theorem 2] it was shown that all solutions of the three-pamateric equation:

$$\left(t^{\lambda}x'\right)'+\frac{\sin(t)}{t^{\mu}}x'+\frac{\cos(t)}{t^{\nu}}x=0,\ t>0,$$

is oscillatory provided $-1\leqslant\lambda<1$, $1\leqslant\mu<\infty$, $-1\leqslant\nu<1$ and $-2\nu+1\geqslant\lambda$. For $\lambda=0$ this damped linear equation contains the two-parametric equation (4.54) as a special case.

• In 1986 Halvorsen and Mingarelli generalized Coppel's Theorem 56 so that the mean-value zero of $q(t)$, i.e. condition (4.35) characterizes the oscillation of equation (4.34): $x''+\lambda q(t)x=0$, as follows.

THEOREM 72. ([37, Theorem 3.1] - from 1986) *Let $q(t)\not\equiv 0$ be an almost periodic function. Equation* (4.34) *is oscillatory for every $\lambda\neq 0$ if and only if* (4.35) *holds.*

Also, this theorem generalizes Stanek's theorem published in [85] from the periodic to the almost periodic case. On the oscillations of several kind of equations with periodic coefficients having mean-value zero, we refer reader to Kwong and Wong [55] - from 2003, Sugie and Matsumura [86] - from 2008, Došlý, Özbekler and Hilscher [21] - from 2012 and references therein.

• In 1989 Wong in his paper [106] studied a general equation (4.1): $x''+q(t)f(x)=0$ in the sub-linear case.

THEOREM 73. ([106] - from 1989) *Let $q\in C([t_0,\infty),\mathbb{R})$. Assume $f(y)$ satisfy the basic condition* (4.2)*, the sub-linear condition* (4.4) *and for some constant $c>0$,*

$$\frac{F''(y)F(y)}{F'^2(y)}\leqslant-\frac{1}{c}\quad \textit{for all } y\neq 0, \quad (4.58)$$

where $F(y)=\int_0^y 1/f(v)dv$. If there exists a positive concave function $\varphi(t)$ such that $\varphi'\geqslant 0$, $\varphi''\leqslant 0$, and

$$\limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\varphi^{\lambda}(\tau)q(\tau)d\tau ds=\infty, \quad (4.59)$$

where $\lambda=1/(1+c)<1$, then sub-linear equation (4.1) *is oscillatory.*

It is clear that condition (4.58) can be writen in the following simpler equivalent form

$$f'(y)F(y)\geqslant\frac{1}{c}\quad \text{for all } y\neq 0. \quad (4.60)$$

Also, in particular for $\varphi(t)\equiv t$, condition (4.59) becomes Kura's condition (4.45) and thus, Theorem 73 generalizes Kura's Theorem 64. Especially for the sub-linear Emden-Fowler equation, we can relate Kura's Theorem 64, Philos's Theorem 66 and Wong's Theorem 73 in this way: condition (4.58) is fulfilled in particular for $f(y)=|y|^{\gamma}\mathrm{sgn}(y)=|y|^{\gamma-1}y$, $f'(y)=\gamma|y|^{\gamma-1}$ and

$c = (1-\gamma)/\gamma$; also, for $\lambda = 1/(c+1)$ we have $\lambda = \gamma$; hence, if we put for $\varphi(t) \equiv 1$ and $\lambda = \gamma$ into (4.59), then Theorem 73 verifies Theorem 64 with $\beta = \gamma$ and Theorem 66 with $\beta = \lambda = \gamma$, since from (4.50) and (4.60) it follows $I_f = 1/(1+c) = \gamma$. Next, Theorem 73 was illustrated with the equation

$$x'' + t^\sigma \sin(t)\,|x|^{1/2}(1+|x|)\mathrm{sgn}x = 0$$

and showed that all its solutions oscillate provided $\sigma > 1/2$.

• We know that Butler in his Theorem 62 proved that Wintner's oscillation criterion (4.23), besides linear equations, also remains valid for the Emden-Fowler equation (4.3) with $\gamma > 1$. What is about oscillation of equation (4.3) with $\gamma > 1$ when the limit in (4.23) does not exists, that is,

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds. \tag{4.61}$$

Obviously condition (4.61) is stronger than Hartman's condition (4.25), because "liminf" is not supposed to be a finite real number. The answer was given in the following Wong's result published in 1989 in his paper [107] that was dedicated to G.J. Butler.

THEOREM 74. ([107] - from 1989) *Let* $\gamma > 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *If* $q(t)$ *satisfies* (4.36) *and* (4.61)*, then the super-linear Emden-Fowler equation* (4.3) *is oscillatory.*

The statement in (4.37) gives a possible connection of Wong's conditions (4.36) and (4.61) with Hartman's condition (4.25).

Furthermore, in [107] Wong generalized his Theorem 70 in a way to replace condition (4.36) by the related one:

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds > -\infty. \tag{4.62}$$

This is presented in the following result.

THEOREM 75. ([107] - from 1989) *Let* $\gamma > 0$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *If* $q(t)$ *satisfies* (4.56) *and* (4.62)*, then general Emden-Fowler equation* (4.3) *is oscillatory.*

We illustrate Theorem 75 on the next example.

EXAMPLE 10. The Emden-Fowler equation

$$x'' + \big(-t^2\sin(t) + 4t\cos(t) + 2\sin(t) + 2\big)|x|^\gamma \mathrm{sgn}(x) = 0,\ t \geqslant t_0,\ \gamma > 0, \tag{4.63}$$

is oscillatory. Indeed, for $q(t) = -t^2\sin(t) + 4t\cos(t) + 2\sin(t) + 2$, we have:

$$\int_{t_0}^{t} q(\tau)d\tau = \int_{t_0}^{t}\big(2\tau\sin(\tau) + \tau^2\cos(\tau) + 2\tau\big)' d\tau = 2t\sin(t) + t^2\cos(t) + 2t + c_0,$$

$$\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds = t^2\sin(t) + t^2 + c_0 t + c_1,$$

where $c_0 = -2t_0\sin(t_0) - t_0^2\cos(t_0) - 2t_0$ and $c_1 \in \mathbb{R}$ depending only on t_0. Hence,

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_{t_0}^{t}(t-s)^2 q(s)ds = c_2 + \limsup_{t\to\infty}\frac{2t}{3} = \infty,$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds = \liminf_{t\to\infty}\big(t(\sin(t)+1)\big) + c_0 + \liminf_{t\to\infty}\frac{c_1}{t} = c_0 > -\infty,$$

where $c_2 \in \mathbb{R}$. Thus, conditions (4.56) with $\alpha = 2$ and (4.62) are fulfilled and hence, by Wong's Theorem 75 we conclude that equation (4.63) is oscillatory. □

• In 1989 Yeh [121] studied the second-order nonlinear perturbed differential equations

$$(r(t)x')' + p(t)x' + f(t,x) = g(t,x,x'),\ t \geqslant t_0, \tag{4.64}$$

and he gave the next oscillation criterion.

THEOREM 76. ([121, Theorem 1] - from 1989) *Let* $r \in C^1([t_0,\infty),\mathbb{R})$ *and* $r(t) > 0$, $q, e \in C([t_0,\infty),\mathbb{R})$ *and* $e(t) > 0$ *on* $[t_0,\infty)$. *Let* $f \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, $g \in C([t_0,\infty)\times\mathbb{R}^2,\mathbb{R})$ *and for all* $y \neq 0$ *and* $v \in \mathbb{R}$,

$$yf(t,y) \geqslant yq(t)f_1(y) \quad \text{and} \quad yg(t,y,v) \leqslant e(t)yg_1(y)g_2(v),$$

where $f_1 \in C^1(\mathbb{R},\mathbb{R})$ *and* $g_1, g_2 \in C(\mathbb{R},\mathbb{R})$ *such that*

$$\begin{aligned}
&yf_1(y) > 0 \quad \text{and} \quad yg_1(y) > 0 \text{ for } y \neq 0,\\
&0 < g_2(v) \leqslant c \text{ for some constant } c \text{ and all } v \in \mathbb{R},\\
&f_1'(y) \geqslant k > 0 \quad \text{and} \quad g_1(y)/f_1(y) \leqslant K \text{ for } y \neq 0 \text{ and some } K \geqslant 0.
\end{aligned}$$

If there exists a function $\varphi \in C^1([t_0,\infty),(0,\infty))$ *such that for some* $\alpha > 0$,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t^\alpha}\int_{t_0}^t (t-s)^{\alpha-2}\varphi(s)\Big[(t-s)^2\big(q(s)-cKe(s)\big)\\
-\frac{1}{4k}r(s)\Big[(t-s)\Big(\frac{p(s)}{r(s)}-\frac{\varphi'(s)}{\varphi(s)}\Big)+\alpha\Big]^2\Big]ds = \infty,
\end{aligned} \tag{4.65}$$

then nonlinear perturbed equation (4.64) *is oscillatory.*

This result extends an integral oscillation criterion due to Grace and Lalli [32, 1980] involving the integral mean of Kamenev-type. On the other hand, in particular for $f(t,y) = q(t)y$ and $g(t,y,v) \equiv 0$ equation (4.64) becomes the damped linear equation (4.51) as well as $k = 1$ and $K = 0$. Hence, it is easy to see that condition (4.65) becomes (4.57) and thus, Theorem 76 generalizes Yan's Theorem 67.

• In 1989 Philos in [76] also studied the oscillation of second-order linear differential equation (4.22): $(r(t)x')' + q(t)x = 0$ with $r(t) \equiv 1$ but in a framework of the so-called Philos functions $H(t,s)$ and $h(t,s)$. The first kind of such result was given in one of the most cited results from the oscillation theory of the second-order differential equations.

THEOREM 77. ([76] - from 1989) *Let* $r(t) \equiv 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *Let* $D = \{(t,s) : t \geqslant s \geqslant t_0\}$ *and* $H : D \to \mathbb{R}$ *be a continuous function such that* $H(t,t) = 0$ *for* $t \geqslant t_0$, $H(t,s) > 0$ *for* $t > s \geqslant t_0$ *and* $\frac{\partial H}{\partial s}$ *be a continuous and nonpositive function on* $D_0 = \{(t,s) : t > s \geqslant t_0\}$ *such that the following function* $h : D_0 \to \mathbb{R}$ *is continuous:*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \text{ for all } (t,s) \in D_0. \tag{4.66}$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \Big[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\Big]ds = \infty, \tag{4.67}$$

then linear equation (4.22) *is oscillatory.*

A classic example for the function $H(t,s)$ satisfying required assumptions of Theorem 77 is $H(t,s)=(t-s)^{\alpha}$, $\alpha\geqslant 2$. This theorem was extended by many authors to several types of the second-order differential equations.

• In 1990 Grace and Lalli in [34] studied the oscillation of the Emden-Fowler equation (4.3): $x''+q(t)|x|^{\gamma}\mathrm{sgn}(x)=0$ with damped term:

$$(r(t)x')'+p(t)x'+q(t)|x|^{\gamma}\mathrm{sgn}(x)=0,\ \gamma>0. \tag{4.68}$$

He extending some previous Wong and Philos results on the oscillation of equation (4.3) in the super-linear case from undapmed to dapmed equation.

THEOREM 78. ([34, Theorem 10] - from 1990) *Let* $r(t)>0$, $r\in C^1([t_0,\infty),\mathbb{R})$, $p,q\in C([t_0,\infty),\mathbb{R})$, *and* $\gamma>1$. *Let there exist a function* $\rho\in C^1([t_0,\infty),(0,\infty))$ *such that:*

$$r(t)\rho'(t)-p(t)\rho(t)\geqslant 0 \quad \textit{and} \quad \big(r(t)\rho'(t)-p(t)\rho(t)\big)'\leqslant 0 \ \textit{for}\ t\geqslant t_0,$$

$$\liminf_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds>-\infty \quad \textit{and} \quad \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)\rho(s)}ds=\infty. \tag{4.69}$$

If

$$\limsup_{t\to\infty}\frac{1}{\int_{t_0}^{t}h(s)ds}\int_{t_0}^{t}\int_{t_0}^{s}h(s)\rho(\tau)q(\tau)d\tau ds=\infty, \tag{4.70}$$

where $h(t)=1/[r(t)\rho(t)]$, *then damped super-linear Emden-Fowler equation* (4.68) *is oscillatory.*

In [34], several integral oscillation criteria were obtained including generalized integral mean with weighted function in the sense of Coles presented above in his Theorem 55 but for more general class of equations: $(r(t)\Psi(x)x')'+p(t)x'+q(t)f(x)=0$, see [34, Theorems 1-8], with many useful examples, see [34, Theorems 1-8].

• In 1992, by combining both Philos's Theorem 66 and Wong's Theorem 73, Wong and Yeh in [113] proved the following oscillation result for general equation (4.1): $x''+q(t)f(x)=0$ in the sub-linear case.

THEOREM 79. ([113] - from 1992) *Let* $q\in C([t_0,\infty),\mathbb{R})$, *and let* $f(y)$ *satisfy* (4.2), (4.4) *and* (4.60). *If there exists a positive concave function* $\varphi(t)$ *such that for some* $\alpha>1$,

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}\varphi^{\lambda}(s)q(s)ds=\infty, \tag{4.71}$$

where $\lambda\in[0,I_f]$ *and* I_f *is from* (4.50), *then sub-linear equation* (4.1) *is oscillatory.*

• In 1992 Philos and Purnaras in [77] studied the oscillation of general super-linear equation (4.1).

THEOREM 80. ([77] - from 1992) *Let* $q\in C([t_0,\infty),\mathbb{R})$ *and let* $f(y)$ *satisfy* (4.2), (4.6) *and*

$$\min\left\{\inf_{y>0}\left[f'(y)\int_{y}^{\infty}\frac{dz}{f(z)}\right],\ \inf_{y<0}\left[f'(y)\int_{y}^{-\infty}\frac{dz}{f(z)}\right]\right\}>1.$$

If $q(t)$ *satisfies* (4.61) *and for some integer* $n\geqslant 1$,

$$\liminf_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds>-\infty, \tag{4.72}$$

then super-linear equation (4.1) *is oscillatory.*

This theorem extends Wong's Theorem 74 from the super-linear Emden-Fowler equation to general super-linear equation (4.1). Here the constant d appearing in Naito's condition (4.7) is not used. It seems that the condition $n \geqslant 1$ is too weak, since as usually $n > 2$ was supposed.

• In 1993 El-Sheik in [25] considered the oscillation of the second-order differential equation

$$(r(t)\Psi(x)x')' + q(t)f(x) = 0,\ t \geqslant t_0, \tag{4.73}$$

where $r(t)$, $\Psi(y)$, $\Psi'(y)$ and $q(t)$ are positive functions.

THEOREM 81. ([25, Theorem 3.2] - from 1993) *Let* $r(t) > 0$ *and* $r, q \in C^1([t_0,\infty),(0,\infty))$. *Assume* $f(y)$ *satisfy* (4.2) *and* $f'(y)/\Psi(y) \geqslant k > 0$ *and* $\Psi'(y) > 0$. *Let*

$$\int_0^\infty \frac{(r(s)q(s))'}{r(s)q(s)}ds < \infty.$$

If for some $n \geqslant 3$ *the Kamenev condition* (4.41) *holds and*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t-s)^{n-3} r(s)ds < \infty, \tag{4.74}$$

then nonlinear equation (4.73) *is oscillatory.*

THEOREM 82. ([25, Theorem 3.3] - from 1993) *Let* $r(t) > 0$, $r, q \in C([t_0,\infty),(0,\infty))$. *Assume* $f(y)$ *satisfy* (4.2) *and there exist* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that* $\eta(t) \geqslant \Psi(y) > 0$, $f'(y)/\eta(t) \geqslant K_1 > 0$, *and* $f'(y) \geqslant 1$ *on* $[t_0,\infty)\times\mathbb{R}/\{0\}$. *Let*

$$R(t) = \int_0^t \frac{1}{r(s)\eta(s)}ds,\ \ t \geqslant t_0.$$

If $R(\infty)$ *exists and for some* $n \geqslant 3$ *the function* $q(t)$ *satisfies*

$$\limsup_{t\to\infty} \frac{1}{R^{n-1}(t)} \int_{t_0}^t \big(R(t) - R(s)\big)^{n-1} q(s)ds = \infty, \tag{4.75}$$

then nonlinear equation (4.73) *is oscillatory.*

Besides equation (4.73), in [25] the oscillation of related differential equation with functional argument $\tau(t)$ was also considered: $(r(t)\Psi(x(t))x'(t))' + q(t)f(x(\tau(t)) = 0$, $t \geqslant t_0$, where $\tau(t) \leqslant t$, see [25, Theorems 4.1 and 4.2].

• In 1993 Wong in his paper [105] studied the oscillation of general equation (4.1): $x'' + q(t)f(x) = 0$ in both sub- and super-linear cases.

THEOREM 83. ([105] - from 1993) *Let* $q \in C([t_0,\infty),\mathbb{R})$. *Assume* $f(y)$ *satisfy* (4.2) *and either sub-linear conditions* (4.4)-(4.5) *or super-linear conditions* (4.6)-(4.7). *If* $q(t)$ *satisfies:*

$$-\infty < \liminf_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^t (t-s)^\alpha q(s)ds < \limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^t (t-s)^\alpha q(s)ds \leqslant \infty, \tag{4.76}$$

for some $\alpha \geqslant 1$, *then general equation* (4.1) *is oscillatory.*

Wong's oscillation criterion (4.76) includes the integral mean (of convolution type) originally considered in Kamenev's Theorem 61 for linear case and Wong's Theorem 70 for general case of the Emden-Fowler equation. It is different from the integral mean appearing in Hartman's Theorem 52. However, the properties of and relation between "lim inf" and "lim sup" in (4.76) are the same as in the Hartman's condition (4.25). We suggest the reader to study the implication or equivalence between Hartman's and Wong's criterions (4.25) and (4.76) respectively.

• In 1994 Manabu Naito in his paper [66] also studied the oscillation of general equation (4.1).

THEOREM 84. ([66, Corollary 3] - from 1994) *Let* $q \in C([t_0,\infty),\mathbb{R})$. *Assume* $f(y)$ *satisfy* (4.2) *and either sub-linear conditions* (4.4)-(4.5) *or super-linear conditions* (4.6)-(4.7). *Let* $\alpha \in \mathbb{R}$ *be such that*

$$\alpha \geqslant 1 \text{ in the sub-linear and } \alpha > d/(d-1) \text{ in the super-linear case,}$$

where the constant d *is from super-linear condition* (4.7). *Then general equation* (4.1) *is oscillatory provided* $q(t)$ *satisfies* (4.56) *or*

$$\liminf_{t\to\infty} \frac{1}{t^\alpha}\int_{t_0}^t (t-s)^\alpha q(s)ds < \limsup_{t\to\infty} \frac{1}{t^\alpha}\int_{t_0}^t (t-s)^\alpha q(s)ds. \tag{4.77}$$

This theorem is closely related to Wong's Theorem 83. If in the sublinear case condition (4.56) holds, then Theorem 84 is a special case of Wong-Yeh's Theorem 79; however, if (4.77) holds, then Theorem 84 generalizes Wong's Theorem 83 since (4.77) is larger than (4.76). In the super-linear case, the parameter α in Theorem 84 depends on the constant d which is not the case in Theorem 83, where $\alpha \geqslant 1$ is a fixed real number.

• In 1995 Horng Jaan Li [58] and in 1996 Yuri V. Rogovchenko [79] studied the Philos-type oscillation criterion for the second-order linear equation (4.22): $(r(t)x')' + q(t)x = 0$.

THEOREM 85. ([58, Theorem 2.1] - from 1995 and [79]- from 1996) *Let* $r \in C^1([t_0,\infty),\mathbb{R})$ *and* $r(t) > 0$, $q \in C([t_0,\infty),\mathbb{R})$. *Let the sets* D, D_0 *and functions* $H(t,s)$, $h(t,s)$ *be defined as in Theorem* 77. *If there exists a function* $\varphi \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \Big[H(t,s)\rho(s) - \frac{1}{4}a(s)r(s)h^2(t,s)\Big]ds = \infty, \tag{4.78}$$

where

$$a(s) = e^{-2\int_{t_0}^s \varphi(\tau)d\tau} \quad \text{and} \quad \rho(s) = a(s)\big[q(s) + r(s)\varphi^2(s) - (r(s)\varphi(s))'\big],$$

then linear equation (4.22) *is oscillatory.*

According to Rogovchenko's observation given in [79], the conflicting hypothesis (C_1) is removed from the original [58, Theorem 2.1] and it is not appearing in Theorem 85. In contrast to Philos's Theorem 77, in Theorem 85 three new functions are included: $\varphi(t)$, $a(t)$ and $\rho(t)$. In particular for $\varphi(t) \equiv 0$, it is clear that $a(t) \equiv 1$ and $\rho(t) = q(t)$. Hence Theorem 85 generalizes Philos's Theorem 77.

If $R(t) = \int_{t_0}^t [1/r(s)]ds$, then putting in Theorem 85 for $H(t,s) = \big[R(t) - R(s)\big]^\lambda$ and $\varphi \equiv 0$, Li in [58, Corollary 2.2] derived the following interesting conclusion.

THEOREM 86. ([58, Corollary 2.2] - from 1995) *Let* $r \in C^1([t_0,\infty),\mathbb{R})$ *and* $r(t)>0$, $q \in C([t_0,\infty),\mathbb{R})$. *If for some* $\lambda > 1$,

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda}(t)} \int_{t_0}^{t} \left[R(t)-R(s)\right]^{\lambda} q(s)ds = \infty, \tag{4.79}$$

then linear equation (4.22) *is oscillatory.*

• In 1998 Wan-Tong Li in his paper [59] considered the oscillation of a general class of second-order differential equation:

$$(r(t)x'^{\sigma})' + q(t)f(x) = 0, \; t \geqslant t_0, \tag{4.80}$$

where σ is any quotient of odd integers. In the case for $\sigma > 1$, he proved an integral oscillation criterion given in [59, Theorem 3.1] not including the integral mean and hence, it is not presented here. However, for $\sigma = 1$ the following Li's criterion generalizes previous Theorem 85 from linear to possible nonlinear case.

THEOREM 87. ([59, Theorem 3.2] - from 1998) *Let* $r \in C^1([t_0,\infty),\mathbb{R})$, $r(t)>0$, $\sigma = 1$ *and* $q \in C([t_0,\infty),\mathbb{R})$. *Let* $f(y)$ *satisfy* (4.2) *and* $f'(y) \geqslant \mu > 0$ *for all* $y \neq 0$. *Let the sets* D, D_0 *and functions* $H(t,s)$, $h(t,s)$ *be defined as in Theorem* 77. *If there exists a function* $\varphi \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s) - \frac{1}{4\mu} a(s)r(s)h^2(t,s)\right] ds = \infty, \tag{4.81}$$

where

$$a(s) = e^{-2\mu\int_{t_0}^{s}\varphi(\tau)d\tau} \quad \textit{and} \quad \rho(s) = a(s)\left[q(s) + \mu r(s)\varphi^2(s) - (r(s)\varphi(s))'\right],$$

then the nonlinear equation (4.80) *is oscillatory.*

Now, it is clear that if $f(y) \equiv y$ and $\mu = 1$, then Theorem 87 includes Theorem 85 as a special case. Analogously to the relation between Theorems 85 and 86, putting for $R(t) = \int_{t_0}^{t} 1/r(s)ds$ and $H(t,s) = \left(R(t)-R(s)\right)^{\lambda}$, from the previous theorem it follows that oscillation criterion (4.79) also holds for equation (4.80) with $\sigma = 1$, see [59, Corollary 3.3] for more details. One of the main results in [59] was illustrated with the differential equation

$$(tx')' + \frac{\lambda}{t\ln^2(t)}(x + x^3) = 0, \; t > 1, \tag{4.82}$$

and showed that (4.82) is oscillatory provided $\lambda > 1/4$. We suggest the reader to compare this result with the related one for the linear equation (4.24) given in Example 5 above.

• In 1999 Qingkai Kong in his paper [51] considered similar kind of nonlinear equations to equation (4.80):

$$(r(t)x'^{\sigma})' + q(t)x^{\sigma} = 0, \; t \geqslant t_0, \tag{4.83}$$

where σ is any quotient of odd integers and $1/r, q$ are locally integrable functions on $[t_0,\infty)$. This equation is of half-linear type, since if $x(t)$ is a solution of (4.83), then clearly $cx(t)$, for any $c \in \mathbb{R}$, is also a solution of (4.83). Analogously to the class $\mathscr{H}$ of functions $H(t,s)$ introduced in (2.53) in Kong's interval oscillation criterion, see Theorem 88 for linear unforced differential equation (2.1), one can introduce its half-linear case by the same (2.53) but with h_1, h_2 satisfying:

$$\frac{\partial H}{\partial t}(t,s) = h_1(t,s)\left(H(t,s)\right)^{\frac{\sigma}{\sigma+1}} \quad \text{and} \quad -\frac{\partial H}{\partial s}(t,s) = h_2(t,s)\left(H(t,s)\right)^{\frac{\sigma}{\sigma+1}}. \tag{4.84}$$

We see that Kong's equality (2.54) is a linear analogous of (4.84) in particular for $\sigma = 1$.

THEOREM 88. ([51, Theorem 3.1] - from 1999) *Let* $r(t) > 0$ *on* $[t_0, \infty)$ *and* $1/r, q \in L_{\mathrm{loc}}([t_0, \infty), \mathbb{R})$. *If there exists a function* $H \in \mathscr{H}$ *with* (4.84) *such that for each* $a \geqslant t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,a)} \int_a^t \left[H(t,s)q(s) - \frac{1}{(\sigma+1)^{(\sigma+1)}} r(s) h_2^{\sigma+1}(t,s)\right] ds = \infty, \tag{4.85}$$

then the half-linear equation (4.83) *is oscillatory.*

We can remark that condition (4.85) is a half-linear generalization of related conditions (4.78) and (4.81) with $r(t) \equiv 1$ and $\varphi \equiv 0$. Also, for $r(t) \equiv 1$ and $\sigma = 1$, condition (4.85) becomes Philos condition (4.67). It seems that Theorem 88 is the first oscillation criterion involving the integral mean for half-linear differential equations. From Theorem 88, Kong derived the following interesting consequence.

THEOREM 89. ([51, Theorem 3.4] - from 1999) *Let* $r(t) \equiv 1$ *and* $q \in L_{\mathrm{loc}}([t_0, \infty), \mathbb{R})$. *Then half-linear equation* (4.83) *is oscillatory provided for each* $a \geqslant t_0$ *and some* $\alpha > \sigma$ *one of the following statements holds:*
i)

$$\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_a^t (t-s)^\alpha q(s) ds = \infty;$$

ii)

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-\sigma}} \int_a^t (s-a)^\alpha q(s) ds > \frac{\alpha^{\sigma+1}}{(\sigma+1)^{(\sigma+1)}(\alpha-\sigma)},$$
$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-\sigma}} \int_a^t (t-s)^\alpha q(s) ds > \frac{\alpha^{\sigma+1}}{(\sigma+1)^{(\sigma+1)}(\alpha-\sigma)};$$

iii)

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-\sigma}} \int_a^t (s-a)^\alpha \left[q(s) + q(2t-s)\right] ds > \frac{2\alpha^{\sigma+1}}{(\sigma+1)^{(\sigma+1)}(\alpha-\sigma)}.$$

Obviously, Kong's condition $i)$ is a half-linear generalization of Kamenev's condition (4.41) and Wong's condition (4.56).

• In 1999 Yuri V. Rogovchenko in his paper [80] studied the oscillation of second-order nonlinear differential equation without delay argument:

$$x''(t) + q(t)f(x(t))g(x'(t)) = 0,\ t \geqslant t_0, \tag{4.86}$$

and with the delay argument $\tau(t)$:

$$x''(t) + q(t)f(x(\tau(t)))g(x'(t)) = 0,\ t \geqslant t_0. \tag{4.87}$$

The oscillation of these two classes of equations have been already considered by Grace in Lalli in [33, Theorems 1 and 2] exploring some integral oscillation criteria not including the integral mean and hence it is not presented here.

THEOREM 90. ([80, Theorem 1] - from 1999) *Let* $q \in C([t_0, \infty), \mathbb{R})$ *and* $q(t) \not\equiv 0$ *on any ray in* $[t_0, \infty)$. *Let* $f(y)$ *satisfy* (4.2), $f'(y) \geqslant K > 0$ *for all* $y \neq 0$, *and* $g(v)$ *is continuous on* $\mathbb{R}$,

$g(v) \geqslant C > 0$ *for all* $v \neq 0$. *Let the sets* D, D_0 *and functions* $H(t,s)$, $h(t,s)$ *be defined as in Theorem* 77. *If there exists a function* $\varphi \in C^1([t_0,\infty),\mathbb{R})$, $\varphi(t) > 0$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)Cq(s)\varphi(s) - \frac{\varphi(s)}{4K}\left(h(t,s) - \frac{\varphi'(s)}{\varphi(s)}\sqrt{H(t,s)}\right)^2 \right] ds = \infty, \tag{4.88}$$

then the nonlinear equation (4.86) *is oscillatory.*

Condition (4.88) can be divided into two parts so that the first one is infinite and the second one is finite in the sense of limit superior, as it was done in the next result.

THEOREM 91. ([80, Corollary 1] - from 1999) *Let* $q \in C([t_0,\infty),\mathbb{R})$ *and* $q(t) \neq 0$ *on any ray in* $[t_0,\infty)$. *Let* $f(y)$ *satisfy* (4.2), $f'(y) \geqslant K > 0$ *for all* $y \neq 0$, *and* $g(v)$ *is continuous on* $\mathbb{R}$, $g(v) \geqslant C > 0$ *for all* $v \neq 0$. *Let the sets* D, D_0 *and functions* $H(t,s)$, $h(t,s)$ *be defined as in Theorem* 77. *If there exists a function* $\varphi \in C^1([t_0,\infty),\mathbb{R})$, $\varphi(t) > 0$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)q(s)\varphi(s)ds = \infty, \tag{4.89}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \varphi(s)\left(h(t,s) - \frac{\varphi'(s)}{\varphi(s)}\sqrt{H(t,s)}\right)^2 ds < \infty, \tag{4.90}$$

then the nonlinear equation (4.86) *is oscillatory.*

Previous theorems were illustrated to the following nonlinear equation:

$$x'' + \frac{1}{(1+\sin^2(t))(1+\cos^2(t))} x(1+x^2)(1+x'^2) = 0$$

which has two oscillatory solutions $x_1(t) = \sin(t)$ and $x_2(t) = \cos(t)$. But, it is not enough to conclude that this equation is oscillatory since for nonlinear equation Sturm's separation theorem does not hold in general. However, Theorem 90 proves that this equation is oscillatory, see [80, Corollary 2]. Related oscillation criteria for the delayed equation (4.87) can be found in [80, Theorem 3 and Corollary 4].

4.3. Summary for the sub-linear case

In this subsection we summarize all results of this section concerning the sub-linear equation (4.1):

year – author's theorem, criteria for $q(t)$:

1971 – Kamenev's Theorem 57, (4.32): $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)d\tau ds = \infty$;

1982 – Kura's Theorem 64, (4.45): $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \tau^{\beta} q(\tau)d\tau ds = \infty$, $\beta \in [0,\gamma]$, $\gamma \in (0,1)$;

1982 – Yeh's Theorem 65, (4.47): $\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} sq(s)ds = \infty$, $n > 2$;

1983 – Philos's Theorem 66, (4.45) with $\beta \in [0, I_f]$ and I_f from (4.50);

1986 – Wong's Theorem 70, (4.36): $-\infty < \liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds < 0$ and
(4.56): $\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha q(s)ds = \infty$, $\alpha > 1$;

1989 – Wong's Theorem 73,
(4.59): $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \varphi^\lambda(\tau)q(\tau)d\tau ds = \infty$, $\lambda = 1/(1+c) < 1$ and $\varphi > 0$, $\varphi' \geqslant 0$, $\varphi'' < 0$;

1989 – Wong's Theorem 75, (4.56) and (4.62): $-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)d\tau ds$;

1992 – Wong and Yeh's Theorem 79, (4.71): $\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha \varphi^\lambda(s)q(s)ds = \infty$,
where $\lambda \in [0, I_f]$ and I_f is from (4.50);

1993 – Wong's Theorem 83,
(4.76): $-\infty < \liminf_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha q(s)ds < \limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha q(s)ds \leqslant \infty$;

1994 – Naito's Theorem 84, (4.56) or
(4.77): $\liminf_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha q(s)ds < \limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-s)^\alpha q(s)ds$;

2000 – Wong's Theorem 49, (4.11): $\limsup_{t\to\infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} h(t,s)q(s)ds = \infty$.

In all previous statements, where it appears, $\alpha > 1$. In particular for $h(t,s) = (t-s)^\alpha$, $\alpha > 1$, condition (4.11) becomes (4.56). Since $h(t,s) = (t-s)^\alpha$ for $\alpha > 1$ satisfies all assumptions of Theorem 49, one can conclude that Theorem 49 generalizes each of previous results in which (4.56) is appearing.

4.4. Summary for the super-linear case

Now, we make a summary concerning the super-linear case in the equation $x'' + q(t)f(x) = 0$. In the case when the nonlinear term $f(y)$ satisfies (4.39), we always point out that the Emden-Fowler super-nonlinearity is not included, see Comment 12.

Year – author's theorem, $f(y)$ includes or not the Emden-Fowler super-nonlinearity, the corresponding criteria for $q(t)$:

1973 – Wong's Theorem 58, the criteria for $q(t)$:
(4.32): $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)d\tau ds = \infty$ and (4.36): $-\infty < \liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds < 0$;

1975 – Onose's Theorem 59, the Emden-Fowler super-nonlinearity is not included,
the criteria for $q(t)$: (4.32) and (4.36);

1978 – Chen's Theorem 60,
the criteria for $q(t)$: (4.32) and (4.36);

1980 – Butler's Theorem 62, the criteria for $q(t)$:
(4.23): $\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds = \infty$;

1980 – Yeh's Theorem 63, the criterion for $q(t)$:
(4.44): $\limsup_{t\to\infty} \frac{1}{n!t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds = \infty$;

1984 – Yan's Theorem 69, the Emden-Fowler super-nonlinearity is not included, the criterion for $q(t)$:
(4.52): $\limsup_{t\to\infty} \frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}s^{\beta}q(s)ds = \infty$, for $\alpha > 1$ and $\beta = 0$;

1989 – Wong's Theorem 74, the criteria for $q(t)$:
(4.36) and (4.61): $\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds$;

1989 – Wong's Theorem 75, the criteria for $q(t)$:
(4.56): $\limsup_{t\to\infty} \frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}q(s)ds = \infty$ and (4.62): $\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(\tau)d\tau ds > -\infty$;

1990 – Grace and Lalli's Theorem 78, the criteria for $q(t)$:
(4.69): $\liminf_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)ds > -\infty$ and
(4.70): $\limsup_{t\to\infty} \frac{1}{\int_{t_0}^{t}\rho^{-1}(s)ds}\int_{t_0}^{t}\int_{t_0}^{s}\frac{\rho(\tau)q(\tau)}{\rho(s)}d\tau ds = \infty$;

1992 – Philos and Purnaras's Theorem 80, the criteria for $q(t)$:
(4.61) and (4.72): $\liminf_{t\to\infty} \frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}q(s)ds > -\infty$;

1993 – Wong's Theorem 83, the criterion for $q(t)$:
(4.76): $-\infty < \liminf_{t\to\infty} \frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}q(s)ds < \limsup_{t\to\infty} \frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}q(s)ds \leqslant \infty$, $\alpha \geqslant 1$;

1994 – Naito's Theorem 84, the criterion for $q(t)$:
(4.56) or (4.77): $\liminf_{t\to\infty} \frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}q(s)ds < \limsup_{t\to\infty} \frac{1}{t^{\alpha}}\int_{t_0}^{t}(t-s)^{\alpha}q(s)ds$, for some $\alpha > d/(d-1)$;

2000 – Wong's Theorem 50, the criteria for $q(t)$:
(4.15): $\limsup_{t\to\infty} \frac{1}{h_1(t,t_0)}\int_{t_0}^{t}h_1(t,s)q(s)ds = \infty$ and
(4.16): $-\infty < \liminf_{t\to\infty} \frac{1}{h_2(t,t_0)}\int_{t_0}^{t}h_2(t,s)q(s)ds < 0$.

5. Wong academic career

In very recently article [16] written by Professor Goong Chen, who is one of Editors-in-Chief of the Journal of Mathematical Analysis and Applications, a dual career by James S.W. Wong is presented: in the mathematics and business. At the end of this article, we present a brief reviews to Wong academic career.

He has written a total of 151 papers. The first 81 papers appeared between 1964-1976 and other 70 papers between 1980-2013. He had the fortune to collaborate with 34 fellow mathematicians over a period of 40 years (1964-2013). Jointly with Professor Man-Kam Kwong, he published more than 19 papers. One of their best papers was published in Journal of Differential Equations 238 (2007), 18–42, resolving the longstanding Kiguradze-Nehari conjecture in the affirmative. This paper is an extension of Wong's paper published in the inaugural issue of Analysis and Applications 1(2003), 71-79, which settled the Kiguaradz conjecture originated from his 1964 paper.

He received a B. Sc. in Physics and Mathematics from Baylor University in 1960 and a Ph. D. in Mathematics from California Institute of Technology in 1964. He was appointed Professor of Mathematics at the University of Iowa in 1970.

In 1974, he returned to Hong Kong, where he was first appointed as Honorary Research Associate in the HKU, Department of Mathematics in 1980 and has served as Honorary Professor since 2004. He has also served on the University Council, the Committee Review Committee, and the Finance Committee since 1996. He has served on the Labour Advisory Board, the Textile Advisory Board, the Council of the Hong Kong Baptist University, the Council for Academic Accreditation, the Research Grants Council, and the Council of the Open University. He was appointed as a Justice of the Peace in 1987.

In June 2012, he received the title of Honorary Editor of the journal "Journal of Mathematical Analysis and Applications", see:
http://www.sciencedirect.com/science/article/pii/S0022247X13004162.

In October 2013, he was among the eight distinguished recipients of the Honorary University Fellowships of the university of Hong-Kong, see:
http://www.hku.hk/press/news_detail_10256.html.

In February 2014, he will receive the title of Honorary Editor of the journal "Differential Equations and Applications".

In May 2014, he will receive a Distinguished Caltech's Alumni Award in California Institute of Technology. James S.W. Wong is the first math Ph.D. to receive this award as mathematics professor since 1966 among a total of 237 recipients, there were only 4 with Ph.D. in Mathematics (3 professors of computer science, including prof. Donald E. Knuth at Stanford, and 1 of electrical engineering).

Simultaneously with his academic career, Dr James S.W. Wong is a successful businessman, industrialist, real estate developer and entrepreneur. He is the Chairman of Hon Kwok Land Investment Co Ltd, Chinney Investments Ltd and Chinney Alliance Group Ltd.

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