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COMPETITIVE GOMPERTZ MODEL OF TWO SPECIES

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Abstract. In this paper, a competitive Gompertz model of two species is proposed. Furthermore, under a certain condition, the existence of monotone traveling wave solutions of this model is shown by the method of super- and subsolutions, which is developed in [6, 15, 16].

Dedicate to my lovely sisters Wen and Chun

1. Introduction

On the basis of the logistic growth equation, the competitive Lotka-Volterra system which describes the interaction between two distinct species takes the following form:

$$\begin{cases} u_t = d_1\, u_{xx} + u\,(\lambda_1 - c_{11}\, u - c_{12}\, v), \\ \\ v_t = d_2\, v_{xx} + v\,(\lambda_2 - c_{21}\, u - c_{22}\, v), \end{cases} \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ stand for the density of the two species u and v, respectively; d_i, λ_i, c_{ii} $(i = 1,2)$, and c_{ij} $(i, j = 1,2.\ i \neq j)$ are the diffusion rates, the intrinsic growth rates, the intra-specific competition rates, and the inter-specific competition rates, which are assumed to be positive parameters, respectively.

In ecology, it is important to determine which species will survive in a competitive system. In order to tackle this problem, we can use traveling wave solutions, which are solutions of the form

$$(u(x,t), v(x,t)) = (u(z), v(z)), \quad z = x - \theta\, t, \tag{1.2}$$

where θ represents the wave velocity of the traveling wave.

Under suitable scalings of the dependent and independent variables, the traveling wave solution $(u(z), v(z))$ of (1.1) satisfies

$$\begin{cases} u_{zz} + \theta\, u_z + \ u\,(1 - u - a_1\, v) = 0, \\ \\ d\, v_{zz} + \theta\, v_z + \lambda\, v\,(1 - a_2\, u - v) = 0, \end{cases} \qquad z \in \mathbb{R}, \tag{1.3}$$

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where d, λ, a_1 and a_2 are positive parameters. We remark that the determination of θ is part of solving system (1.3). A lot of attention has been paid to studying the existence of traveling wave solutions for (1.1). For instance, see [4, 6, 8, 9, 10, 11, 12, 13, 15, 16, 20] and references cited therein. In particular, in [7, 18, 19], exact traveling wave solutions of (1.1) were constructed by applying judicious ansätz for solutions.

Under certain situations, it is appropriate to use other growth equations rather than using the logistic growth equation. For instance, the Gompertz growth equation provides a good fit for the growth of tumors and microorganisms. We derive the Gompertz growth equation here. Suppose that $u = u(t)$ is the size of the tumor and $r = r(t)$ is the tumor growth rate. The Gompertz growth law can be described by the following differential equations

$$\begin{cases} \dfrac{du}{dt} = r(t)\,u(t), \\[2mm] \dfrac{dr}{dt} = -a\,r(t), \\[2mm] u(0) = u_0,\ r(0) = r_0, \end{cases}$$

where $a > 0$ is a constant. This system can be solved to give

$$u(t) = u_0\, e^{\frac{r_0}{a}(1-e^{-at})}.$$

It turns out that u satisfies the following ODE

$$\frac{du}{dt} = a\,u\ln\left(\frac{u_0\exp(r_0\,a^{-1})}{u}\right), \tag{1.4}$$

which is the so-called Gompertz growth equation. We note that the density of species tends to increase more rapidly at low density under the Gompertz growth equation than under the logistic growth equation.

Based on the Gompertz growth equation (1.4), we propose the following competitive Gompertz model of two species [22]:

$$\begin{cases} u_t = d_1\,u_{xx} + u\ln\left(\dfrac{1}{u + c_{12}\,v + \alpha_1} + \beta_1\right), \\[2mm] v_t = d_2\,v_{xx} + v\ln\left(\dfrac{1}{c_{21}\,u + v + \alpha_2} + \beta_2\right), \quad x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{1.5}$$

where $u = u(x,t)$ and $v = v(x,t)$ denote the densities of species u and v, respectively; d_1 and d_2 are the diffusion rates which are assumed to be positive constants; α_1, α_2, β_1 and β_2 are positive parameters and are related by

$$\beta_1 = 1 - \frac{1}{1+\alpha_1}, \quad \beta_2 = 1 - \frac{1}{\gamma+\alpha_2}, \quad \alpha_1 \geqslant 1, \quad \alpha_2 \geqslant 1, \tag{1.6}$$

where γ is a positive constant. It is readily seen that (1.5) is a competitive system in the sense that

$$\frac{\partial f(u,v)}{\partial v} > 0, \quad \frac{\partial g(u,v)}{\partial u} > 0 \tag{1.7}$$

for $u,v>0$, where

$$f(u,v)=u\ln\Big(\frac{1}{u+c_{12}v+\alpha_1}+\beta_1\Big)\quad\text{and}\quad g(u,v)=v\ln\Big(\frac{1}{c_{21}u+v+\alpha_2}+\beta_2\Big).$$

For simplicity, choose $\alpha_1=1$ and $\alpha_2=\alpha$. Under the condition (1.6), the corresponding diffusionless system of (1.5) is

$$\begin{cases} u_t=u\ln\Big(\dfrac{1}{u+c_{12}v+1}+\dfrac{1}{2}\Big),\\ v_t=v\ln\Big(\dfrac{1}{c_{21}u+v+\alpha}+1-\dfrac{1}{\gamma+\alpha}\Big), \quad t>0, \end{cases} \tag{1.8}$$

which has four equilibrium points

$$\mathscr{E}_1=(0,0),\quad \mathscr{E}_2=(1,0),\quad \mathscr{E}_3=(0,\gamma),\quad \mathscr{E}_4=(u^*,v^*), \tag{1.9}$$

where

$$u^*=\frac{-1+\gamma c_{12}}{-1+c_{12}c_{21}},\quad v^*=\frac{-\gamma+c_{21}}{-1+c_{12}c_{21}}. \tag{1.10}$$

It is easy to see that the asymptotic behavior of solutions (u,v) of (1.8) with initial conditions $u(0)$, $v(0)>0$ can be classified into four cases according to the relations among c_{12}, c_{21}, and γ.

THEOREM 1.1. (Local stability of equilibria) *Three cases are monostable while the other case is bistability:*

(i) *If* $\gamma<\min(c_{21},1/c_{12})$, *then* $\lim_{t\to\infty}(u(t),v(t))=(1,0)$.

(ii) *If* $c_{21}<\gamma<1/c_{12}$, *then* $\lim_{t\to\infty}(u(t),v(t))=(u^*,v^*)$.

(iii) *If* $1/c_{12}<\gamma<c_{21}$, *then* $\lim_{t\to\infty}(u(t),v(t))=$ *either* $(1,0)$ *or* $(0,\gamma)$ *depending on the initial conditions.*

(iv) *If* $\gamma>\max(c_{21},1/c_{12})$, *then* $\lim_{t\to\infty}(u(t),v(t))=(0,\gamma)$.

Proof. The proof is elementary and hence omitted.

We remark that the local stability of equilibria in (1.8) does not depend on the value of the parameter α due to the fact that the null line

$$\ln\Big(\frac{1}{c_{21}u+v+\alpha}+1-\frac{1}{\gamma+\alpha}\Big)=0 \tag{1.11}$$

turns out to be

$$c_{21}u+v=\gamma, \tag{1.12}$$

in which α does not appear.

Motivated by the method of super- and subsolutions applied in [6, 15, 16], we establish the existence of traveling wave solutions

$$(u(x,t),v(x,t)) = (u(z),v(z)), \quad z := x-ct, \tag{1.13}$$

for

$$\begin{cases} u_t = u_{xx} + u \ln\left(\dfrac{1}{u+c_{12}v+1} + \dfrac{1}{2}\right), \\ v_t = v_{xx} + v \ln\left(\dfrac{1}{c_{21}u+v+\alpha} + 1 - \dfrac{1}{\gamma+\alpha}\right), \quad x \in \mathbb{R}, \quad t > 0 \end{cases} \tag{1.14}$$

by mean of Theorem 4.2 in [21]. Traveling wave solutions $(u(z),v(z))$ satisfy

$$\begin{cases} u_{zz} + c\,u_z + u \ln\left(\dfrac{1}{u+c_{12}v+1} + \dfrac{1}{2}\right) = 0, \\ v_{zz} + c\,v_z + v \ln\left(\dfrac{1}{c_{21}u+v+\alpha} + 1 - \dfrac{1}{\gamma+\alpha}\right) = 0, \quad z \in \mathbb{R}. \end{cases} \tag{1.15}$$

When v is absent in (1.5), (1.5) becomes

$$u_t = d_1 u_{xx} + u \ln\left(\frac{1}{u+\alpha_1} + \beta_1\right). \tag{1.16}$$

To simplify the problem, we choose $d_1 = 1$, $\alpha_1 = \alpha$ and $\beta_1 = \frac{\alpha}{1+\alpha}$ so that following Gompertz equation with diffusion is obtained:

$$u_t = u_{xx} + u \ln\left(\frac{1}{u+\alpha} + \frac{\alpha}{1+\alpha}\right), \quad x \in \mathbb{R},\ t > 0. \tag{1.17}$$

It is noted that due to the choice $\beta_1 = \frac{\alpha}{1+\alpha}$ in the nonlinear term

$$u \ln\left(\frac{1}{u+\alpha} + \beta_1\right),$$

$u = 0$ and $u = 1$ are the two zeros of $u \ln\left(\frac{1}{u+\alpha} + \beta_1\right)$, that is, $f(0) = f(1) = 0$.

The remainder of this paper is organized as follows. In Section 2, some well-known results, including the existence of traveling wave solutions for the Fisher equation as well as minimal speed of traveling wave solutions are presented without proof. Applying the results in Section 2, Section 3 is devoted to the existence and stability of traveling wave solutions for (1.17). In addition, minimal speed of traveling wave solutions for (1.17) is given there. By combining the method of super- and subsolutions developed in [6, 15, 16] with Theorem 4.2 in [21], we show in Section 4 the existence of monotone traveling wave solutions which connect the two equilibrium points $(0,\gamma)$ and (u^*,v^*) at infinity for (1.14), under the condition $c_{21} < \gamma < 1/c_{12}$. Finally, we conclude the present paper with some remarks in Section 5.

2. Preliminaries

The following proposition asserts the existence of front solutions for the nonlinear diffusion equation and the estimate of the minimal speed for which the front solutions exist.

PROPOSITION 2.1. (*Existence*) *There exists a* $c^* > 0$ *such that if* $c \geqslant c^*$, *the following nonlinear diffusion equation*

$$u_t = u_{xx} + f(u), x \in \mathbb{R}, t > 0 \tag{2.1}$$

with the Fisher-type nonlinear term satisfying

$$f \in \mathscr{C}^1([0,1]),\ f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0,\ f(u) > 0 \text{ for all } u \in (0,1) \tag{2.2}$$

has a unique (up to a translation) monotonic traveling wave solution

$$u(x,t) = U(x - ct), \quad U(-\infty) = 1, \quad U(\infty) = 0, \tag{2.3}$$

where c *is the propagating speed.*

PROPOSITION 2.2. (*Minimal speed*) *The speed of propagation* c *in Proposition* 2.1 *has a lower bound* c^* *which is bounded above and below by* [1]

$$2\sqrt{f'(0)} \leqslant c^* \leqslant 2\sup\left\{ \sqrt{\frac{f(u)}{u}} \,\middle|\, u \in (0,1) \right\}. \tag{2.4}$$

Let $c_0 = 2\sqrt{f'(0)}$, *then we have* [17]

- *If* $\frac{2}{u^2}\int_0^u f(s)\,ds \leqslant f'(0)$ *for all* $u \in (0,1]$ $\Rightarrow$ $c^* = c_0$ *(linear selection)*
- *If* $\frac{1}{2}\int_0^1 f(u)\,du \geqslant f'(0)$ $\Rightarrow$ $c^* > c_0$ *(nonlinear selection)*

For the case $c^* > c_0$, c^* *is characterized by the following variational formulation* [2]

$$c^* = \sup\left\{ \frac{2\int_0^1 \sqrt{fgh}\,du}{\int_0^1 g\,du} \,\middle|\, g \in \mathfrak{D} \right\}, \tag{2.5}$$

where $\mathfrak{D}$ *is the space defined by*

$$\mathfrak{D} = \left\{ g \in \mathscr{C}^1\big((0,1)\big) \mid g \geqslant 0, h \equiv -g' > 0 \text{ in } (0,1), \int_0^1 g(u)\,du < \infty \right\}, \tag{2.6}$$

or [5]

$$c^* = \inf_{\rho \in \mathfrak{B}} \sup_{0<u<1} \left\{ \rho'(u) + \frac{f(u)}{\rho(u)} \right\}, \tag{2.7}$$

where $\mathfrak{B}$ *is the space defined by*

$$\mathfrak{B} = \left\{ \rho \in \mathscr{C}^1\big([0,1]\big) \mid \rho > 0 \text{ in } (0,1),\ \rho(1) = 0,\ \rho'(1) < 0 \right\}. \tag{2.8}$$

3. Gompertz Growth Equation

In this section, the existence of traveling wave solutions for (1.17) and minimal speed can be established using the results in Section 2.

3.1. Existence of traveling wave solutions and minimal speed

THEOREM 3.1. (*Existence*) *Suppose that* $\alpha > 0$ *is a constant. There exists a* $c^* > 0$ *such that if* $c \geqslant c^*$, *then the following equation with generalized Gompertz growth*

$$u_t = u_{xx} + u \ln\left(\frac{1}{u+\alpha} + \frac{\alpha}{1+\alpha}\right), \quad x \in \mathbb{R},\ t > 0$$

has a unique (up to a translation) monotonic traveling front solution $u(x,t) = U(x-ct)$ *with* $U(\infty) = 0$ *and* $U(-\infty) = 1$.

Proof. Let $f(u) = u \ln(\frac{1}{u+\alpha} + \frac{\alpha}{1+\alpha})$. Then it is easy to see the following properties of f hold:

- $f \in \mathscr{C}^1([0,1])$.
- $f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0$.
- $f(u) > 0,\ \forall\ u \in (0,1)$.

Thus, we have the desired result by Proposition 2.1.

THEOREM 3.2. (*Minimal speed*) *The minimal speed* c^* *in Theorem* 3.1 *is explicitly given as a function of* α *by*

$$c^* = 2\sqrt{\ln\left[1 + \frac{1}{\alpha\,(1+\alpha)}\right]}. \tag{3.1}$$

Proof. Let $f(u) = u \ln(\frac{1}{u+\alpha} + \frac{\alpha}{1+\alpha})$. Since a straightforward calculation shows that

$$f'(0) = \ln\left[1 + \frac{1}{\alpha\,(1+\alpha)}\right], \tag{3.2}$$

it suffices to prove by Proposition 2.2 that

$$\frac{2}{u^2}\int_0^u f(s)\,ds \leqslant f'(0) \text{ for all } u \in (0,1]. \tag{3.3}$$

Indeed, the difference $\frac{2}{u^2}\int_0^u f(s)\,ds - f'(0)$ is

$$\frac{2}{u^2}\int_0^u f(s)\,ds - f'(0)$$

$$
\begin{aligned}
&= \frac{1+\alpha}{u\alpha} - \ln\Big(\frac{1+\alpha+\alpha^2}{\alpha+\alpha^2}\Big) + \ln\Big(1 - \frac{1}{1+\alpha} + \frac{1}{u+\alpha}\Big) \\
&\quad + \frac{\alpha^2}{u^2}\ln(1+\frac{u}{\alpha}) + \frac{1}{u^2\alpha^2}\big(1+\alpha+\alpha^2\big)^2 \ln\Big[\frac{1+\alpha+\alpha^2}{1+\alpha\,(1+u+\alpha)}\Big] \\
&:= g(u,\alpha).
\end{aligned} \tag{3.4}
$$

It is easy to see that for all $\alpha > 0$

$$
\lim_{u\to 0} g(u,\alpha) = 0 \tag{3.5}
$$

and for all $\alpha > 0$, $u \in (0,1]$

$$
\begin{aligned}
\frac{\partial g(u,\alpha)}{\partial u} &= -\frac{2\alpha^2}{u^3}\ln\Big(1+\frac{u}{\alpha}\Big) - \frac{2}{u^2\alpha}(1+\alpha) \\
&\qquad - 2(1+\alpha+\alpha^2)\ln\Big(1+\frac{u\alpha}{1+\alpha+\alpha^2}\Big) < 0
\end{aligned} \tag{3.6}
$$

These facts imply that for all $\alpha > 0$, $u \in (0,1]$

$$
g(u,\alpha) < 0. \tag{3.7}
$$

This completes the proof of the theorem.

It is clear from Theorem 3.2 that $c^* > 0$ can be arbitrarily small if the parameter α is chosen sufficiently large. This property for the generalized Gompertz model is quite different from that for the Fisher's equation $u_t = u_{xx} + u(1-u)$. The minimal speed for which a monotonic front solution joining $u = 1$ with $u = 0$ exists is known to be 2.

REMARK 3.1. If f is concave, then the minimal speed $c^* = 2\sqrt{f'(0)}$ ([1], [14]).

3.2. Stability of Traveling Waves

We employ the technique adapted by Canosa [3] to study the stability of traveling waves for (1.17). It turns out that these traveling waves are asymptotically stable in some sense.

THEOREM 3.3. (Stability) *Under certain small perturbations, the traveling wave solutions in Theorem 3.1 are asymptotically stable.*

Proof. To show the desired result, rewrite (1.17) in the moving frame

$$
u_t = u_{zz} + c\,u_z + u\ln\Big(\frac{1}{u+\alpha} + \frac{\alpha}{1+\alpha}\Big), \quad z = x - ct, \quad t > 0. \tag{3.8}
$$

For simplicity, let $\beta = \frac{\alpha}{1+\alpha}$. Here a small perturbation $V(z,t)$ which vanishes outside a bounded interval $(-L,L)$, that is,

$$
V(z,t) = 0, \quad \text{for} \quad |z| \geqslant L, L > 0, \tag{3.9}
$$

is added to the traveling wave solution $U(z)$. Then $U(z)+V(z,t)$ is inserted to (3.8) to give

$$V_t = U_{zz} + V_{zz} + cU_z + cV_z + U \ln\left(\frac{1}{U+V+\alpha} + \beta\right) + V \ln\left(\frac{1}{U+V+\alpha} + \beta\right). \quad (3.10)$$

Since $U(z)$ is a traveling wave that solves

$$U_{zz} + cU_z + U \ln\left(\frac{1}{U+\alpha} + \beta\right) = 0, \quad (3.11)$$

it follows that (3.10) is reduced to

$$V_t = V_{zz} + cV_z + U \ln\left[\frac{U+\alpha+\beta(U+\alpha)(U+V+\alpha)}{U+V+\alpha+\beta(U+\alpha)(U+V+\alpha)}\right] + V \ln\left(\frac{1}{U+V+\alpha} + \beta\right), \quad (3.12)$$

of which the corresponding linearized equation is

$$V_t = V_{zz} + cV_z + \left[\ln\left(\frac{1}{U+\alpha} + \beta\right) - \gamma\right] V, \quad (3.13)$$

where $\gamma = \frac{U}{U+\alpha+\beta(U+\alpha)^2}$. We then look for a solution in the form

$$V(z,t) = v(z)e^{-\lambda t}, \quad (3.14)$$

where $v(z)$ satisfies

$$\begin{cases} v'' + cv' + \left[\lambda + \ln\left(\frac{1}{U+\alpha} + \beta\right) - \gamma\right] v = 0, \\ v(-L) = v(L) = 0, \end{cases} \quad (3.15)$$

and λ is the eigenvalue of the problem (3.15). To solve this eigenvalue problem, we apply the transformation $v(z) = w(z)e^{-\frac{c}{2}z}$ to get

$$\begin{cases} w'' + \left[\lambda - \left(\frac{c^2}{4} - \ln\left(\frac{1}{U+\alpha} + \beta\right) + \gamma\right)\right] w = 0, \\ w(-L) = w(L) = 0. \end{cases} \quad (3.16)$$

It is a well-known result that if $\frac{c^2}{4} - \ln\left(\frac{1}{U+\alpha} + \beta\right) + \gamma > 0$, then all the eigenvalues λ of (3.16) are positive. Indeed, by Theorem 3.2 we have

$$\frac{c^2}{4} - \ln\left(\frac{1}{U+\alpha} + \beta\right) + \gamma \geqslant \frac{c^{*2}}{4} - \ln\left(\frac{1}{U+\alpha} + \beta\right) + \gamma \quad (3.17)$$

$$= \ln\left[1 + \frac{1}{\alpha\,(1+\alpha)}\right] - \ln\left(\frac{1}{U+\alpha} + \frac{\alpha}{1+\alpha}\right) + \gamma$$
$$\geqslant \ln\left[1 + \frac{1}{\alpha\,(1+\alpha)}\right] - \ln\left(\frac{1}{\alpha} + \frac{\alpha}{1+\alpha}\right) + \gamma$$
$$= \gamma := \frac{U}{U+\alpha+\beta(U+\alpha)^2} > 0$$

since $0 \leqslant U(z) \leqslant 1$ for all $z \in \mathbb{R}$ and c^* is the minimal speed in Theorem 3.2. This establishes the stability of the traveling wave solutions which were previously shown in Theorem 3.1.

4. Traveling Waves in Competitive Gompertz Model of Two Species

Sometimes, a *competition* model can be transformed into a *cooperation* model by means of a suitable change of variables. This fact is essential in proving the following existence theorem for (1.15) under the monostable assumption $c_{21} < \gamma < 1/c_{12}$.

THEOREM 4.1. (Existence of waves under $c_{21} < \gamma < 1/c_{12}$) *Under the hypothesis $c_{21} < \gamma < 1/c_{12}$, (1.14) has a monotonic traveling wave solution $(u(x,t), v(x,t)) = (U(z), V(z))\,(z := x - ct)$ which joins $(U,V)(\infty) = (0,\gamma)$ with $(U,V)(-\infty) = (u^*, v^*)$ for any $c \geqslant 2\sqrt{\ln\frac{3}{2}}$, where c is the propagating speed of wave.*

Proof. System (1.15) is transformed into

$$\begin{cases} (u_1)_{zz} + c\,(u_1)_z + u_1 \ln\left(\dfrac{1}{u_1 + c_{12}(\gamma - u_2) + 1} + \dfrac{1}{2}\right) = 0, \\ (u_2)_{zz} + c\,(u_2)_z + (u_2 - \gamma) \ln\left(\dfrac{1}{c_{21}\,u_1 + (\gamma - u_2) + \alpha} + 1 - \dfrac{1}{\gamma + \alpha}\right) = 0, \quad z \in \mathbb{R}. \end{cases} \tag{4.1}$$

by the change of variables

$$\begin{cases} u_1(z) = u(z), \\ u_2(z) = \gamma - v(z), \quad z \in \mathbb{R}. \end{cases} \tag{4.2}$$

To construct upper solutions of (4.1), let $\bar{u}_1(z) = \bar{u}(z)$ and $\bar{u}_2(z) = \gamma\bar{u}(z)$, where $0 \leqslant \bar{u} \leqslant 1$ solves for any $c \geqslant 2\sqrt{\ln\frac{3}{2}}$

$$\begin{cases} \bar{u}_{zz} + c\,\bar{u}_z + \bar{u} \ln\left(\dfrac{1}{\bar{u}+1} + \dfrac{1}{2}\right) = 0, \quad z \in \mathbb{R}, \\ \bar{u}(-\infty) = 1, \quad \bar{u}(\infty) = 0. \end{cases} \tag{4.3}$$

Consider $(\bar{u}_1, u_2)$ and $(u_1, \bar{u}_2)$, where u_1 satisfies $0 \leqslant u_1 \leqslant \bar{u}_1$ and u_2 satisfies $0 \leqslant u_2 \leqslant \bar{u}_2$. Then $(\bar{u}_1, u_2)$ and $(u_1, \bar{u}_2)$ form a pair of upper solutions for (4.1). Indeed, one readily verifies that due to $\gamma - u_2 \geqslant \gamma - \bar{u}_2 = \gamma(1-\bar{u}) \geqslant 0$, we have

$$\begin{aligned}
&(\bar{u}_1)_{zz} + c\,(\bar{u}_1)_z + \bar{u}_1 \ln\Big(\frac{1}{\bar{u}_1 + c_{12}\,(\gamma - u_2) + 1} + \frac{1}{2}\Big) \\
&= \bar{u}_1 \Big[\ln\Big(\frac{1}{\bar{u}_1 + c_{12}\,(\gamma - u_2) + 1} + \frac{1}{2}\Big) - \ln\Big(\frac{1}{\bar{u}_1 + 1} + \frac{1}{2}\Big)\Big] \leqslant 0. \qquad (4.4)
\end{aligned}$$

On the other hand, it is easy to verify that the following inequality is true.

$$\begin{aligned}
&(\bar{u}_2)_{zz} + c\,(\bar{u}_2)_z + (\bar{u}_2 - \gamma) \ln\Big(\frac{1}{c_{21}\,u_1 + (\gamma - \bar{u}_2) + \alpha} + 1 - \frac{1}{\gamma + \alpha}\Big) \\
&= \gamma(\bar{u}_{zz} + c\,\bar{u}_z) + (\gamma\bar{u} - \gamma) \ln\Big(\frac{1}{c_{21}\,u_1 + (\gamma - \gamma\bar{u}) + \alpha} + 1 - \frac{1}{\gamma + \alpha}\Big) \\
&= -\gamma\bar{u} \ln\Big(\frac{1}{\bar{u} + 1} + \frac{1}{2}\Big) - \gamma(1-\bar{u}) \ln\Big(\frac{1}{c_{21}\,u_1 + \gamma(1-\bar{u}) + \alpha} + 1 - \frac{1}{\gamma + \alpha}\Big) \leqslant 0 \qquad (4.5)
\end{aligned}$$

since by means of the fact $0 \leqslant \bar{u} \leqslant 1$ and the hypothesis of monostable condition $c_{21} < \gamma < 1/c_{12}$ we clearly have

$$\ln\Big(\frac{1}{\bar{u}+1} + \frac{1}{2}\Big) \geqslant \ln\Big(\frac{1}{1+1} + \frac{1}{2}\Big) = 0 \qquad (4.6)$$

and

$$\begin{aligned}
\ln\Big(\frac{1}{c_{21}\,u_1 + \gamma(1-\bar{u}) + \alpha} + 1 - \frac{1}{\gamma+\alpha}\Big) &\geqslant \ln\Big(\frac{1}{\gamma\bar{u} + \gamma(1-\bar{u}) + \alpha} + 1 - \frac{1}{\gamma+\alpha}\Big) \\
&= \ln\Big(\frac{1}{\gamma+\alpha} + 1 - \frac{1}{\gamma+\alpha}\Big) = 0. \qquad (4.7)
\end{aligned}$$

Let

$$F_1(u_1,u_2) = u_1 \ln\Big(\frac{1}{u_1 + c_{12}\,(\gamma - u_2) + 1} + \frac{1}{2}\Big), \qquad (4.8)$$

$$F_2(u_1,u_2) = (u_2 - \gamma) \ln\Big(\frac{1}{c_{21}\,u_1 + (\gamma - u_2) + \alpha} + 1 - \frac{1}{\gamma+\alpha}\Big). \qquad (4.9)$$

We clearly have for $i = 1,2$ that

$$F_i\Big(s, \frac{c_{21}}{2}s\Big) > 0 \qquad (4.10)$$

for sufficiently small $s > 0$. Indeed, it is readily to verify that

$$\begin{aligned}
F_1\Big(s, \frac{c_{21}}{2}s\Big) &= s \ln\Big(\frac{1}{s + c_{12}\,(\gamma - \frac{c_{21}}{2}s) + 1} + \frac{1}{2}\Big) \\
&= s \ln\Big(\frac{1}{s(1 - \frac{1}{2}c_{12}\,c_{21}) + c_{12}\,\gamma + 1} + \frac{1}{2}\Big) > 0 \qquad (4.11)
\end{aligned}$$

for sufficiently small $s > 0$. The last inequality above holds since by hypothesis $c_{21} < \gamma < 1/c_{12}$

$$s(1 - \frac{1}{2}c_{12}c_{21}) + c_{12}\gamma < 1 \tag{4.12}$$

for sufficiently small $s > 0$. Also, one readily verifies that

$$\begin{aligned} F_2\Big(s, \frac{c_{21}}{2}s\Big) &= (\frac{c_{21}}{2}s - \gamma)\ln\Big(\frac{1}{c_{21}s + (\gamma - \frac{c_{21}}{2}s) + \alpha} + 1 - \frac{1}{\gamma+\alpha}\Big) \\ &= (\frac{c_{21}}{2}s - \gamma)\ln\Big(\frac{1}{\frac{c_{21}}{2}s + \gamma + \alpha} + 1 - \frac{1}{\gamma+\alpha}\Big) > 0 \end{aligned} \tag{4.13}$$

for sufficiently small $s > 0$. Let $\overrightarrow{F} = (F_1, F_2)$, $\overrightarrow{w_+} = (0,0)$, $\overrightarrow{w_-} = (u^*, \gamma - v^*)$ and $\mathfrak{K}$ be the class which contains all the functions $\overrightarrow{\rho}(z) = (\rho_1(z), \rho_2(z))$ satisfying

$$\mathfrak{K} = \Big\{\overrightarrow{\rho} \in \mathscr{C}^2(-\infty,\infty) \mid \overrightarrow{\rho} \text{ is monotonically decreasing with } \lim_{z\to\pm\infty}\overrightarrow{\rho}(z) = \overrightarrow{w_\pm}\Big\}. \tag{4.14}$$

The existence of the upper solution $\overrightarrow{\rho} = (\bar{u}_1, \bar{u}_2)$ satisfying (4.4) and (4.5) gives that $c \geqslant \omega^*$, where

$$\omega^* = \inf_{\overrightarrow{\rho}\in\mathfrak{K}} \sup_{z,i}\Big\{\frac{\rho_i''(z) + F_i(\overrightarrow{\rho}(z))}{-\rho_i'(z)}\Big\}. \tag{4.15}$$

We can now employ Theorem 4.2 in [21] to obtain the existence of solution $\overrightarrow{u}(z) = (u_1(z), u_2(z))$ for (4.1) with $u_1(z)$, $u_2(z)$ monotonically decreasing in z for $-\infty < z < +\infty$ and $\lim_{z\to\pm\infty}\overrightarrow{u}(z) = \overrightarrow{w_\pm}$. Finally, set

$$\begin{cases} u(z) = u_1(z), \\ v(z) = \gamma - u_2(z), \quad z \in \mathbb{R}. \end{cases} \tag{4.16}$$

Then $(u(z), v(z))$ is a solution of (1.15) with $(u,v)(+\infty) = (0,\gamma)$ and $(u,v)(-\infty) = (u^*, v^*)$. This completes the proof.

Changing the roles of u and v, we obtain the following analogous result.

COROLLARY 4.2. (*Existence of waves under the monostable condition*) *Under the hypothesis* $c_{21} < \gamma < 1/c_{12}$, (1.14) *has a monotonic traveling wave solution*

$$(u(x,t), v(x,t)) = (U(z), V(z)) \ (z := x - ct),$$

which joins $(U,V)(\infty) = (1,0)$ *with* $(U,V)(-\infty) = (u^*, v^*)$ *for any* $c \geqslant 2\sqrt{\ln\frac{3}{2}}$, *where* c *is the propagating speed of wave.*

REMARK 4.1. From the proof of Theorem 4.1, we see that in fact $(\bar{u}_1, \bar{u}_2)$ also forms a pair of upper solutions for (1.15) in the sense that

$$\begin{cases} (\bar{u}_1)_{zz} + c\,(\bar{u}_1)_z + \bar{u}_1 \ln\left(\dfrac{1}{\bar{u}_1 + c_{12}\,(\gamma - \bar{u}_2) + 1} + \frac{1}{2}\right) \leqslant 0, \\ (\bar{u}_2)_{zz} + c\,(\bar{u}_2)_z + (\bar{u}_2 - \gamma) \ln\left(\dfrac{1}{c_{21}\,\bar{u}_1 + (\gamma - \bar{u}_2) + \alpha} + 1 - \dfrac{1}{\gamma + \alpha}\right) \leqslant 0, \quad z \in \mathbb{R}. \end{cases} \tag{4.17}$$

(4.1) is a cooperative system, which is verified by a straightforward calculation.

5. Concluding Remarks

Under the condition $c_{21} < \gamma < 1/c_{12}$, we proved in Theorem 4.1 the existence of monotone traveling wave solutions which connect the two equilibrium points $(0, \gamma)$ and (u^*, v^*) at infinity for (1.14). For the existence of traveling wave solutions under the symmetric cases $\gamma < \min(c_{21}, 1/c_{12})$ and $\gamma > \max(c_{21}, 1/c_{12})$, it suffices to consider only one of the two cases. Therefore, an open problem to be solved is the existence of traveling wave solutions for (1.14) under the two symmetric cases together with the bistable case $1/c_{12} < \gamma < c_{21}$. On the other hand, the method of super- and subsolutions we adapted to show the existence was originally developed in [6, 15, 16]. Accordingly, it is shown in this paper that this method has broader application, in particular, it can be applied under the non-polynomial Gompertz nonlinearities. In addition, further investigations also include how to establish the stability of the traveling wave solutions.

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