Differential
Equations
& Applications
Volume 6, Number 2 (2014), 187–208
doi:10.7153/dea-06-09

ON THE EXISTENCE OF ASYMPTOTICALLY STABLE SOLUTIONS FOR A MIXED FUNCTIONAL INTEGRAL EQUATION IN N VARIABLES

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(*Communicated by Jurang Yan*)

Abstract. The aim of this paper is to study the existence of asymptotically stable solutions for a mixed functional integral equation in N variables. This is done by using a fixed point theorem of Krasnosel'skii type in the Fréchet space and the new integral inequalities with explicit estimate. In order to illustrate the results obtained here, an example is given.

1. Introduction

In this paper, we establish the existence of asymptotically stable solutions for the mixed functional integral equation in N variables of the form

$$u(x) = V\left(x, u(x), \int_{B_x} V_1\left(x, y, u(\sigma_1(y))\right) dy\right) + \int_{\mathbb{R}_+^N} F\left(x, y, u(\sigma_2(y))\right) dy, \tag{1.1}$$

where $x \in \mathbb{R}_+^N = \{(x_1, ..., x_N) \in \mathbb{R}^N : x_1 \geqslant 0, ..., x_N \geqslant 0\}$,

$$V : \mathbb{R}_+^N \times E^2 \to E,\ V_1 : \Delta \times E \to E;\ F : \mathbb{R}_+^{2N} \times E \to E,$$
$$\sigma_1,\ \sigma_2 : \mathbb{R}_+^N \to \mathbb{R}_+^N \text{ are continuous},$$
$$\Delta = \{(x,y) \in \mathbb{R}_+^{2N} : y \in B_x\},\ B_x = [0, x_1] \times ... \times [0, x_N],$$
$$\text{the functions } \sigma_1,\ \sigma_2 : \mathbb{R}_+^N \to \mathbb{R}_+^N \text{ are continuous with } \sigma_1(x) \in B_x,\ \forall x \in \mathbb{R}_+^N,$$
$$E \text{ is a Banach space with norm } |\cdot|.$$

It is well known that, nonlinear integral equations and nonlinear functional integral equations have been some topics of great interest in the field of nonlinear analysis for a long time. Since the pioneering work of Volterra up to our days, integral equations have attracted the interest of scientists not only because of their mathematical context but also because of their miscellaneous applications in various fields of science and technology, see [18]. The special cases of (1.1) occur in mechanics, population dynamics, engineering systems, the theory of "adiabatic tubular chemical reactors", etc. For the details of such problems, it can be found in, for example, Corduneanu [6]

Mathematics subject classification (2010): 47H10, 45G10, 47N20, 65J15.
Keywords and phrases: fixed point theorem of Krasnosel'skii type, mixed functional integral equation in N variables, contraction mapping, completely continuous, asymptotically stable solution.

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or Deimling [7]. It also can be found some applications of integral or integrodifferential equations to various problems occurring in contemporary research, such as the following integrodifferential equation is encountered in the mathematical description of coagulation process [6], under certain simplifying assumptions

$$\begin{aligned} f(t,x) &= f_0(x) + \frac{1}{2}\int_0^t \int_0^x \phi(x-y,y) f(s,x-y) f(s,y)\,dyds \\ &\quad - \int_0^t \int_0^\infty f(s,x)\phi(x,y) f(s,y)\,dyds. \end{aligned}$$

In general, existence results of integral equations have been obtained via the fundamental methods in which the fixed point theorems are often applied, see [1] – [18] and the references given therein. Recently, using the technique of the measure of noncompactness and the Darbo fixed point theorem, Z. Liu et al. [10] have proved the existence and asymptotic stability of solutions for the equation

$$x(t) = f\left(t,\, x(t),\, \int_0^t u(t,s,x(a(s)),x(b(s)))\,ds\right),\ t \in \mathbb{R}_+.$$

In [2], using a fixed point theorem of Krasnosel'skii, Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the following integral equation

$$u(t) = q(t) + \int_0^t K(t,s,u(s))ds + \int_0^\infty G(t,s,u(s))ds,\ t \in \mathbb{R}_+,$$

where the functions given with real values are supposed to be continuous satisfying suitable conditions. In case the Banach space E is arbitrary, recently in [14], [15], the existence of asymptotically stable solutions to the following integral equations

$$x(t) = q(t) + f(t,x(t)) + \int_0^t V(t,s,x(s))ds + \int_0^\infty G(t,s,x(s))ds,\ t \in \mathbb{R}_+,$$

or

$$\begin{aligned} u(x,y) &= q(x,y) + f(x,y,u(x,y)) + \int_0^x \int_0^y V(x,y,s,t,u(s,t))\,dsdt \\ &\quad + \int_0^\infty \int_0^\infty F(x,y,s,t,u(s,t))\,dsdt,\ (x,y) \in \mathbb{R}_+^2, \end{aligned}$$

also have been proved by using the fixed point theorem of Krasnosel'skii type as follows.

THEOREM 1. [12]. *Let $(X, |\cdot|_n)$ be a Fréchet space and let $U,\ C : X \to X$ be two operators. Assume that*

(i) U is a $k-$contraction operator, $k \in [0,1)$ (depending on n), with respect to a family of seminorms $\|\cdot\|_n$ equivalent with the family $|\cdot|_n$;

(ii) C is completely continuous;

(iii) $\lim\limits_{|x|_n \to \infty} \frac{|Cx|_n}{|x|_n} = 0,\ \forall n \in \mathbb{N}$.

Then $U + C$ has a fixed point.

In [11], Lungu and Rus established some results relative to existence, uniqueness, integral inequalities and data dependence for solutions of the following functional Volterra-Fredholm integral equation in two variables with deviating argument in a Banach space by Picard operators technique

$$u(x,y) = g(x,y,h(u)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u(s,t))\, dsdt,\ (x,y) \in \mathbb{R}_+^2.$$

In [16], based on the applications of the Banach fixed point theorem coupled with Bielecki type norm and the integral inequality with explicit estimates, B. G. Pachpatte studied some basic properties of solutions of the Fredholm type integral equation in two variables as follows

$$u(x,y) = f(x,y) + \int_0^a \int_0^b g(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t))\, dtds.$$

With the same methods, in [17], the existence, uniqueness and other properties of solutions of certain Volterra integral and integrodifferential equations in two variables were considered.

Applying the Banach fixed point theorem, in [8], El-Borai et al. have proved the existence of a unique solution of a nonlinear integral equation of type Volterra-Hammerstein in $n-$dimensional of the form

$$\mu\phi(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(t,\tau)K(x,y)\gamma(\tau,y,\phi(y,\tau))\, dyd\tau,$$

where $x = (x_1,\dots,x_n),\ y = (y_1,\dots,y_n);\ \mu,\ \lambda$ are constants. After that, in [1], M. A. Abdou et al. investigated the following mixed nonlinear integral equation of the second kind in $n-$dimensional

$$\begin{aligned}\mu\phi(x,t) = \lambda \int_\Omega k(x,y)\gamma(t,y,\phi(y,t))\, dy + \lambda \int_0^t \int_\Omega G(t,\tau)k(x,y)\gamma(\tau,y,\phi(y,\tau))\, dyd\tau \\ +\lambda \int_0^t F(t,\tau)\phi(x,\tau)d\tau + f(x,t),\end{aligned}$$

where $x = (x_1,\dots,x_n),\ y = (y_1,\dots,y_n)$. Also using the Banach fixed point theorem, the existence of a unique solution of this equation was proved.

This paper consists of four sections and the existence of solutions, the existence of asymptotically stable solutions for (1.1) will be presented in sections 2, 3. The main tools are Theorem 1.1, Propositions 2.1, 2.2, and the new integral inequalities, see Lemma 3.1 as below. Finally, we give an illustrated example. The results obtained here may be considered as the generalizations of those in [10], [14].

2. The Existence of Solutions

Let $X = C(\mathbb{R}_+^N;E)$ be the space of all continuous functions on $\mathbb{R}_+^N$ to E which be equipped with the numerable family of seminorms

$$|u|_n = \sup_{x\in[0,n]^N} |u(x)|,\ n \geqslant 1.$$

Then X is complete with the metric

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{|u-v|_n}{1+|u-v|_n}$$

and X is the Fréchet space [13]. Consider in X the other family of seminorms $\|\cdot\|_n$ defined by

$$\|u\|_n = |u|_{\gamma_n} + |u|_{h_n}, \; n \geqslant 1,$$

where

$$\begin{aligned} |u|_{\gamma_n} &= \sup_{x\in[0,n]^N,\, |x|_1 \leqslant \gamma_n} |u(x)|, \\ |u|_{h_n} &= \sup_{x\in[0,n]^N,\, |x|_1 \geqslant \gamma_n} e^{-h_n(|x|_1-\gamma_n)} |u(x)|, \\ |x|_1 &= x_1 + \ldots + x_N, \end{aligned}$$

$\gamma_n \in (0,n)$ and $h_n > 0$ are arbitrary numbers. $\|\cdot\|_n$ and $|\cdot|_n$ are equivalent because

$$e^{-h_n(nN-\gamma_n)} |u|_n \leqslant \|u\|_n \leqslant 2|u|_n, \; \forall u \in X, \; \forall n \geqslant 1.$$

Based on the construct of such $(X, \; |\cdot|_n)$, the following are valid. It is useful to prove existence of solutions for some nonlinear functional integral equations in the space of all continuous functions on $\mathbb{R}_+^N$ with values in a general Banach space. So is to (1.1). For an exhaustive knowledge, we refer the reader to the work [6], wherein many methods in order to solve of nonlinear integral equations in abstract spaces can be found and applications were given.

PROPOSITION 2.1. (Banach, see [3], [4]) *Let $(X, \; |\cdot|_n)$ be a Fréchet space and let $\Phi : X \to X$ be an L_n-contraction on X with respect to a family of seminorms $\|\cdot\|_n$ equivalent with $|\cdot|_n$. Then Φ has a unique fixed point in X.*

The details of the proof can be found in Appendix of [13].

PROPOSITION 2.2. *Let $X = C(\mathbb{R}_+^N; E)$ be the Fréchet space defined as above and A be a subset of X. For each $n \in \mathbb{N}$, let $X_n = C([0,n]^N; E)$ be the Banach space of all continuous functions $u : [0,n]^N \to E$ with the norm*

$$|u|_n = \sup_{x\in[0,n]^N} |u(x)|$$

and $A_n = \{u|_{[0,n]^N} : u \in A\}$.

The set A in X is relatively compact if and only if for each $n \in \mathbb{N}$, A_n is equicontinuous in X_n and for every $x \in [0,n]^N$, the set $A_n(x) = \{u(x) : u \in A_n\}$ is relatively compact in E.

The proof of Proposition 2.2 is similar to that in Appendix of [12], it follows from the Ascoli-Arzela's Theorem, (see [9], p. 211).

We suppose that the following hypotheses are satisfied:

(A_1) There exist a constant $L \in [0,1)$ and a continuous function $\omega_0 : \mathbb{R}_+^N \to \mathbb{R}_+$ such that

$$|V(x;u,v) - V(x;\bar{u},\bar{v})| \leqslant L|u-\bar{u}| + \omega_0(x)|v-\bar{v}|,\ \forall x \in \mathbb{R}_+^N,\ \forall u,v,\bar{u},\bar{v} \in E;$$

(A_2) There exists a continuous function $\omega_1 : \Delta \to \mathbb{R}_+$ such that

$$|V_1(x,y;u) - V_1(x,y;\bar{u})| \leqslant \omega_1(x,y)|u-\bar{u}|,\ \forall (x,y) \in \Delta,\ \forall u,\bar{u} \in E;$$

(A_3) F is completely continuous with the property: for all bounded subsets I_1, I_2 of $\mathbb{R}_+^N$ and for any bounded subset J of E, for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\forall x,\bar{x} \in I_1,\ |x-\bar{x}|_1 < \delta \Longrightarrow |F(x,y;u) - F(\bar{x},y;u)| < \varepsilon,\ \forall (y,u) \in I_2 \times J;$$

(A_4) There exists a continuous function $\omega_2 : \mathbb{R}_+^{2N} \to \mathbb{R}_+$ such that for each bounded subset I of $\mathbb{R}_+^N$, the following conditions hold:

$$\int_{\mathbb{R}_+^N} \sup_{x\in I} \omega_2(x,y)dy < \infty,$$

and

$$|F(x,y;u)| \leqslant \omega_2(x,y), \forall (x,y;u) \in I \times \mathbb{R}_+^N \times E;$$

(A_5) $\sigma_1,\ \sigma_2 \in C(\mathbb{R}_+^N;\mathbb{R}_+^N)$, with $\sigma_1(x) \in B_x$, for all $x \in \mathbb{R}_+^N$ and

$$\lim_{\eta\to 0_+} \int_{y\in[0,n]^N,\ |\sigma_1(y)|_1 \leqslant \eta} dy = 0,\ \forall n \in \mathbb{N}.$$

REMARK 1. – Assumption (A_5) can be valid when $\sigma_1(x) = x$ or $\sigma_1(x) = ax$, $0 < a < 1$. Indeed,

(i) In the case of $\sigma_1(x) = x$. We have

$$\begin{aligned}
&\int_{y\in[0,n]^N,\ |y|_1\leqslant\eta} dy \\
&\quad \leqslant \int_{\mathbb{R}_+^N,\ |y|_1\leqslant\eta} dy = \underset{\mathbb{R}_+^N,\ y_1+\ldots+y_N\leqslant\eta}{\int\ldots\int} dy_1\ldots dy_N \\
&\quad = \int_0^\eta dy_1 \int_0^{\eta-y_1} dy_2 \ldots \int_0^{\eta-y_1-y_2-\ldots-y_{N-1}} dy_N \\
&\quad = \int_0^\eta dy_1 \int_0^{\eta-y_1} dy_2 \ldots \int_0^{\eta-y_1-y_2-\ldots-y_{N-2}} (\eta-y_1-y_2-\ldots-y_{N-1})\, dy_{N-1} \\
&\quad = \int_0^\eta dy_1 \int_0^{\eta-y_1} dy_2 \ldots \int_0^{\eta-y_1-y_2-\ldots-y_{N-3}} \frac{(\eta-y_1-y_2-\ldots-y_{N-2})^2}{2} dy_{N-2} \\
&\quad = \int_0^\eta dy_1 \int_0^{\eta-y_1} dy_2 \ldots \int_0^{\eta-y_1-y_2-\ldots-y_{N-4}} \frac{(\eta-y_1-y_2-\ldots-y_{N-3})^3}{3!} dy_{N-3} \\
&\quad = \int_0^\eta dy_1 \int_0^{\eta-y_1} dy_2 \ldots \int_0^{\eta-y_1-y_2-\ldots-y_{N-5}} \frac{(\eta-y_1-y_2-\ldots-y_{N-4})^4}{4!} dy_{N-4}
\end{aligned}$$

$$\vdots$$

$$= \int_0^{\eta} dy_1 \int_0^{\eta - y_1} dy_2 \ldots \int_0^{\eta - y_1 - y_2 - \ldots - y_{N-k-1}} \frac{(\eta - y_1 - y_2 - \ldots - y_{N-k})^k}{k!} dy_{N-k}$$

$$\vdots$$

$$= \int_0^{\eta} dy_1 \int_0^{\eta - y_1} \frac{(\eta - y_1 - y_2)^{N-2}}{(N-2)!} dy_2$$

$$= \int_0^{\eta} \frac{(\eta - y_1)^{N-1}}{(N-1)!} dy_1 = \frac{\eta^N}{N!} \to 0, \text{ as } \eta \to 0_+.$$

(ii) In the case of $\sigma_1(x) = ax,\ 0 < a < 1,$ it is clear that

$$\int_{y \in [0,n]^N,\ |ay|_1 \leqslant \eta} dy = \int_{y \in [0,n]^N,\ |y|_1 \leqslant \frac{\eta}{a}} dy \leqslant \frac{\left(\frac{\eta}{a}\right)^N}{N!} = \frac{\eta^N}{N! a^N} \to 0 \text{ as } \eta \to 0_+.$$

THEOREM 2. *Let* $(A_1) - (A_5)$ *hold. Then the equation (1.1) has a solution on* $\mathbb{R}_+^N$.

Proof. First, we consider the equation

$$u(x) = V\left(x, u(x), \int_{B_x} V_1(x, y, u(\sigma_1(y)))\, dy\right),\ x \in \mathbb{R}_+^N, \tag{2.1}$$

and show that it has a unique solution $u = \xi$. Writing (2.1) in the form

$$u(x) = \Phi u(x),\ x \in \mathbb{R}_+^N, \tag{2.2}$$

where

$$\Phi u(x) = V\left(x, u(x), \int_{B_x} V_1(x, y, u(\sigma_1(y)))\, dy\right),\ (x, u) \in \mathbb{R}_+^N \times X. \tag{2.3}$$

For all $u,\ v \in X$, assumptions $(A_1),\ (A_2)$ lead to

$$|\Phi u(x) - \Phi v(x)| \leqslant L|u(x) - v(x)| + \omega_0(x) \int_{B_x} \omega_1(x, y) |u(\sigma_1(y)) - v(\sigma_1(y))|\, dy.$$

Let $n \in \mathbb{N}$ be fixed. For all $x \in [0,n]^N,\ |x|_1 \leqslant \gamma_n$, with $\gamma_n \in (0, n)$ chosen later. By $\sigma_1(x) \in B_x,\ \forall x \in [0,n]^N$, we can estimate as follows

$$|\Phi u(x) - \Phi v(x)| \leqslant L|u - v|_{\gamma_n} + \bar{\omega}_n \left(\frac{|x|_1}{N}\right)^N |u - v|_{\gamma_n} \leqslant \left(L + \bar{\omega}_n \frac{\gamma_n^N}{N^N}\right) |u - v|_{\gamma_n},$$

in which

$$\begin{cases} \bar{\omega}_n = \tilde{\omega}_{0n} \tilde{\omega}_{1n},\ \ \tilde{\omega}_{0n} = \sup\{\omega_0(x) : x \in [0,n]^N\}, \\ \tilde{\omega}_{1n} = \sup\{\omega_1(x, y) : (x, y) \in \Delta_n\}, \\ \Delta_n = \{(x, y) \in [0,n]^{2N} : y \in B_x\}. \end{cases}$$

So

$$|\Phi u-\Phi v|_{\gamma_n} \leqslant \left(L+\bar{\omega}_n \frac{\gamma_n^N}{N^N}\right)|u-v|_{\gamma_n}.$$

Furthermore, for all $x \in [0,n]^N,\ |x|_1 \geqslant \gamma_n,$ we have

$$\begin{aligned}|\Phi u(x)-\Phi v(x)| &\leqslant L|u(x)-v(x)|+\bar{\omega}_n\int_{B_x}|u(\sigma_1(y))-v(\sigma_1(y))|\,dy\\ &= L|u(x)-v(x)|+\int_{B_x,\ |\sigma_1(y)|_1\geqslant\gamma_n}|u(\sigma_1(y))-v(\sigma_1(y))|\,dy\\ &\quad+\int_{B_x,\ |\sigma_1(y)|_1\leqslant\gamma_n}|u(\sigma_1(y))-v(\sigma_1(y))|\,dy.\end{aligned}$$

By the inequalities

$$\begin{cases}0<e^{-h_n(|x|_1-\gamma_n)}\leqslant 1,\ \forall x\in[0,n]^N,\ |x|_1\geqslant\gamma_n,\\ |\sigma_1(x)|_1\leqslant|x|_1,\ \forall x\in\mathbb{R}_+^N,\\ |\sigma_1(y)|_1\leqslant|y|_1\leqslant|x|_1,\ \forall y\in B_x,\end{cases}$$

with $h_n>0$ is also chosen later, we get

$$\begin{aligned}&|\Phi u(x)-\Phi v(x)|\,e^{-h_n(|x|_1-\gamma_n)}\\ &\leqslant Le^{-h_n(|x|_1-\gamma_n)}|u(x)-v(x)|\\ &\quad+\bar{\omega}_n e^{-h_n(|x|_1-\gamma_n)}[\int_{B_x,\ |\sigma_1(y)|_1\geqslant\gamma_n}|u(\sigma_1(y))-v(\sigma_1(y))|\,dy\\ &\quad+\int_{B_x,\ |\sigma_1(y)|_1\leqslant\gamma_n}|u(\sigma_1(y))-v(\sigma_1(y))|\,dy]\\ &\leqslant L|u-v|_{h_n}\\ &\quad+\bar{\omega}_n\int_{B_x,\ |\sigma_1(y)|_1\geqslant\gamma_n}e^{h_n(|\sigma_1(y)|_1-|x|_1)}e^{-h_n(|\sigma_1(y)|_1-\gamma_n)}|u(\sigma_1(y))-v(\sigma_1(y))|\,dy\\ &\quad+\bar{\omega}_n e^{-h_n(|x|_1-\gamma_n)}\int_{B_x,\ |\sigma_1(y)|_1\leqslant\gamma_n}|u(\sigma_1(y))-v(\sigma_1(y))|\,dy\\ &\leqslant L|u-v|_{h_n}+\bar{\omega}_n|u-v|_{h_n}\int_{B_x,\ |\sigma_1(y)|_1\geqslant\gamma_n}e^{h_n(|\sigma_1(y)|_1-|x|_1)}dy\\ &\quad+\bar{\omega}_n e^{-h_n(|x|_1-\gamma_n)}|u-v|_{\gamma_n}\int_{B_x,\ |\sigma_1(y)|_1\leqslant\gamma_n}dy\\ &\equiv L|u-v|_{h_n}+\bar{\omega}_n|u-v|_{h_n}J_1+\bar{\omega}_n e^{-h_n(|x|_1-\gamma_n)}|u-v|_{\gamma_n}J_2.\end{aligned}$$

On the other hand

$$\begin{aligned}J_1 &= \int_{B_x,\ |\sigma_1(y)|_1\geqslant\gamma_n}e^{h_n(|\sigma_1(y)|_1-|x|_1)}dy\\ &\leqslant \int_{B_x,\ |\sigma_1(y)|_1\geqslant\gamma_n}e^{h_n(|y|_1-|x|_1)}dy\leqslant\int_{B_x}e^{h_n(|y|_1-|x|_1)}dy\end{aligned}$$

$$\begin{aligned}
&= e^{-h_n|x|_1}\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_N} e^{h_n(y_1+y_2+\ldots+y_N)}dy_1dy_2\ldots dy_N\\
&= e^{-h_n|x|_1}\int_0^{x_1} e^{h_ny_1}dy_1\int_0^{x_2} e^{h_ny_2}dy_2\ldots\int_0^{x_N} e^{h_ny_N}dy_N\\
&= \frac{1}{h_n^N}e^{-h_n|x|_1}\left(e^{h_nx_1}-1\right)\left(e^{h_nx_2}-1\right)\ldots\left(e^{h_nx_N}-1\right)\\
&= \frac{1}{h_n^N}\left(1-e^{-h_nx_1}\right)\left(1-e^{-h_nx_2}\right)\ldots\left(1-e^{-h_nx_N}\right)\leqslant\frac{1}{h_n^N}.
\end{aligned}$$

By (A_5), for all $x\in[0,n]^N$, the following property is valid

$$J_2=\int_{B_x,\ |\sigma_1(y)|_1\leqslant\gamma_n} dy\leqslant\int_{y\in[0,n]^N,\ |\sigma_1(y)|_1\leqslant\gamma_n} dy=\varphi_n(\gamma_n)\to 0, \text{ as } \gamma_n\to 0_+.$$

Hence

$$\begin{aligned}
&|\Phi u(x)-\Phi v(x)|\,e^{-h_n(|x|_1-\gamma_n)}\\
&\quad\leqslant L\,|u-v|_{h_n}+\bar{\omega}_n\,|u-v|_{h_n}\int_{B_x,\ |\sigma_1(y)|_1\geqslant\gamma_n} e^{h_n(|\sigma_1(y)|_1-|x|_1)}dy\\
&\qquad\quad+\bar{\omega}_ne^{-h_n(|x|_1-\gamma_n)}\,|u-v|_{\gamma_n}\int_{B_x,\ |\sigma_1(y)|_1\leqslant\gamma_n} dy\\
&\quad\equiv L\,|u-v|_{h_n}+\bar{\omega}_n\,|u-v|_{h_n}J_1+\bar{\omega}_ne^{-h_n(|x|_1-\gamma_n)}\,|u-v|_{\gamma_n}J_2\\
&\quad\leqslant L\,|u-v|_{h_n}+\bar{\omega}_n\,|u-v|_{h_n}\frac{1}{h_n^N}+\bar{\omega}_n\,|u-v|_{\gamma_n}\varphi_n(\gamma_n)\\
&\quad=\left(L+\bar{\omega}_n\frac{1}{h_n^N}\right)|u-v|_{h_n}+\bar{\omega}_n\varphi_n(\gamma_n)\,|u-v|_{\gamma_n}.
\end{aligned}$$

So

$$|\Phi u-\Phi v|_{h_n}\leqslant\left(L+\bar{\omega}_n\frac{1}{h_n^N}\right)|u-v|_{h_n}+\bar{\omega}_n\varphi_n(\gamma_n)\,|u-v|_{\gamma_n}.$$

Consequently,

$$\begin{aligned}
\|\Phi u-\Phi v\|_n&=|\Phi u-\Phi v|_{\gamma_n}+|\Phi u-\Phi v|_{h_n}\\
&\leqslant\left(L+\bar{\omega}_n\frac{\gamma_n^N}{N^N}\right)|u-v|_{\gamma_n}+\left(L+\bar{\omega}_n\frac{1}{h_n^N}\right)|u-v|_{h_n}+\bar{\omega}_n\varphi_n(\gamma_n)\,|u-v|_{\gamma_n}\\
&=\left(L+\bar{\omega}_n\frac{\gamma_n^N}{N^N}+\bar{\omega}_n\varphi_n(\gamma_n)\right)|u-v|_{\gamma_n}+\left(L+\bar{\omega}_n\frac{1}{h_n^N}\right)|u-v|_{h_n}\\
&\leqslant L_n\,\|u-v\|_n,
\end{aligned}\tag{2.4}$$

where

$$L_n=\max\left\{L+\bar{\omega}_n\frac{\gamma_n^N}{N^N}+\bar{\omega}_n\varphi_n(\gamma_n),\ L+\bar{\omega}_n\frac{1}{h_n^N}\right\}.$$

Choosing h_n, γ_n such that

$$h_n>\sqrt[N]{\frac{1}{1-L}\bar{\omega}_n}\quad\text{and}\quad 0<\gamma_n<n,\ 0<\frac{\gamma_n^N}{N^N}+\varphi_n(\gamma_n)<\frac{1-L}{\bar{\omega}_n},$$

then $L_n < 1$, so Φ is a L_n- contraction operator on $(X, \|\cdot\|_n)$. Apply Proposition 2.1, Φ has a unique fixed point $u = \xi$. Hence, $u = \xi$ is a unique solution of (2.1). With the transformation $u = v + \xi$, we can write the equation (1.1) in the form

$$v(x) = Uv(x) + Cv(x),\ x \in \mathbb{R}_+^N, \tag{2.5}$$

where

$$\begin{cases} Uv(x) = -\xi(x) + V\left(x, v(x) + \xi(x), \int_{B_x} V_1(x,y,v(\sigma_1(y)) + \xi(\sigma_1(y)))\,dy\right), \\ Cv(x) = \int_{\mathbb{R}_+^N} F(x,y,v(\sigma_2(y)) + \xi(\sigma_2(y))\,)\,dy,\ x \in \mathbb{R}_+^N. \end{cases} \tag{2.6}$$

The operator U is a L_n-contraction, with respect to a family of seminorms $\|\cdot\|_n$. Indeed, fixed an arbitrary positive integer $n \in \mathbb{N}$. We have

$$\begin{aligned} Uv(x) - U\tilde{v}(x) = {} & V\left(x, v(x) + \xi(x), \int_{B_x} V_1(x,y,v(\sigma_1(y)) + \xi(\sigma_1(y)))\,dy\right) \\ & - V\left(x, \tilde{v}(x) + \xi(x), \int_{B_x} V_1(x,y,\tilde{v}(\sigma_1(y)) + \xi(\sigma_1(y)))\,dy\right). \end{aligned}$$

For all $x \in [0,n]^N$, $|x|_1 \leqslant \gamma_n$,

$$\begin{aligned} |Uv(x) - U\tilde{v}(x)| & \leqslant L|v(x) - \tilde{v}(x)| + \omega_0(x)\int_{B_x} \omega_1(x,y)\,|v(\sigma_1(y)) - \tilde{v}(\sigma_1(y))|\,dy \\ & \leqslant \left(L + \bar{\omega}_n \frac{\gamma_n^N}{N^N}\right)|v - \tilde{v}|_{\gamma_n}, \end{aligned}$$

leads to

$$|Uv - U\tilde{v}|_{\gamma_n} \leqslant \left(L + \bar{\omega}_n \frac{\gamma_n^N}{N^N}\right)|v - \tilde{v}|_{\gamma_n}.$$

For all $x \in [0,n]^N$, $|x|_1 \geqslant \gamma_n$,

$$|Uv(x) - U\tilde{v}(x)|\,e^{-h_n(|x|_1 - \gamma_n)} \leqslant \bar{\omega}_n \varphi_n(\gamma_n)\,|v - \tilde{v}|_{\gamma_n} + \left(L + \bar{\omega}_n \frac{1}{h_n^N}\right)|v - \tilde{v}|_{h_n},$$

so

$$|Uv - U\tilde{v}|_{h_n} \leqslant \bar{\omega}_n \varphi_n(\gamma_n)\,|v - \tilde{v}|_{\gamma_n} + \left(L + \bar{\omega}_n \frac{1}{h_n^N}\right)|v - \tilde{v}|_{h_n}.$$

This implies that

$$\|Uv - U\tilde{v}\|_n \leqslant L_n \|v - \tilde{v}\|_n.$$

and then U is a L_n-contraction operator with respect to $\|\cdot\|_n$.

The operator $C : X \to X$ is completely continuous. It can be proved by using $(A_3) - (A_5)$, via the dominated convergence theorem and Proposition 2.2. The details are as follows.

(i) For any $v_0 \in X$, let $\{v_m\}$ be a sequence in X such that $\lim\limits_{m\to\infty} v_m = v_0$.

Let $n \in \mathbb{N}$ be fixed. For any given $\varepsilon > 0$, because

$$\int_{\mathbb{R}_+^N} \sup_{x \in [0,n]^N} \omega_2(x,y)dy < \infty,$$

there exists $T_n \in \mathbb{N}$ such that

$$\int_{\mathbb{R}_+^N \setminus S_n} \omega_2(x,y)dy \leqslant \int_{\mathbb{R}_+^N \setminus S_n} \sup_{x \in [0,n]^N} \omega_2(x,y)dy < \frac{\varepsilon}{4}, \ \forall x \in [0,n]^N. \tag{2.7}$$

where $S_n = \{y \in \mathbb{R}_+^N : y_1^2 + y_2^2 + \ldots + y_N^2 \leqslant T_n^2\}$.

Put $K = \{v_m(\sigma_2(y)) + \xi(\sigma_2(y)) : y \in S_n, \ m \in \mathbb{Z}_+\}$, then K is compact in E. Indeed, let $\left\{(v_{m_j} + \xi)(\sigma_2(y_j))\right\}_j$ be a sequence in K. We can assume that $\lim_{j \to \infty} y_j = y_0$ and that $\lim_{j \to \infty} v_{m_j} + \xi = v_0 + \xi$. We have

$$\begin{aligned}
&\left|(v_{m_j} + \xi)(\sigma_2(y_j)) - (v_0 + \xi)(\sigma_2(y_0))\right| \\
&\quad \leqslant \left|(v_{m_j} + \xi)(\sigma_2(y_j)) - (v_0 + \xi)(\sigma_2(y_j))\right| \\
&\qquad + \left|(v_0 + \xi)(\sigma_2(y_j)) - (v_0 + \xi)(\sigma_2(y_0))\right| \\
&\quad \leqslant \left|v_{m_j} - v_0\right|_{T_n} + \left|(v_0 + \xi)(\sigma_2(y_j)) - (v_0 + \xi)(\sigma_2(y_0))\right|,
\end{aligned}$$

which shows that $\lim_{j \to \infty} (v_{m_j} + \xi)(\sigma_2(y_j)) = (v_0 + \xi)(\sigma_2(y_0))$ in E. It means that K is compact in E.

For $\varepsilon > 0$ be given as above, by F is continuous on the compact set $[0,n]^N \times S_n \times K$, there exists $\delta > 0$ such that for every $u, \ \bar{u} \in K, \ |u - \bar{u}| < \delta$,

$$|F(x,y;u) - F(x,y;\bar{u})| < \frac{\varepsilon}{2\text{mes}(S_n)}, \ \forall (x,y) \in [0,n]^N \times S_n.$$

By $\lim_{m \to \infty} \sup_{y \in S_n} |(v_m + \xi)(\sigma_2(y)) - (v_0 + \xi)(\sigma_2(y))| = 0$, there exists m_0 such that for $m > m_0$,

$$|(v_m + \xi)(\sigma_2(y)) - (v_0 + \xi)(\sigma_2(y))| < \delta, \ \forall y \in S_n.$$

Hence, for all $x \in [0,n]^N$, for all $m > m_0$, we obtain

$$\begin{aligned}
|Cv_m(x) - Cv_0(x)| &\leqslant \int_{S_n} |F(x,y;(v_m + \xi)(\sigma_2(y))) - F(x,y;(v_0 + \xi)(\sigma_2(y)))|dy \\
&\quad + 2\int_{\mathbb{R}_+^N \setminus S_n} \omega_2(x,y)dy \\
&< \text{mes}(S_n) \times \frac{\varepsilon}{2\text{mes}(S_n)} + 2\frac{\varepsilon}{4} = \varepsilon,
\end{aligned}$$

so $|Cv_m - Cv_0|_n < \varepsilon$, for all $m > m_0$, it means that C is continuous.

(ii) Let Ω be a bounded subset of X. We have to prove that for $n \in \mathbb{N}$,

(a) The set $(C\Omega)_n$ is equicontinuous in X_n.

(b) For every $x \in [0,n]^N$, the set $(C\Omega)_n(x) = \{Cv|_{[0,n]^N}(x) : v \in \Omega\}$ is relatively compact in E. Let $n \in \mathbb{N}$ be fixed. Consider any $\varepsilon > 0$ given. Then, (2.7) holds with $T_n \in \mathbb{N}$ as above.

Proof of (a) : For any $v \in \Omega$, for all $x,\ \bar{x} \in [0,n]^N$,

$$\begin{aligned}|Cv(x) - Cv(\bar{x})| \leqslant \int_{S_n} \Big| F\big(x,y;(v+\xi)(\sigma_2(y))\big) \\ - F\big(\bar{x},y;(v+\xi)(\sigma_2(y))\big)\Big| dy \\ + \int_{\mathbb{R}_+^N \setminus S_n} (\omega_2(x,y) + \omega_2(\bar{x},y))\, dy. \quad (2.8)\end{aligned}$$

According to (2.7), (2.8) and (A_4), $(C\Omega)_n$ is equicontinuous on X_n.

Proof of (b) : Let $\{Cv_k|_{[0,n]^N}(x)\}_k$, $v_k \in \Omega$, be a sequence in $(C\Omega)_n(x)$. We need show that there exists a convergent subsequence of $\{Cv_k|_{[0,n]^N}(x)\}_k$.

Put

$$S = \{(v+\xi)(\sigma_2(y)) : v \in \Omega,\ y \in S_n\}.$$

Then S is bounded in E and consequently the set $F([0,n]^N \times S_n \times S)$ is relatively compact in E, since F is completely continuous.

The sequence $\{F(x,y;(v_k+\xi)(\sigma_2(y)))\}_k$ belongs to $F([0,n]^N \times S_n \times S)$, so there exists a subsequence $\left\{F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right)\right\}_j$ and $\Psi(x,y) \in E$, such that

$$\left|F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right) - \Psi(x,y)\right| \to 0 \text{ as } j \to \infty. \quad (2.9)$$

Morever, by (A_5), we get

$$\left|F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right)\right| \leqslant \omega_2(x,y),\ \forall (x,y) \in [0,n]^N \times S_n.$$

Thus

$$\begin{aligned}\Big|F(x,y;(v_{k_j}+\xi)(\sigma_2(y))) - \Psi(x,y)\Big| \\ \leqslant \Big|F(x,y;(v_{k_j}+\xi)(\sigma_2(y)))\Big| + |\Psi(x,y)| \\ \leqslant\ 2\omega_2(x,y),\ \ \forall (x,y) \in [0,n]^N \times S_n, \quad (2.10)\end{aligned}$$

$$\omega_2(x,\cdot) \in L^1(S_n).$$

Using the dominated convergence theorem, (2.9) and (2.10) lead to

$$\int_{S_n} \left|F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right) - \Psi(x,y)\right| dy \to 0, \text{ as } j \to \infty.$$

It means that, for given $\varepsilon > 0$, there exists j_0 such that for $j > j_0$,

$$\int_{S_n} \left|F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right) - \Psi(x,y)\right| dy < \frac{\varepsilon}{2}.$$

Consequently, for $j > j_0$,

$$\begin{aligned}\left|Cv_{k_j}(x) - \int_{S_n} \Psi(x,y)dy\right| &= \left|\int_{\mathbb{R}_+^N} F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right)dy - \int_{S_n} \Psi(x,y)dy\right| \\ &\leqslant \left|\int_{S_n} F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right)dy - \int_{S_n} \Psi(x,y)dy\right| \\ &\quad + \left|\int_{\mathbb{R}_+^N \setminus S_n} F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right)dy\right| \\ &\leqslant \int_{S_n} \left|F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right) - \Psi(x,y)\right|dy \\ &\quad + \int_{\mathbb{R}_+^N \setminus S_n} \left|F\left(x,y;(v_{k_j}+\xi)(\sigma_2(y))\right)\right|dy \\ &\leqslant \frac{\varepsilon}{2} + \int_{\mathbb{R}_+^N \setminus S_n} \omega_2(x,y)dy < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

Note that $\left\{Cv_{k_j}(x)\right\}_j$ is a subsequence of $\{Cv_k(x)\}_k$. Then, $(C\Omega)_n(x)$ is relatively compact in E.

Applying Proposition 2.2 , $C(\Omega)$ is relatively compact in X. Therefore, C is completely continuous.

Finally, we show that $\forall n \in \mathbb{N}$,

$$\lim_{|v|_n \to \infty} \frac{|Cv|_n}{|v|_n} = 0. \tag{2.11}$$

By (A_4), for all $x \in [0,n]^N$, we get

$$|Cv(x)| \leqslant \int_{\mathbb{R}_+^N} |F(x,y;(v+\xi)(\sigma_2(y)))|dy \leqslant \int_{\mathbb{R}_+^N} \sup_{x\in[0,n]^N} \omega_2(x,y)dy < \infty.$$

It follows that (2.11) holds.

Applying Theorem 1, the operator $U+C$ has a fixed point v in X. So, Equation (1.1) has a solution $u = v+\xi$ on $\mathbb{R}_+^N$. Theorem 2 is proved.

3. The Existence of Asymptotically Stable Solutions

Based on the notion of asymptotically stable solutions to the functional equation mentioned in [2] with citations and notes, we use the following definition and also note that it is stated on spaces of functions defined on $\mathbb{R}_+^N$ not necessarily bounded.

DEFINITION. A function $\tilde{u}$ is said to be an *asymptotically stable solution* of (1.1) if for any solution u of (1.1),

$$\lim_{|x|_1 \to +\infty} |u(x) - \tilde{u}(x)| = 0. \tag{3.1}$$

In this section, we assume $(A_1)-(A_5)$ hold. Then, by Theorem 2, (1.1) has a solution on $\mathbb{R}_+^N$. On the other hand, if u is a solution of (1.1) then $v=u-\xi$ satisfies (2.5). This implies that for all $x\in\mathbb{R}_+^N$,

$$|v(x)|\leqslant|Uv(x)|+|Cv(x)|, \tag{3.2}$$

where $Uv(x),\ Cv(x)$ as in (2.6). Using $(A_1)-(A_5)$ and note that

$$\xi(x)=V\left(x,\xi(x),\int_{B_x}V_1(x,y,\xi(\sigma_1(y)))\,dy\right), \tag{3.3}$$

we obtain for all $x\in\mathbb{R}_+^N$,

$$|v(x)|\leqslant L|v(x)|+\omega_0(x)\int_{B_x}\omega_1(x,y)|v(\sigma_1(y))|dy+\int_{\mathbb{R}_+^N}\omega_2(x,y)dy. \tag{3.4}$$

It implies that

$$|v(x)|\leqslant\int_{B_x}r(x,y)|v(\sigma_1(y))|dy+a(x), \tag{3.5}$$

with

$$a(x)=\frac{1}{1-L}\int_{\mathbb{R}_+^N}\omega_2(x,y)dy,\ r(x,y)=\frac{1}{1-L}\omega_0(x)\omega_1(x,y). \tag{3.6}$$

We need prove the following auxiliary result.

LEMMA 1. *Let $w,\ a\in C(\mathbb{R}_+^N;\mathbb{R}_+)$, and $r\in C(\Delta;\mathbb{R}_+)$, $r(x,y)\leqslant r(x,0)\leqslant r(0,0)$, $\forall y\in B_x,\ \forall x\in\mathbb{R}_+^N$, and $\sigma_1\in C(\mathbb{R}_+^N;\mathbb{R}_+^N)$, $\sigma_1(x)\in B_x,\ \forall x\in\mathbb{R}_+^N$. If*

$$w(x)\leqslant\int_{B_x}r(x,y)w(\sigma_1(y))dy+a(x), \tag{3.7}$$

for all $x\in\mathbb{R}_+^N$, then

$$\text{(i)}\ \ w(x)\leqslant\bar{a}(x)+\bar{r}(x)\sum_{k=0}^{\infty}\frac{\left(\bar{r}(0)x_1\ldots x_N\right)^k}{(k!)^N}\int_{B_x}\bar{a}(y)dy, \tag{3.8}$$

$$\text{(ii)}\ \ w(x)\leqslant\bar{a}(x)+\bar{r}(x)\exp\left(\bar{r}(0)x_1\ldots x_N\right)\int_{B_x}\bar{a}(y)dy,$$

for all $x\in\mathbb{R}_+^N$, where $\bar{a}(x),\ \bar{r}(x)$ are defined by

$$\bar{a}(x)=a(x)+a(\sigma_1(x)),\ \bar{r}(x)=r(x,0)+r(\sigma_1(x),0). \tag{3.9}$$

Proof.
By $B_{\sigma_1(x)}\subset B_x,\ r(x,y)\leqslant r(x,0),\ \forall y\in B_x,\ \forall x\in\mathbb{R}_+^N$,

$$w(x)\leqslant a(x)+r(x,0)\int_{B_x}w(\sigma_1(y))dy, \tag{3.10}$$

$$\begin{aligned} w(\sigma_1(x)) &\leqslant a(\sigma_1(x)) + r(\sigma_1(x),0)\int_{B_{\sigma_1(x)}} w(\sigma_1(y))dy \\ &\leqslant a(\sigma_1(x)) + r(\sigma_1(x),0)\int_{B_x} w(\sigma_1(y))dy, \end{aligned} \tag{3.11}$$

then

$$\bar{w}(x) \leqslant \bar{a}(x) + \bar{r}(x)\int_{B_x} w(\sigma_1(y))dy \leqslant \bar{a}(x) + \bar{r}(x)\int_{B_x} \bar{w}(y)dy, \tag{3.12}$$

where

$$\bar{w}(x) = w(x) + w(\sigma_1(x)). \tag{3.13}$$

Put

$$A\bar{w}(x) = \bar{r}(x)\int_{B_x} \bar{w}(y)dy,\ \forall \bar{w} \in C(\mathbb{R}_+^N;\mathbb{R}_+). \tag{3.14}$$

It follows from (3.12) and (3.14) that

$$\begin{aligned} \bar{w}(x) &\leqslant \bar{a}(x) + A\bar{w}(x) \leqslant \bar{a}(x) + A\left(\bar{a} + A\bar{w}\right)(x) \\ &= \bar{a}(x) + A\bar{a}(x) + A^2\bar{w}(x) \leqslant \ldots \leqslant \bar{a}(x) + \sum_{k=0}^{n-1} A^{k+1}\bar{a}(x) + A^{n+1}\bar{w}(x). \end{aligned} \tag{3.15}$$

By induction, we obtain

$$A^{k+1}\bar{a}(x) \leqslant \bar{r}(x)\frac{\left(\bar{r}(0)x_1\ldots x_N\right)^k}{(k!)^N}\int_{B_x} \bar{a}(y)dy. \tag{3.16}$$

So

$$\begin{aligned} w(x) &\leqslant \bar{w}(x) \leqslant \bar{a}(x) + \sum_{k=0}^{n-1} A^{k+1}\bar{a}(x) + A^{n+1}\bar{w}(x) \\ &\leqslant \bar{a}(x) + \bar{r}(x)\sum_{k=0}^{n-1}\frac{\left(\bar{r}(0)x_1\ldots x_N\right)^k}{(k!)^N}\int_{B_x}\bar{a}(y)dy + \frac{\left(\bar{r}(0)x_1\ldots x_N\right)^n}{(n!)^N}\bar{r}(x)\int_{B_x}\bar{w}(y)dy. \end{aligned} \tag{3.17}$$

For $X_0 > 0$ is given, we have

$$\left|\frac{\left(\bar{r}(0)x_1\ldots x_N\right)^k}{(k!)^N}\right| \leqslant \frac{\left(\bar{r}(0)X_0^N\right)^k}{(k!)^N},\ \forall x \in [0,X_0]^N,\ \forall k \in \mathbb{N}. \tag{3.18}$$

The positive series $\sum_{k=0}^{\infty}\frac{\left(\bar{r}(0)X_0^N\right)^k}{(k!)^N}$ converges and then $\sum_{k=0}^{\infty}\frac{\left(\bar{r}(0)x_1\ldots x_N\right)^k}{(k!)^N}$ converges uniformly on $[0,X_0]^N$. By the continuity of the function $x \longmapsto \frac{\left(\bar{r}(0)x_1\ldots x_N\right)^k}{(k!)^N}$ on $[0,X_0]^N$, the sum of the series $\sum_{k=0}^{\infty}\frac{\left(\bar{r}(0)x_1\ldots x_N\right)^k}{(k!)^N}$ is continuous on $[0,X_0]^N$. On the other hand, $X_0 > 0$ is arbitrary, so the sum of this series is continuous on $\mathbb{R}_+^N$. Note that

$$\frac{\left(\bar{r}(0)x_1\ldots x_N\right)^n}{(n!)^N} \to 0 \text{ as } n \to \infty \text{ for all } x \in \mathbb{R}_+^N,$$

it implies from (3.17) that

$$w(x) \leqslant \bar{w}(x) \leqslant \bar{a}(x) + \bar{r}(x) \sum_{k=0}^{\infty} \frac{(\bar{r}(0)x_1 \dots x_N)^k}{(k!)^N} \int_{B_x} \bar{a}(y)dy, \text{ for all } x \in \mathbb{R}_+^N. \tag{3.19}$$

The inequality (3.8)(i) is proved. Then the inequality (3.8)(ii) follows by

$$0 \leqslant \frac{(\bar{r}(0)x_1 \dots x_N)^k}{(k!)^N} \leqslant \frac{(\bar{r}(0)x_1 \dots x_N)^k}{k!}, \ \forall x \in \mathbb{R}_+^N. \tag{3.20}$$

Consequently

$$\sum_{k=0}^{\infty} \frac{(\bar{r}(0)x_1 \dots x_N)^k}{(k!)^N} \leqslant \sum_{k=0}^{\infty} \frac{(\bar{r}(0)x_1 \dots x_N)^k}{k!} = \exp\left(\bar{r}(0)x_1 \dots x_N\right), \ \forall x \in \mathbb{R}_+^N. \tag{3.21}$$

Thus

$$w(x) \leqslant \bar{w}(x) \leqslant \bar{a}(x) + \bar{r}(x) \exp\left(\bar{r}(0)x_1 \dots x_N\right) \int_{B_x} \bar{a}(y)dy, \text{ for all } x \in \mathbb{R}_+^N. \tag{3.22}$$

Using (3.8) (ii) with $w(x) = |v(x)|$, we obtain

$$|v(x)| \leqslant \bar{a}(x) + \bar{r}(x) \exp\left(\bar{r}(0)x_1 \dots x_N\right) \int_{B_x} \bar{a}(y)dy, \tag{3.23}$$

for all $x \in \mathbb{R}_+^N$, where $\bar{r}(x)$, $\bar{a}(x)$ are defined by

$$\begin{aligned} \bar{a}(x) &= a(x) + a(\sigma_1(x)), \\ \bar{r}(x) &= r(x,0) + r(\sigma_1(x),0), \\ a(x) &= \frac{1}{1-L} \int_{\mathbb{R}_+^N} \omega_2(x,y)dy, \\ r(x,y) &= \frac{1}{1-L} \omega_0(x)\omega_1(x,y), \end{aligned} \tag{3.24}$$

Then, we obtain the main theorem in this section.

THEOREM 3. *Let $(A_1)-(A_5)$ hold. If*

$$\lim_{|x|_1 \to +\infty} \left[\bar{a}(x) + \bar{r}(x) \exp\left(\bar{r}(0)x_1 \dots x_N\right) \int_{B_x} \bar{a}(y)dy \right] = 0, \tag{3.25}$$

where $\bar{r}(x)$, $\bar{a}(x)$ are defined as in (3.24), then every solution u to (1.1) is an asymptotically stable solution.

Furthermore,

$$\lim_{|x|_1 \to +\infty} |u(x) - \xi(x)| = 0. \tag{3.26}$$

Remark 2. We present an example in which $\omega_0,\ \omega_1,\ \omega_2,\ \sigma_1$ satisfying the assumption (3.25) given as above. Let

$$\begin{cases} \omega_1(x,y) = \frac{\sqrt{(1-L)\alpha_1}}{\sqrt{1+\beta_1 \exp\left(\gamma_1 |x|_1^N\right)+\beta_2 |y|_1^{\lambda_1}}}, \\ \omega_0(x) = \omega_1(x,0) = \frac{\sqrt{(1-L)\alpha_1}}{\sqrt{1+\beta_1 \exp\left(\gamma_1 |x|_1^N\right)}}, \\ \omega_2(x,y) = \frac{\exp(-\gamma_2 |x|_1)}{1+|y|_2^{\lambda_2}}, \\ |y|_2 = \sqrt{y_1^2+\ldots+y_N^2},\ \sigma_1(x) = \theta_1 x, \end{cases}$$

where $\alpha_1,\ \beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2,\ \lambda_1,\ \lambda_2,\ \theta_1$ are positive constants with $\lambda_1 > N,\ \lambda_2 > N,$ $0 < \theta_1 \leqslant 1,\ \ \gamma_1 > \frac{2\alpha_1}{N^N(1+\beta_1)\theta_1}$.

(i) Calculating the functions $r(x,y),\ a(x)$:

$$\begin{cases} r(x,y) = \frac{1}{1-L}\omega_0(x)\omega_1(x,y) = \frac{\alpha_1}{\sqrt{1+\beta_1 \exp\left(\gamma_1 |x|_1^N\right)+\beta_2 |y|_1^{N+1}}\sqrt{1+\beta_1 \exp\left(\gamma_1 |x|_1^N\right)}}; \\ r(x,y) \leqslant r(x,0) = \frac{\alpha_1}{1+\beta_1 \exp\left(\gamma_1 |x|_1^N\right)} \leqslant r(0,0) = \frac{\alpha_1}{1+\beta_1}, \\ a(x) = \frac{1}{1-L}\int_{\mathbb{R}_+^N} \omega_2(x,y)dy = \frac{\exp(-\gamma_2 |x|_1)}{1-L}\int_{\mathbb{R}_+^N} \frac{dy}{1+|y|_2^{\lambda_2}} \\ \qquad = \frac{\exp(-\gamma_2 |x|_1)}{1-L}\omega_N \int_0^\infty \frac{r^{N-1}dy}{1+r^{\lambda_2}} = \alpha_2 \exp\left(-\gamma_2 |x|_1\right), \\ \alpha_2 = \frac{\omega_N}{1-L}\int_0^\infty \frac{r^{N-1}dy}{1+r^{\lambda_2}}, \end{cases}$$

where ω_N is the area of unit sphere in $\mathbb{R}^N$.

(ii) Calculating the functions $\bar{a}(x),\ \int_{B_x} \bar{a}(y)dy,\ \bar{r}(x)$:

$$\begin{aligned} \bar{a}(x) = a(x) + a(\sigma_1(x)) &= \alpha_2 \left[\exp\left(-\gamma_2 |x|_1\right) + \exp\left(-\gamma_2 |\theta_1 x|_1\right)\right] \\ &\leqslant 2\alpha_2 \exp\left(-\theta_1\gamma_2 |x|_1\right) \to 0,\ \text{as}\ |x|_1 \to +\infty; \end{aligned}$$

$$\int_{B_x} \bar{a}(y)dy \leqslant \frac{2\alpha_2}{(\theta_1\gamma_2)^N}(1-e^{-\theta_1\gamma_2 x_1})\ldots(1-e^{-\theta_1\gamma_2 x_N}) \leqslant \frac{2\alpha_2}{(\theta_1\gamma_2)^N},\ \forall x \in \mathbb{R}_+^N;$$

$$\begin{aligned} \bar{r}(x) = r(x,0) + r(\sigma_1(x),0) &= \frac{\alpha_1}{1+\beta_1 \exp\left(\gamma_1 |x|_1^N\right)} + \frac{\alpha_1}{1+\beta_1 \exp\left(\gamma_1 |\theta_1 x|_1^N\right)} \\ &\leqslant \frac{2\alpha_1}{1+\beta_1 \exp\left(\ \theta_1\gamma_1 |x|_1^N\right)} \leqslant \frac{2\alpha_1}{\beta_1}\exp\left(-\theta_1\gamma_1 |x|_1^N\right); \end{aligned}$$

and

$$\bar{r}(0) = \frac{2\alpha_1}{1+\beta_1}.$$

It follows that

$$\bar{r}(x)\exp\left(\bar{r}(0)x_1\ldots x_N\right) \leqslant \frac{2\alpha_1}{\beta_1}\exp\left(-\theta_1\gamma_1|x|_1^N\right)\exp\left(\frac{2\alpha_1}{1+\beta_1}\frac{|x|_1^N}{N^N}\right)$$
$$= \frac{2\alpha_1}{\beta_1}\exp\left(-\left[\theta_1\gamma_1-\frac{2\alpha_1}{N^N(1+\beta_1)}\right]|x|_1^N\right)\to 0,$$

as $|x|_1 \to +\infty$, since $\gamma_1 > \frac{2\alpha_1}{N^N(1+\beta_1)\theta_1}$. Then (3.25) holds.

4. An Example

Let us illustrate the results obtained by means of an example. Let $E = C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $v : [0,1] \to \mathbb{R}$ with the norm

$$\|v\| = \sup_{0\leqslant t\leqslant 1}|v(t)|,\ v\in E.$$

Then, for all $u \in X = C(\mathbb{R}_+^2;E)$, for any $x \in \mathbb{R}_+^2$, $u(x)$ is an element of E and we denote

$$u(x)(t) = u(x,t),\ 0\leqslant t\leqslant 1.$$

Consider (1.1) in form

$$u(x) = V\left(x,u(x),\int_{B_x}V_1\left(x,y,u(\sigma_1(y))\right)dy\right) + \int_{\mathbb{R}_+^2}F\left(x,y,u(\sigma_2(y))\right)dy,\ x\in\mathbb{R}_+^2, \tag{4.1}$$

where $\sigma_i(x) = \bar{\sigma}_i x$, $0 < \bar{\sigma}_i \leqslant 1$, $i = 1,2$; $B_x = [0,x_1]\times[0,x_2]$. Giving the continuous functions V, V_1, F as follows.

(i) Function $V : \mathbb{R}_+^2 \times E^2 \to E$,

$$V(x,u,v)(t) = 2(1-k_1)u_*(x,t) + k_1|u(t)| + e^{-\gamma|x|_1^2}|v(t)|,\ 0\leqslant t\leqslant 1,\ (x,u,v)\in\mathbb{R}_+^2\times E^2,$$

with $u_*(x,t) = \frac{1}{t+e^{|x|_1}}$ and γ, k_1 are given constants such that $0 < k_1 < 1$, $\gamma > \frac{\pi}{(1-k_1)\bar{\sigma}_1^2}$.

(ii) Function $V_1 : \Delta\times E \to E$,

$$V_1(x,y,u)(t) = e^{-2|y|_1}u_*(x,t)\sin\left(\pi\frac{u(t)}{u_*(\sigma_1(y),t)}\right),$$

$0\leqslant t\leqslant 1$, $(x,y,u)\in\Delta\times E$, $\Delta = \{(x,y)\in\mathbb{R}_+^2\times\mathbb{R}_+^2 : y\in B_x\}$.

(iii) Function $F : \mathbb{R}_+^2\times\mathbb{R}_+^2\times E \to E$,

$$F(x,y,u)(t) = 4(k_1-1)e^{-2|y|_1}u_*(x,t)\sin\left(\frac{\pi}{2}\int_0^1\frac{u(s)}{u_*(\sigma_2(y),s)}ds\right),$$

$0 \leqslant t \leqslant 1,\ (x,y,u) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E.$

We prove that $(A_1)-(A_5)$ hold. It is easy to see that (A_5) holds, see Remark 1.

– Assumption (A_1) holds, by for all $(x,u,v),\ (x,\bar{u},\bar{v}) \in \mathbb{R}_+^2 \times E^2,\ \forall t \in [0,1]$,

$$\|V(x,u,v)-V(x,\bar{u},\bar{v})\| \leqslant k_1 \|u-\bar{u}\| + \omega_0(x)\|v-\bar{v}\|,\ \forall (x,u,v),\ (x,\bar{u},\bar{v}) \in \mathbb{R}_+^2 \times E^2,$$

with $\omega_0(x) = e^{-\gamma |x|_1^2}$.

– Assumption (A_2) holds, for all $(x,y,u),\ (x,y,\bar{u}) \in \Delta \times E,\ \Delta = \{(x,y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : y \in B_x\},\ \forall t \in [0,1]$,

$$\begin{aligned}
|V_1(x,y,u)(t)-V_1(x,y,\bar{u})(t)| &\leqslant e^{-2|y|_1} u_*(x,t) \frac{\pi}{u_*(\sigma_1(y),t)} |u(t)-\bar{u}(t)| \\
&\leqslant \pi e^{-2|y|_1} \frac{t+e^{|\sigma_1(y)|_1}}{t+e^{|x|_1}} \|u-\bar{u}\| \\
&\leqslant \pi e^{-|y|_1} \frac{te^{-|y|_1}+e^{|\sigma_1(y)|_1-|y|_1}}{t+e^{|x|_1}} \|u-\bar{u}\| \\
&\leqslant 2\pi e^{-|x|_1-|y|_1} \|u-\bar{u}\| \\
&= \omega_1(x,y)\|u-\bar{u}\|,
\end{aligned}$$

in which

$$\omega_1(x,y) = 2\pi e^{-|x|_1-|y|_1}.$$

– Assumption (A_3) is also fulfilled. Indeed, First, we can show $F : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E \to E$ is continuous.

Next, we show $F : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E \to E$ is compact. Let B is bounded in $\mathbb{R}_+^2 \times \mathbb{R}_+^2 \times E$, we deduce from

$$F(x,y,u)(t) = 4(k_1-1)e^{-2|y|_1} u_*(x,t) \sin\left(\frac{\pi}{2}\int_0^1 \frac{u(s)}{u_*(\sigma_2(y),s)} ds\right),$$

$$\begin{aligned}
\|F(x,y,u)\| &\leqslant \omega_2(x,y) = 4(1-k_1)e^{-|x|_1-2|y|_1} \\
&\leqslant 4(1-k_1) \equiv M,\ \forall (x,y,u) \in B,
\end{aligned}$$

that $F(B)$ is uniformly bounded in E. For all $t_1,\ t_2 \in [0,1],\ \forall (x,y,u) \in B$,

$$\begin{aligned}
&F(x,y,u)(t_1)-F(x,y,u)(t_2) \\
&\quad = 4(k_1-1)e^{-2|y|_1} \frac{t_2-t_1}{\left(t_1+e^{|x|_1}\right)\left(t_2+e^{|x|_1}\right)} \sin\left(\frac{\pi}{2}\int_0^1 \frac{u(s)}{u_*(\sigma_2(y),s)} ds\right),
\end{aligned}$$

so

$$\begin{aligned}
|F(x,y,u)(t_1)-F(x,y,u)(t_2)| &\leqslant 4(1-k_1)e^{-2|y|_1} \frac{|t_2-t_1|}{\left(t_1+e^{|x|_1}\right)\left(t_2+e^{|x|_1}\right)} \\
&\leqslant 4(1-k_1)|t_2-t_1|,
\end{aligned}$$

it implies that $F(B)$ is equicontinuous.

Finally, for all bounded subsets $I_1,\ I_2$ of $\mathbb{R}_+^2$ and for any bounded subset J of E, for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\forall x, \bar{x} \in I_1, |x-\bar{x}|_1 < \delta \Longrightarrow |F(x,y;u) - F(\bar{x},y;u)| < \varepsilon,\ \forall (y,u) \in I_2 \times J;$$

We get the above property since

$$\|F(x,y,u) - F(\bar{x},y,u)\| \leqslant 4(1-k_1)\,|x-\bar{x}|_1\,,$$

$\forall x,\ \bar{x} \in I_1,\ \forall (y,u) \in I_2 \times J$. Indeed,

$$\begin{aligned}
&F(x,y,u)(t) - F(\bar{x},y,u)(t)\\
&\quad = 4\,(k_1-1)\,e^{-2|y|_1}[u_*(x,t) - u_*(\bar{x},t)]\sin\left(\frac{\pi}{2}\int_0^1 \frac{u(s)}{u_*(\sigma_2(y),s)}ds\right)\\
&\quad = 4\,(k_1-1)\,e^{-2|y|_1}\,\frac{e^{|\bar{x}|_1} - e^{|x|_1}}{(t+e^{|x|_1})(t+e^{|\bar{x}|_1})}\sin\left(\frac{\pi}{2}\int_0^1 \frac{u(s)}{u_*(\sigma_2(y),s)}ds\right),
\end{aligned}$$

so

$$\begin{aligned}
|F(x,y,u)(t) - F(\bar{x},y,u)(t)| &\leqslant 4(1-k_1)e^{-2|y|_1}\frac{\left|e^{|\bar{x}|_1} - e^{|x|_1}\right|}{(t+e^{|x|_1})(t+e^{|\bar{x}|_1})}\\
&\leqslant 4(1-k_1)e^{-2|y|_1}\,\left||\bar{x}|_1 - |x|_1\right|\\
&\leqslant 4(1-k_1)\,|\bar{x}-x|_1\,.
\end{aligned}$$

Assumption (A_4) is also clearly, by for all bounded subset $I \subset \mathbb{R}_+^2$, $\forall (x,y;u) \in I \times \mathbb{R}_+^2 \times E$, $\forall t \in [0,1]$,

$$\begin{aligned}
|F\,(x,y;u)\,(t)| \leqslant 4\,(1-k_1)\,e^{-2|y|_1}u_*(x,t) &\leqslant \frac{4(1-k_1)e^{-2|y|_1}}{t+e^{|x|_1}}\\
&\leqslant 4(1-k_1)e^{-|x|_1-2|y|_1} = \omega_2(x,y),\\
\int_{\mathbb{R}_+^2}\sup_{x\in I}\omega_2(x,y)dy \leqslant 4(1-k_1)\int_{\mathbb{R}_+^2} e^{-2|y|_1}dy &= 1-k_1 < \infty,
\end{aligned}$$

since $\int_{\mathbb{R}_+^2} e^{-2|y|_1}dy = \frac{1}{4}$.

On the other hand, the condition (3.25) is true. Indeed,

$$\begin{aligned}
\omega_0(x) &= e^{-\gamma|x|_1^2},\\
\omega_1(x,y) &= 2\pi e^{-|x|_1-|y|_1},\\
\omega_2(x,y) &= 4(1-k_1)e^{-|x|_1-2|y|_1}.
\end{aligned}$$

(i) $\bar{a}(x) \to 0$ as $|x|_1 \to +\infty$:

$$a(x) = \frac{1}{1-k_1}\int_{\mathbb{R}_+^2}\omega_2(x,y)dy$$

$$= \frac{1}{1-k_1} 4(1-k_1)e^{-|x|_1} \int_{\mathbb{R}_+^2} e^{-2|y|_1} dy = e^{-|x|_1}, \ \forall x \in \mathbb{R}_+^2;$$

and

$$\bar{a}(x) = a(x) + a(\sigma_1(x)) = e^{-|x|_1} + e^{-\bar{\sigma}_1 |x|_1} \to 0.$$

(ii) $\bar{r}(x)\exp(\bar{r}(0)x_1x_2)\int_{B_x}\bar{a}(y)dy \to 0$ as $|x|_1 \to +\infty$:

(ii$_1$) $\int_{B_x}\bar{a}(y)dy$ is bounded:

$$\begin{aligned}
\int_{B_x}\bar{a}(y)dy &= \int_{B_x} e^{-|y|_1}dy + \int_{B_x} e^{-\bar{\sigma}_1|y|_1}dy \\
&= (1-e^{-x_1})(1-e^{-x_2}) + \frac{1}{\bar{\sigma}_1^2}(1-e^{-\bar{\sigma}_1 x_1})(1-e^{-\bar{\sigma}_1 x_2}) \\
&\leqslant 1 + \frac{1}{\bar{\sigma}_1^2} \leqslant \bar{C}_3,
\end{aligned}$$

(ii$_2$) $\bar{r}(x)\exp(\bar{r}(0)x_1x_2) \to 0$ as $|x|_1 \to +\infty$:

$$\begin{aligned}
r(x,y) &= \frac{1}{1-k_1}\omega_0(x)\omega_1(x,y) = \frac{2\pi}{1-k_1}e^{-\gamma|x|_1^2-|x|_1-|y|_1}, \\
\bar{r}(x) &= r(x,0)+r(\sigma_1(x),0) = \frac{2\pi}{1-k_1}\left[e^{-\gamma|x|_1^2-|x|_1} + e^{-\gamma\bar{\sigma}_1^2|x|_1^2-\bar{\sigma}_1|x|_1}\right] \\
&\leqslant \frac{4\pi}{1-k_1}e^{-\gamma\bar{\sigma}_1^2|x|_1^2}, \\
\bar{r}(0) &= \frac{4\pi}{1-k_1}, \\
\bar{r}(x)\exp(\bar{r}(0)x_1x_2) &\leqslant \frac{4\pi}{1-k_1}e^{-\gamma\bar{\sigma}_1^2|x|_1^2}\exp\left(\bar{r}(0)\frac{1}{4}|x|_1^2\right) \\
&= \frac{4\pi}{1-k_1}e^{-\gamma\bar{\sigma}_1^2|x|_1^2}\exp\left(\frac{\pi}{1-k_1}|x|_1^2\right) \\
&= \frac{4\pi}{1-k_1}\exp\left[-\left(\gamma - \frac{\pi}{(1-k_1)\bar{\sigma}_1^2}\right)\bar{\sigma}_1^2|x|_1^2\right] \to 0
\end{aligned}$$

as $|x|_1 \to +\infty$, since $\gamma - \frac{\pi}{(1-k_1)\bar{\sigma}_1^2} > 0$.

The result is

$$\bar{r}(x)\exp(\bar{r}(0)x_1x_2)\int_{B_x}\bar{a}(y)dy \to 0 \ \text{ as } |x|_1 \to +\infty,$$

(3.25) follows. Theorems 3 holds for (4.1). For more details, it is not difficult to show that the following equation

$$\xi(t) = V\left(x, \xi(x), \int_{B_x} V_1(x,y,\xi(\sigma_1(y)))\,dy\right), \ x \in \mathbb{R}_+^2$$

has a unique solution ξ defined by

$$\xi : \mathbb{R}_+^2 \to E, \ \ \xi(x)(t) = \xi(x,t) = \frac{2}{t + e^{|x|_1}}, \ \forall t \in [0,1], \tag{4.2}$$

and

$$u_* : \mathbb{R}_+^2 \to E, \ \ u_*(x)(t) = u_*(x,t) = \frac{1}{t + e^{|x|_1}}, \ \forall t \in [0,1], \tag{4.3}$$

is the solution of (4.1). Furthermore

$$\lim_{|x|_1 \to \infty} \|u_*(x) - \xi(x)\| = \lim_{|x|_1 \to \infty} e^{-|x|_1} = 0.$$

Consequently, ξ and u_* as in (4.2), (4.3) are asymptotically stable solutions of (4.1).

Acknowledgements. The authors wish to express their sincere thanks to the referees for the suggestions and valuable comments. The authors are also extremely grateful for the support given by Vietnam National University HoChiMinh City (VNU-HCM) under Grant no. **B2013-18-05.**

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(Received May 17, 2013)

(Revised November 30, 2013)

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