Differential
Equations
& Applications
Volume 6, Number 2 (2014), 255–265 doi:10.7153/dea-06-12

NONTRIVIAL SOLUTIONS FOR SYSTEMS OF STURM–LIOUVILLE BOUNDARY VALUE PROBLEMS

JOHN R. GRAEF, SHAPOUR HEIDARKHANI AND LINGJU KONG

(*Communicated by Philip Korman*)

Abstract. Sufficient conditions are established for the existence of at least one nontrivial classical solution to the boundary value system with Sturm-Liouville boundary conditions

$$\begin{cases} -(\phi_{p_i}(u_i'(x)))' = \lambda F_{u_i}(x,u_1,\dots,u_n)h_i(u_i'(x)) \quad \text{in } (a,b), \\ \alpha_i u_i(a) - \beta_i u_i'(a) = 0, \quad \gamma_i u_i(b) + \sigma_i u_i'(b) = 0, \end{cases} \qquad i = 1,\dots,n.$$

The analysis is based on variational methods and critical point theory.

1. Introduction and preliminary results

In this paper, we study the existence of at least one nontrivial classical solutions to the boundary value system with Sturm-Liouville boundary conditions

$$\begin{cases} -(\phi_{p_i}(u_i'(x)))' = \lambda F_{u_i}(x,u_1,\dots,u_n)h_i(u_i'(x)) \quad \text{in } (a,b), \\ \alpha_i u_i(a) - \beta_i u_i'(a) = 0, \quad \gamma_i u_i(b) + \sigma_i u_i'(b) = 0, \end{cases} \qquad i = 1,\dots,n, \tag{1.1}$$

where λ is a positive parameter, $p_i > 1$, $\phi_{p_i}(t) = |t|^{p_i-2}t$, α_i, $\gamma_i \geqslant 0$, β_i, $\sigma_i > 0$, $h_i : \mathbb{R} \to [0,\infty)$ is a bounded and continuous function with $\inf_{t\in\mathbb{R}} h_i(t) > 0$ for $i = 1,\dots,n$. In addition, $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1,t_2,\dots,t_n) \to F(x,t_1,t_2,\dots,t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in [a,b]$, F_{t_i} is continuous in $[a,b]\times\mathbb{R}^n$ for $i = 1,\dots,n$, and $F(x,0,\dots,0) = 0$ for all $x \in [a,b]$, where F_{t_i} denotes the partial derivative of F with respect to t_i. By a *classical solution* of system (1.1), we mean a function $u = (u_1,\dots,u_n)$ such that, for $i = 1,\dots,n$, $u_i(x) \in C^1[a,b]$, $\phi_{p_i}(u_i'(x)) \in C^1[a,b]$, and $u_i(x)$ satisfies (1.1).

Problems of Sturm-Liouville type have been widely investigated for some time. For some recent work, see [3, 6, 7, 8, 9, 10, 12, 13] and the references therein. In this paper, we establish some sufficient conditions under which system (1.1) has at least one nontrivial classical solution. Our approach is to use variational methods; the main tool is a local minimum theorem established in [4], which is recalled below. This lemma and variations of it have frequently been used to obtain multiplicity results for

Mathematics subject classification (2010): 34B15, 47J10.
Keywords and phrases: nontrivial solutions, Sturm-Liouville boundary conditions, critical point theory.

Paper DEA-06-12

nonlinear problems of a variational nature. See, for example, [1, 2, 4, 5, 11] and the references therein.

We now recall the local minimum theorem that appeared in [4]. For a given nonempty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define

$$\eta(r_1, r_2) = \inf_{v \in \Phi^{-1}(r_1, r_2)} \frac{\sup_{u \in \Phi^{-1}(r_1, r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} \tag{1.2}$$

and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{\Phi(v) - r_1} \tag{1.3}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$. In what follows, we let X^* denote the dual space of X.

LEMMA 1.1. ([4, Theorem 5.1]) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Let $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that*

$$\eta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in (1/\rho(r_1, r_2), 1/\eta(r_1, r_2))$, there exists $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Throughout this paper, we let X be the Cartesian product of n Sobolev spaces $W^{1,p_i}([a,b])$, $i = 1, \dots, n$, i.e., $X = W^{1,p_1}([a,b]) \times \dots \times W^{1,p_n}([a,b])$, endowed with the norm

$$||u|| = ||(u_1, \dots, u_n)|| = \sum_{i=1}^{n} ||u_i||_1, \quad u = (u_1, \dots, u_n) \in X,$$

where

$$||u_i||_1 = \left(\int_a^b (|u_i'(x)|^{p_i} + |u_i(x)|^{p_i}) dx \right)^{1/p_i}, \quad i = 1, \dots, n.$$

Then, X is a reflexive real Banach space.

For $i = 1, \dots, n$ and $v \in L^{p_i}([a,b])$, we introduce the notations

$$\begin{aligned}
&||v||_{L^{p_i}} = \left(\int_a^b |v(t)|^{p_i} dt \right)^{1/p_i}, \\
&m_i = \inf_{t \in \mathbb{R}} h_i(t), \quad M_i = \sup_{t \in \mathbb{R}} h_i(t), \\
&\underline{m} = \min\{m_i : i = 1, \dots, n\}, \quad \overline{M} = \max\{m_i : i = 1, \dots, n\}, \\
&\underline{p} = \min\{p_i : i = 1, \dots, n\}, \qquad \overline{p} = \max\{p_i : i = 1, \dots, n\},
\end{aligned}$$

so we have $\overline{M} \geqslant \underline{m} > 0$. We also let q_i be the conjugates of p_i, i.e., $1/p_i + 1/q_i = 1$.

For all $s \in \mathbb{R}$, let

$$J_i(s) = \int_0^s \frac{(p_i - 1)|\delta|^{p_i - 2}}{h_i(\delta)} d\delta, \quad i = 1, \ldots, n.$$

For each $u = (u_1, ..., u_n) \in X$, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) = \sum_{i=1}^{n} \left[\int_a^b \left(\int_0^{u'(x)} J_i(s) ds \right) dx + \frac{\beta_i}{\alpha_i} \int_0^{\frac{\alpha_i u_i(a)}{\beta_i}} J_i(s) ds + \frac{\sigma_i}{\gamma_i} \int_0^{-\frac{\gamma_i u_i(b)}{\sigma_i}} J_i(s) ds \right] \quad (1.4)$$

and

$$\Psi(u) = \int_a^b F(x, u_1(x), \ldots, u_n(x)) dx. \quad (1.5)$$

A simple calculation shows that

$$\frac{1}{\overline{M}\,\overline{p}} \sum_{i=1}^{n} \left(||u_i'||_{L^{p_i}}^{p_i} + \frac{\alpha_i^{p_i - 1}}{\beta_i^{p_i - 1}} |u_i(a)|^{p_i} + \frac{\gamma_i^{p_i - 1}}{\sigma_i^{p_i - 1}} |u_i(b)|^{p_i} \right) \leqslant \Phi(u) \leqslant \frac{1}{\underline{m}\,\underline{p}} \sum_{i=1}^{n} \left(||u_i'||_{L^{p_i}}^{p_i} + \frac{\alpha_i^{p_i - 1}}{\beta_i^{p_i - 1}} |u_i(a)|^{p_i} + \frac{\gamma_i^{p_i - 1}}{\sigma_i^{p_i - 1}} |u_i(b)|^{p_i} \right). \quad (1.6)$$

DEFINITION 1.1. We say that a function $u = (u_1, \ldots, u_n) \in X$ is a *weak solution* of system (1.1) if

$$\sum_{i=1}^{n} \left[\int_a^b J_i(u_i'(x)) v_i'(x) dx + J_i \left(\frac{\alpha_i u_i(a)}{\beta_i} \right) v_i(a) - J_i \left(\frac{-\gamma_i u_i(b)}{\sigma_i} \right) v_i(b) - \lambda \int_a^b F_{u_i}(x, u_1(x), \ldots, u_n(x)) dx \right] = 0$$

for any $v = (v_1, \ldots, v_n) \in X$.

The following lemma was proved in [7] (also see [1, Lemma 2.1]).

LEMMA 1.2. ([7, Lemma 2.1]) *A weak solution to* (1.1) *coincides with a classical solution to* (1.1).

LEMMA 1.3. ([8, Lemma 2.2]) *Let the functionals* $\Phi, \Psi : X \to \mathbb{R}$ *be defined by* (1.4) *and* (1.5). *Then,*

(a) Φ *is sequentially weakly lower semicontinuous, continuous,* $\lim_{||u|| \to \infty} \Phi(u) = \infty$, *and its derivative at the point* $u = (u_1, \ldots, u_n) \in X$ *is the functional* $\Phi'(u)$ *given by*

$$\Phi'(u)(v) = \sum_{i=1}^{n} \left[\int_a^b J_i(u_i'(x)) v_i'(x) dx + J_i \left(\frac{\alpha_i u_i(a)}{\beta_i} \right) v_i(a) - J_i \left(\frac{-\gamma_i u_i(b)}{\sigma_i} \right) v_i(b) \right]$$

for every $v = (v_1, \dots, v_n) \in X$.

(b) Ψ *is sequentially weakly upper semicontinuous and its derivative at the point* $u = (u_1, \dots, u_n) \in X$ *is the functional* $\Psi'(u)$ *given by*

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx$$

for every $v = (v_1, \dots, v_n) \in X$.

REMARK 1.1. By Definition 1.1 and Lemmas 1.2 and 1.3, we see that $u \in X$ is a critical point of $\Phi - \lambda\Psi$ if and only u is a classical solution of system (1.1).

LEMMA 1.4. ([8, Lemma 2.3]) *Assume that, for* $u = (u_1, \dots, u_n) \in X$, *there exists* $r > 0$ *such that* $\Phi(u) \leqslant r$. *Then, we have*

$$\max_{x \in [a,b]} \sum_{i=1}^n |u_i(x)| \leqslant \sum_{i=1}^n \left(\sqrt[p_i]{\frac{\beta_i^{p_i-1}}{\alpha_i^{p_i-1}} \overline{M} \overline{p} r} + \sqrt[p_i]{\overline{M} \overline{p} r}\, (b-a)^{\frac{1}{q_i}} \right). \tag{1.7}$$

LEMMA 1.5. *Assume that*

(H) *Either* $\underline{p} \geqslant 2$ *or* $\overline{p} < 2$.

Then, $\Phi' : X \to X^*$ *admits a continuous inverse on* X^*.

The scalar case of Lemma 1.5 with Dirichlet boundary conditions was proved in [1, Corollary 2.5]. The system case of Lemma 1.5 with Dirichlet boundary conditions was proved in [8, Lemma 2.1]. For the general system case with Sturm-Liouville boundary conditions, the proof is essentially the same. The details are left to the reader.

In this paper, we always assume that the condition (H) holds without further mention.

In the next section, we present our results and their proofs.

2. Main results

For any $\vartheta > 0$, let

$$Q(\vartheta) = \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^n |t_i| \leqslant \vartheta \sum_{i=1}^n \left(\sqrt[p_i]{\frac{\beta_i^{p_i-1}}{\alpha_i^{p_i-1}}} + (b-a)^{\frac{1}{q_i}} \right) \right\},$$

and for any $u = (u_1, \dots, u_n) \in X$, let

$$\Theta_u = \sum_{i=1}^n \left(||u_i'||_{L^{p_i}}^{p_i} + \frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} |u_i(a)|^{p_i} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}} |u_i(b)|^{p_i} \right). \tag{2.1}$$

For a given constant $\nu \geqslant 0$ and a function $u = (u_1, \ldots, u_n) \in X$ with

$$\frac{\nu^{\underline{p}}}{\overline{M}\overline{p}} \neq \frac{\Theta_u}{\underline{m}\underline{p}} \quad \text{and} \quad \frac{\nu^{\overline{p}}}{\overline{M}\overline{p}} \neq \frac{\Theta_u}{\underline{m}\underline{p}},$$

we define

$$a_u(\nu) = \frac{\int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu)} F(x,t_1,\ldots,t_n)dx - \int_a^b F(x,u_1(x),\ldots,u_n(x))dx}{\frac{\nu^{\underline{p}}}{\overline{M}\overline{p}} - \frac{1}{\underline{m}\underline{p}}\Theta_u} \tag{2.2}$$

if $\nu \geqslant 1$, and

$$a_u(\nu) = \frac{\int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu)} F(x,t_1,\ldots,t_n)dx - \int_a^b F(x,u_1(x),\ldots,u_n(x))dx}{\frac{\nu^{\overline{p}}}{\overline{M}\overline{p}} - \frac{1}{\underline{m}\underline{p}}\Theta_u} \tag{2.3}$$

if $\nu < 1$.

THEOREM 2.1. *Assume that there exist two constants $\nu_1 \geqslant 0$ and $\nu_2 > 0$, and a function $w = (w_1, \ldots, w_n) \in X$ such that*

(A1) $\nu_1^{\underline{p}} < \Theta_w$ *and* $\Theta_w/(\underline{m}\underline{p}) < \nu_2^{\underline{p}}/(\overline{M}\overline{p})$ *if* $\nu_2 \geqslant 1$*; and* $\nu_1^{\overline{p}} < \Theta_w$ *and* $\Theta_w/(\underline{m}\underline{p}) < \nu_2^{\overline{p}}/(\overline{M}\overline{p})$ *if* $\nu_2 < 1$*;*

(A2) $a_w(\nu_2) < a_w(\nu_1)$.

Then, for each $\lambda \in (1/a_w(\nu_1),\ 1/a_w(\nu_2))$, system (1.1) has at least one nontrivial classical solution $u_0 = (u_{01}, \ldots, u_{0n}) \in X$ such that $r_1 < \Phi(u_0) < r_2$, where Φ is defined by (1.4) and

$$r_1 = \begin{cases} \dfrac{\nu_1^{\underline{p}}}{\overline{M}\overline{p}}, & \nu_1 \geqslant 1, \\[2ex] \dfrac{\nu_1^{\overline{p}}}{\overline{M}\overline{p}}, & \nu_1 < 1, \end{cases} \qquad r_2 = \begin{cases} \dfrac{\nu_2^{\underline{p}}}{\overline{M}\overline{p}}, & \nu_2 \geqslant 1, \\[2ex] \dfrac{\nu_2^{\overline{p}}}{\overline{M}\overline{p}}, & \nu_2 < 1. \end{cases} \tag{2.4}$$

Proof. Let the functionals Φ, $\Psi : X \to \mathbb{R}$ be defined by (1.4) and (1.5), respectively. In view of Lemmas 1.3 and 1.5, it is east to see that Φ and Ψ satisfy all the regularity assumptions given in Lemma 1.1.

From (1.6) and (2.1), it follows that $\Theta_w/(\overline{M}\overline{p}) \leqslant \Phi(w) \leqslant \Theta_w/(\underline{m}\underline{p})$. Then, by (A1) and (2.4), we have

$$r_1 < \Phi(w) < r_2. \tag{2.5}$$

From (1.7), we see that

$$\Phi^{-1}(-\infty, r_2) \subseteq \left\{ (u_1, \ldots, u_n) \in X : \sum_{i=1}^n |u_i(x)| \leqslant \nu_2 \sum_{i=1}^n \left(\sqrt[p_i]{\frac{\beta_i^{p_i-1}}{\alpha_i^{p_i-1}}} + (b-a)^{\frac{1}{q_i}} \right) \text{ for } x \in [a,b] \right\}.$$

Then,

$$\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r_2)} \Psi(u) = \sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r_2)} \int_a^b F(x,u_1(x),\ldots,u_n(x))dx$$
$$\leqslant \int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu_2)} F(x,t_1,\ldots,t_n)dx.$$

Therefore, from (1.2) and (2.2)–(2.5), we obtain that

$$\begin{aligned}\eta(r_1,r_2) &\leqslant \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)-\Psi(w)}{r_2-\Phi(w)}\\ &\leqslant \frac{\int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu_2)} F(x,t_1,\ldots,t_n)dx-\Psi(w)}{r_2-\Phi(w)}\\ &\leqslant a_w(\nu_2).\end{aligned}$$

Note that (A2) implies that $a_w(\nu_1)>0$, and so

$$\Psi(w) \geqslant \int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu_1)} F(x,t_1,\ldots,t_n)dx.$$

Then, by similar reasoning, we also have

$$\begin{aligned}\rho(r_1,r_2) &\geqslant \frac{\Psi(w)-\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{\Phi(w)-r_1}\\ &\geqslant \frac{\Psi(w)-\int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu_1)} F(x,t_1,\ldots,t_n)dx}{\Phi(w)-r_1}\\ &\geqslant a_w(\nu_1).\end{aligned}$$

Thus, from (A2),

$$\eta(r_1,r_2) \leqslant a_w(\nu_2) < a_w(\nu_1) \leqslant \rho(r_1,r_2).$$

Hence, Lemma 1.1 implies that, for each $\lambda \in (1/a_w(\nu_1),\ 1/a_w(\nu_2))$, $\Phi(u)-\lambda\Psi(u)$ has at least one nontrivial critical point $u_0=(u_{01},\ldots,u_{0n})$ in X satisfying $r_1<\Phi(u_0)<r_2$. Invoking Remark 1.1 completes the proof of the theorem.

In Theorem 2.1, the conditions (A1)–(A3) are related to the function $w\in X$. A different function $w\in X$ would lead to different conditions. For example, for some $\tau>0$, by taking $w(t)=(\tau(x-a),\ldots,\tau(x-a))\in X$ and $w(t)=(\tau,\ldots,\tau)\in X$, respectively, from Theorem 2.1, we have the following two results.

COROLLARY 2.1. *Assume that there exist three constants* $\nu_1\geqslant 0$, $\nu_2\geqslant 1$, *and* $\tau>0$ *such that*

$$\nu_1^{\underline{p}} < \sum_{i=1}^n \left(\tau^{p_i}+\frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}(\tau(b-a))^{p_i}\right), \quad \frac{1}{m\underline{p}}\sum_{i=1}^n \left(\tau^{p_i}+\frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}(\tau(b-a))^{p_i}\right) < \frac{\nu_2^{\underline{p}}}{M\overline{p}}$$

and

$$a_{2,\tau} := \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\nu_2)} F(x,t_1,\dots,t_n)dx - \int_a^b F(x,\tau(x-a),\dots,\tau(x-a))dx}{\frac{\nu_2^{\underline{p}}}{\overline{M}\,\overline{p}} - \frac{1}{m\underline{p}}\sum_{i=1}^n \left(\tau^{p_i} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}(\tau(b-a))^{p_i}\right)}$$

$$< a_{1,\tau} := \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\nu_1)} F(x,t_1,\dots,t_n)dx - \int_a^b F(x,\tau(x-a),\dots,\tau(x-a))dx}{\frac{\nu_1^{\underline{p}}}{\overline{M}\,\overline{p}} - \frac{1}{m\underline{p}}\sum_{i=1}^n \left(\tau^{p_i} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}(\tau(b-a))^{p_i}\right)}.$$

Then, for each $\lambda \in (1/a_{1,\tau},\ 1/a_{2,\tau})$*, system* (1.1) *has at least one nontrivial classical solution* $u_0 = (u_{01},\dots,u_{0n}) \in X$ *such that* $r_1 < \Phi(u_0) < r_2$*, where* Φ *is defined by* (1.4)*, and* r_1 *and* r_2 *are defined by* (2.4)*.*

COROLLARY 2.2. *Assume that there exist three constants* $\nu_1 \geqslant 0$*,* $\nu_2 \geqslant 1$*, and* $\tau > 0$ *such that*

$$\nu_1^{\underline{p}} < \sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}\right)\tau^{p_i}, \quad \frac{1}{m\underline{p}}\sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}\right)\tau^{p_i} < \frac{\nu_2^{\underline{p}}}{\overline{M}\,\overline{p}},$$

and

$$b_{2,\tau} := \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\nu_2)} F(x,t_1,\dots,t_n)dx - \int_a^b F(x,\tau,\dots,\tau)dx}{\frac{\nu_2^{\underline{p}}}{\overline{M}\,\overline{p}} - \frac{1}{m\underline{p}}\sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}\right)\tau^{p_i}}$$

$$< b_{1,\tau} := \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\nu_1)} F(x,t_1,\dots,t_n)dx - \int_a^b F(x,\tau,\dots,\tau)dx}{\frac{\nu_1^{\underline{p}}}{\overline{M}\,\overline{p}} - \frac{1}{m\underline{p}}\sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}\right)\tau^{p_i}}.$$

Then, for each $\lambda \in (1/b_{1,\tau},\ 1/b_{2,\tau})$*, system* (1.1) *has at least one nontrivial classical solution* $u_0 = (u_{01},\dots,u_{0n}) \in X$ *such that* $r_1 < \Phi(u_0) < r_2$*, where* Φ *is defined by* (1.4)*, and* r_1 *and* r_2 *are defined by* (2.4)*.*

By choosing $\nu_1 = 0$ and $\nu_2 = \nu$, the following two results follow directly from Corollaries 2.1 and 2.2.

COROLLARY 2.3. *Assume that there exist two constants* $\nu \geqslant 1$ *and* $\tau > 0$ *such that*

$$\frac{1}{m\underline{p}}\sum_{i=1}^n \left(\tau^{p_i} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}(\tau(b-a))^{p_i}\right) < \frac{\nu^{\underline{p}}}{\overline{M}\,\overline{p}}$$

and

$$c_{2,\tau} := \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\nu)} F(x,t_1,\dots,t_n)dx - \int_a^b F(x,\tau(x-a),\dots,\tau(x-a))dx}{\frac{\nu^{\underline{p}}}{\overline{M}\,\overline{p}} - \frac{1}{m\underline{p}}\sum_{i=1}^n \left(\tau^{p_i} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}}(\tau(b-a))^{p_i}\right)}$$

$$< c_{1,\tau} := \frac{\int_a^b F(x, \tau(x-a), \ldots, \tau(x-a))dx}{\frac{1}{\underline{mp}} \sum_{i=1}^n \left(\tau^{p_i} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}} (\tau(b-a))^{p_i} \right)}. \tag{2.6}$$

Then, for each $\lambda \in (1/c_{1,\tau},\ 1/c_{2,\tau})$, *system* (1.1) *has at least one nontrivial classical solution* $u_0 = (u_{01}, \ldots, u_{0n}) \in X$ *such that* $0 < \Phi(u_0) < \nu^{\underline{p}}/(\overline{M}\overline{p})$, *where* Φ *is defined by* (1.4).

COROLLARY 2.4. *Assume that there exist two constants* $\nu \geqslant 1$ *and* $\tau > 0$ *such that*

$$\frac{1}{\underline{mp}} \sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}} \right) \tau^{p_i} < \frac{\nu^{\underline{p}}}{\overline{M}\overline{p}}$$

and

$$d_{2,\tau} := \frac{\int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu)} F(x, t_1, \ldots, t_n)dx - \int_a^b F(x, \tau, \ldots, \tau)dx}{\frac{\nu^{\underline{p}}}{\overline{M}\overline{p}} - \frac{1}{\underline{mp}} \sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}} \right) \tau^{p_i}}$$

$$< d_{1,\tau} := \frac{\int_a^b F(x, \tau, \ldots, \tau)dx}{\frac{1}{\underline{mp}} \sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}} \right) \tau^{p_i}}. \tag{2.7}$$

Then, for each $\lambda \in (1/d_{1,\tau},\ 1/d_{2,\tau})$, *system* (1.1) *has at least one nontrivial classical solution* $u_0 = (u_{01}, \ldots, u_{0n}) \in X$ *such that* $0 < \Phi(u_0) < \nu^{\underline{p}}/(\overline{M}\overline{p})$, *where* Φ *is defined by* (1.4).

REMARK 2.1. It is easy to see that (2.6) is equivalent to

$$\int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu)} F(x, t_1, \ldots, t_n)dx < \frac{\underline{mp}\nu^{\underline{p}} \int_a^b F(x, \tau(x-a), \ldots, \tau(x-a))dx}{\overline{M}\overline{p} \sum_{i=1}^n \left(\tau^{p_i} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}} (\tau(b-a))^{p_i} \right)},$$

and that (2.7) is equivalent to

$$\int_a^b \sup_{(t_1,\ldots,t_n)\in Q(\nu)} F(x, t_1, \ldots, t_n)dx < \frac{\underline{mp}\nu^{\underline{p}} \int_a^b F(x, \tau, \ldots, \tau)dx}{\overline{M}\overline{p} \sum_{i=1}^n \left(\frac{\alpha_i^{p_i-1}}{\beta_i^{p_i-1}} + \frac{\gamma_i^{p_i-1}}{\sigma_i^{p_i-1}} \right) \tau^{p_i}}.$$

In applications, the above equivalent forms are easier to verify.

We now present a simple version of Corollary 2.4 with $n = 1$. Let $\alpha,\ \gamma$ be two nonnegative constants, $\beta,\ \sigma$ be two positive constants, and $p > 1$. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous and $h : \mathbb{R} \to [0, \infty)$ be continuous with

$$0 < m := \inf_{t\in\mathbb{R}} h(t) \leqslant M := \inf_{t\in\mathbb{R}} h(t) < \infty.$$

Let F be the function defined by

$$F(x,t) = \int_0^t f(x,s)ds \quad \text{for each } (x,t) \in [a,b]\times\mathbb{R}.$$

For any $\vartheta > 0$, set

$$W(\vartheta) = \left\{ t \in \mathbb{R} \,:\, |t| \leqslant \vartheta \left(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + (b-a)^{\frac{1}{q}} \right) \right\},$$

where $1/p + 1/q = 1$.

The following result is an obvious consequence of Corollary 2.4.

COROLLARY 2.5. *Assume that there exist two constants $\nu \geqslant 1$ and $\tau > 0$ such that*

$$\left(\frac{\alpha^{p-1}}{\beta^{p-1}} + \frac{\gamma^{p-1}}{\sigma^{p-1}} \right) \tau^p < \frac{m\nu^p}{M} \tag{2.8}$$

and

$$k_{2,\tau} := \frac{\int_a^b \sup_{t\in W(\nu)} F(x,t)dx - \int_a^b F(x,\tau)dx}{\frac{\nu^p}{Mp} - \frac{1}{mp}\left(\frac{\alpha^{p-1}}{\beta^{p-1}} + \frac{\gamma^{p-1}}{\sigma^{p-1}} \right)\tau^p} < k_{1,\tau} := \frac{\int_a^b F(x,\tau)dx}{\frac{1}{mp}\left(\frac{\alpha^{p-1}}{\beta^{p-1}} + \frac{\gamma^{p-1}}{\sigma^{p-1}} \right)\tau^p}. \tag{2.9}$$

Then, for each $\lambda \in (1/k_{1,\tau},\ 1/k_{2,\tau})$, the problem

$$\begin{cases} -(\phi_p(u'))' = \lambda f(x,u)h(u') \quad \text{in } (a,b), \\ \alpha u(a) - \beta u'(a) = 0, \ \ \gamma u(b) + \sigma u'(b) = 0, \end{cases} \tag{2.10}$$

has at least one nontrivial classical solution $u \in W^{1,p}([a,b])$ such that $0 < \Phi_1(u) < \nu^{\underline{p}}/(\overline{M}\,\overline{p})$, where

$$\Phi_1(u) = \left[\int_a^b \int_0^{u'(x)} \int_0^s \frac{(p-1)|\delta|^{p-2}}{h(\delta)} d\delta ds dx + \frac{\beta}{\alpha}\int_0^{\frac{\alpha u(a)}{\beta}} \int_0^s \frac{(p-1)|\delta|^{p-2}}{h(\delta)} d\delta ds \right.$$
$$\left. + \frac{\sigma}{\gamma}\int_0^{-\frac{\gamma u(b)}{\sigma}} \int_0^s \frac{(p-1)|\delta|^{p-2}}{h(\delta)} d\delta ds \right].$$

REMARK 2.2. It is easy to see that (2.9) is equivalent to

$$\int_a^b \sup_{t\in W(\nu)} F(x,t)dx < \frac{m\nu^p \int_a^b F(x,\tau)dx}{M\left(\frac{\alpha^{p-1}}{\beta^{p-1}} + \frac{\gamma^{p-1}}{\sigma^{p-1}} \right)\tau^p}, \tag{2.11}$$

which is easier to verify in applications.

Results corresponding to Corollaries 2.1–2.3 can be formulated similarly and we leave this to the interested reader.

Finally, as an example, we present the following special case of Corollary 2.5.

COROLLARY 2.6. *Let $f_1 : [a,b] \to (0,\infty)$ and $f_2 : \mathbb{R} \to [0,\infty)$ be continuous functions such that $\lim_{t\to 0^+} f_2(t)/t^{p-1} = \infty$. Let $h : \mathbb{R} \to [0,\infty)$ be a continuous function such that*

$$0 < m := \inf_{t\in\mathbb{R}} h(t) \leqslant M := \inf_{t\in\mathbb{R}} h(t) < \infty.$$

Then, for each

$$\lambda \in \left(0,\ \frac{1}{Mp\int_a^b f_1(x)dx} \sup_{\nu\geqslant 1} \frac{\nu^p}{\int_0^{\kappa\nu} f_2(\xi)d\xi}\right),$$

where

$$\kappa = \sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + (b-a)^{\frac{1}{q}},$$

the problem

$$\begin{cases} -(\phi_p(u'))' = \lambda f_1(x) f_2(u) h(u') & \text{in } (a,b), \\ \alpha u(a) - \beta u'(a) = 0, \quad \gamma u(b) + \sigma u'(b) = 0, \end{cases} \tag{2.12}$$

has at least one nontrivial classical solution $u \in W^{1,p}([a,b])$.

Proof. Let $f(x,t) = f_1(x)f_2(t)$. Clearly, (2.12) is a special case of (2.10). For fixed λ as in the conclusion, there exists a constant $\nu \geqslant 1$ such that

$$\lambda < \frac{1}{Mp\int_a^b f_1(x)dx} \frac{\nu^p}{\int_0^{\kappa\nu} f_2(\xi)d\xi} = \frac{\frac{\nu^p}{Mp}}{\int_a^b f_1(x)dx \sup_{t\in W(\nu)} \int_0^t f_2(\xi)d\xi}.$$

Since $\lim_{t\to 0^+} \frac{f_2(t)}{t^{p-1}} = \infty$, we have $\lim_{t\to 0^+} \frac{\int_0^t f_2(\xi)d\xi}{t^p} = \infty$. Then, for the above ν, in view of the fact that $F(x,t) = f_1(x)\int_0^t f_2(s)ds$, we see that there exists $\tau^* > 0$ such that (2.8) and (2.11) hold, and

$$\lambda < \frac{\frac{\nu^p}{Mp} - \frac{1}{mp}\left(\frac{\alpha^{p-1}}{\beta^{p-1}} + \frac{\gamma^{p-1}}{\sigma^{p-1}}\right)\tau^p}{\int_a^b f_1(x)dx \sup_{t\in W(\nu)} \int_0^t f_2(\xi)d\xi - \int_a^b f_1(x)dx \int_0^\tau f_2(s)ds} = \frac{1}{k_{1,\tau}}$$

for all $\tau \in (0,\tau^*)$. By Remark 2.2, (2.9) holds if $\tau \in (0,\tau^*)$. Note that $\lim_{\tau\to 0^+} k_{2,\tau} = \infty$. The conclusion then follows from Corollary 2.5.

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(Received October 7, 2013)

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403
USA
e-mail: John-Graef@utc.edu

Shapour Heidarkhani
Department of Mathematics
Faculty of Sciences, Razi University
Kermanshah 67149
Iran
e-mail: s.heidarkhani@razi.ac.ir

Lingju Kong
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403
USA
e-mail: Lingju-Kong@utc.edu