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EXISTENCE RESULTS FOR SOME NONLOCAL PROBLEMS

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Abstract. We consider nonlocal elliptic boundary value problems of the form

$$-\mathrm{div}\,(\mathscr{A}(x,u)\nabla u) = \lambda f(u)$$

with Dirichlet boundary conditions where $\mathscr{A}$ is a nonlocal function. We prove existence of nontrivial positive solutions if the graph of the non linear function f is of single positive loop type. Methods of approximation and Schauder fixed point theorem are the main tools to be used here.

1. Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^d$ with smooth boundary. We denote by $\mathscr{A}$, a nonlocal function defined on $\Omega \times L^p(\Omega)$, $p \geqslant 1$, with values in $\mathbb{R}$ such that

$$x \mapsto \mathscr{A}(x,u) \text{ is measurable } \forall u \in L^p(\Omega). \tag{1.1}$$

There exists two constants a_0, a_∞ such that

$$0 < a_0 \leqslant \mathscr{A}(x,u) \leqslant a_\infty \quad \text{a.e. } x \in \Omega,\ \forall u \in L^p(\Omega). \tag{1.2}$$

We further assume that the operator

$$\tilde{T} : L^p(\Omega) \to L^\infty(\Omega)$$

defined by,

$$\tilde{T}(u)(x) = \mathscr{A}(x,u) \quad \text{is continuous.} \tag{1.3}$$

The operator $\tilde{T}$ makes sense because of (1.2).

We are interested in the following problem

$$\begin{cases} -\mathrm{div}\,(\mathscr{A}(x,u)\nabla u) = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where λ is a positive parameter. We will consider existence results for the problem (1.4) when f satisfies one of the following two conditions:

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(a) Let $\theta > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function satisfying

$$\begin{aligned} & f'(0) > 0, \\ & f(t) > 0 \ \forall t \in (0,\theta) \quad \text{and} \quad f(t) = 0 \ \text{otherwise}, \\ & t \mapsto f(t)/t \ \text{is strictly decreasing in} \ (0,\theta]. \end{aligned} \tag{1.5}$$

A typical example of such function would be $f(t) = \sin t$ on $0 \leqslant t \leqslant \pi$ or 0 otherwise.

(b) $f(t) = t$. Clearly in this case we are considering an eigenvalue problem for the nonlocal operator.

Here $\partial\Omega$ denotes the boundary of Ω. Clearly the nature of the problem (1.4) is non-variational. In [4], *Chipot* and *Corrêa* have studied the existence results of the problem

$$-\mathscr{A}(x,u) \triangle u = \lambda f(u), \tag{1.6}$$

with Dirichlet boundary condition. Recently in [11], *Chipot* and *Roy* proved the existence of n solutions if the nonlinear function f has n positive loops, for the same problem (2.16). They also considered asymptotic behavior of the soultions as the parameter $\lambda \to \infty$. However due to lack of maximum principle (which was available in [4] and [11]) for our operator, the situation becomes much more difficult. Similar problems in a local framework were well studied, we refer to [3, 13, 15, 16]. Problems of such kind in nonlocal settings were considered in [4, 5, 6, 8, 9, 10, 12]. Similar issues were also studied in the frame work of asymptotic behavior of parabolic equations (see [6] and [8]).

The paper is organized as follows. In the next section we prove existence of nontrivial solution with f satisfying conditions in (1.5). At the end of this section we will present various examples of local and nonlocal operators $\mathscr{A}$ from the point of view of application. In the last section we study eigenvalue problem for the same nonlocal operator.

2. Some generalizations

In this section we study existence of nontrivial solution for the problem (1.4) when f satisfies (1.5). The condition $t \mapsto \frac{f(t)}{t}$ is strictly decreasing is generally assumed to get existence of unique solution for semilinear problems [1].

Let $g : [0,\theta] \to \mathbb{R}$ be any strictly decreasing function such that $g(\theta) = 0$. Define $f(t) := tg(t)$. It can be easily checked that such a f satisfies the conditions in (1.5).

A solution of (1.4) is understood in weak sense, i.e. a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \mathscr{A}(x,u)\nabla u \cdot \nabla \phi = \lambda \int_\Omega f(u)\phi, \quad \forall \phi \in H_0^1(\Omega). \tag{2.1}$$

Our main result of this section is the following:

THEOREM 1. *Under the assumptions* (1.1)*,* (1.2)*,* (1.3) *and the first condition in* (1.5) *then the problem* (1.4) *admits a positive solution if* $\lambda > \frac{a_\infty \lambda_1}{f'(0)}$.

The theorem will be proved as a consequense of several Lemmas. We recall some basic definitions and properties about mollifiers.

Define for all $u \in L^p(\Omega)$,

$$\mathscr{A}_n(x,u) = \mathscr{A}(x,u) * \psi_{\frac{1}{n}},$$

where $\mathscr{A}(x,u)$ is extended by a_0 outside Ω, $\psi_{\frac{1}{n}}$ is the standard mollifier and "$*$" denotes the operation of mollification. From the definition of the operation of mollification

$$\mathscr{A}_n(x,u) := \int_{B(0,\frac{1}{n})} \mathscr{A}(x-y,u)\psi_{\frac{1}{n}}(y)dy = \int_{\Omega} \mathscr{A}(y,u)\psi_{\frac{1}{n}}(x-y)dy.$$

Let us recall the definition of standard mollifier $\psi_{\frac{1}{n}}$. Define $\psi \in C^\infty(\mathbb{R}^d)$ by

$$\psi(x) := \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geqslant 1, \end{cases}$$

the constant C is chosen such that $\int_{\mathbb{R}^d} \psi = 1$. For each positive integer n, set

$$\psi_{\frac{1}{n}}(x) = n^d \psi(nx).$$

LEMMA 1. *For each* $u \in L^p(\Omega)$ *it holds* $\forall n$,

$$a_0 \leqslant \mathscr{A}_n(x,u) \leqslant a_\infty \quad a.e.\ x \in \Omega. \tag{2.2}$$

Proof. By definition of $\mathscr{A}_n$,

$$\mathscr{A}_n(x,u) = \int_{B(0,\frac{1}{n})} \mathscr{A}(x-y,u)\psi_{\frac{1}{n}}(y)dy \leqslant a_\infty \int_{B(0,\frac{1}{n})} \psi_{\frac{1}{n}}(y)dy = a_\infty.$$

As $\mathscr{A}$ is extended by a_0 outside Ω, the other inequality also holds similarly.

LEMMA 2. *For each fixed* n *and* $x \in \Omega$*, the mapping* $u \to \mathscr{A}_n(x,u)$ *is continuous from* $L^p(\Omega)$ *to* $\mathbb{R}$.

Proof. Let $w_m \to w$ in $L^p(\Omega)$, then for fixed x and n,

$$\begin{aligned} |\mathscr{A}_n(x,w_m) - \mathscr{A}_n(x,w)| &\leqslant \int_{\Omega} |\mathscr{A}(y,w_m) - \mathscr{A}(y,w)|\psi_{\frac{1}{n}}(x-y)dy \\ &\leqslant |\Omega|||\psi_{\frac{1}{n}}||_\infty\, ||\mathscr{A}(x,w_m) - \mathscr{A}(x,w)||_\infty. \end{aligned}$$

The lemma then follows from (1.3).

Consider the problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}_n(x,u)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

First we will prove existence of nontrivial solution for the above problem and then pass through the limit as $n \to \infty$, to get existence results for the problem (2.1).

For fixed $w \in L^2(\Omega)$, consider the problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}_n(x,w)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

For fixed $w \in L^2(\Omega)$, consider the energy functional $J_w^n : H_0^1(\Omega) \to \mathbb{R}$ associated to (2.4), given by

$$J_w^n[u] = \frac{1}{2}\int_\Omega \mathscr{A}_n(x,w)|\nabla u|^2 - \lambda \int_\Omega F(u),$$

where $F(t) = \int_0^t f(s)ds$. Put

$$m_w^n = \inf_{u \in H_0^1(\Omega)} J_w^n[u]. \tag{2.5}$$

From the standard results of calculus of variation, we know that m_w^n is attained by some function $u_w^n \in H_0^1(\Omega)$, that is

$$m_w^n = J_w^n[u_w^n] \tag{2.6}$$

and the same function solves (2.4) weakly. The function u_w^n may not be unique.

Fix an $\varepsilon > 0$ small enough such that $f'(0) - \varepsilon > 0$. Such a choice of ε is possible since it is assumed that $f'(0) > 0$. Again using $f'(0) > 0$, $u_1 > 0$ and that $u_1 \in L^\infty(\Omega)$ it is possible to find small $t_\varepsilon > 0$ such that

$$f(t_\varepsilon u_1) \geqslant (f'(0) - \varepsilon)t_\varepsilon u_1 \tag{2.7}$$

where u_1 is as in (3.2).

LEMMA 3. *If $\delta > 0$ be a fixed small positive number and $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \varepsilon}$. Then $\forall n$ and $\forall w \in L^2(\Omega)$,*

$$-\lambda F(\theta)|\Omega| \leqslant m_w^n \leqslant -\frac{\delta t_\varepsilon^2}{2} \tag{2.8}$$

where $\varepsilon, t_\varepsilon$ is as in (2.7) and $|\Omega|$ denotes the d- dimensional Lebesgue measure of the set Ω.

Proof. For fixed $u \in H_0^1(\Omega)$, we have from (1.5) and the definition of F,

$$F(u) \leqslant F(\theta).$$

Since $\mathscr{A}_n > 0$, we have

$$J_w^n[u] \geqslant -\lambda F(\theta)|\Omega|.$$

Now the left hand side inequality in (2.8) follows since u is arbitrary in the above inequality. For the other side of the inequality, first we estimate the term $F(t_\varepsilon u_1)$ by using (1.5) and (2.7).

$$F(t_\varepsilon u_1) = \int_0^{t_\varepsilon u_1} f(s)ds = \int_0^{t_\varepsilon u_1} \frac{f(s)}{s} s ds \geqslant \frac{f(t_\varepsilon u_1)}{t_\varepsilon u_1} \int_0^{t_\varepsilon u_1} s ds$$
$$\geqslant \frac{f(t_\varepsilon u_1) t_\varepsilon u_1}{2} \geqslant (f'(0) - \varepsilon) \frac{t_\varepsilon^2 u_1^2}{2}. \quad (2.9)$$

Using $u = t_\varepsilon u_1$ in (2.5) along with (2.9), we get

$$m_w^n \leqslant J_w^n[t_\varepsilon u_1] \leqslant \frac{a_\infty t_\varepsilon^2}{2} \int_\Omega |\nabla u_1|^2 - \lambda \int_\Omega F(t_\varepsilon u_1) \leqslant \frac{\lambda_1 a_\infty t_\varepsilon^2}{2} - \frac{\lambda t_\varepsilon^2}{2} (f'(0) - \varepsilon).$$

Now if we choose $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \varepsilon}$, we have

$$m_w^n \leqslant -\frac{\delta t_\varepsilon^2}{2}.$$

This finishes the proof of the lemma.

REMARK 1. Last lemma says us that for $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \varepsilon}$, u_w^n is nontrivial, since $m_w^n < 0$.

LEMMA 4. *If* $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \varepsilon}$ *holds then*

$$0 < u_w^n \leqslant \theta, \quad \text{a.e. } x \in \Omega,$$

where u_w^n *is as in* (2.6).

Proof. Since $f \geqslant 0$ and $\lambda > 0$ we have

$$\operatorname{div}(\mathscr{A}_n(x,w)\nabla u_w^n) \leqslant 0.$$

Hence from Strong maximum principle we have, either $u_w^n > 0$ a.e. $x \in \Omega$ or $u \equiv 0$. From the last remark, we know that u_w^n is nontrivial. Hence we can conclude that $u_w^n > 0$ a.e. $x \in \Omega$.
For the other side of the inequality, let us assume that $u_w^n > \theta$ on a set of positive measure in Ω. Define $v \in H_0^1(\Omega)$ by

$$v = u_w^n \wedge \theta$$

where $a \wedge b = \min\{a,b\}$. Clearly $0 \leqslant v \leqslant \theta$ a.e. $x \in \Omega$. Moreover

$$
\begin{aligned}
J_w^n[v] = \frac{1}{2}\left\{\int_{\{u_w^n \leqslant \theta\}} \mathscr{A}_n(x,w)|\nabla u_w^n|^2 + \int_{\{u_w^n \geqslant \theta\}} \mathscr{A}_n(x,w)|\nabla \theta|^2\right\} \\
- \lambda\left\{\int_{\{u_w^n \leqslant \theta\}} F(u_w^n) + \int_{\{u_w^n > \theta\}} F(\theta)\right\} \\
= \frac{1}{2}\int_\Omega \mathscr{A}_n(x,w)|\nabla u_w^n|^2 - \lambda\int_\Omega F(u_w^n) - \frac{1}{2}\int_{\{u_w^n > \theta\}} \mathscr{A}_n(x,w)|\nabla u_w^n|^2 \\
+ \lambda\int_{\{u_w^n > \theta\}} \{F(u_w^n) - F(\theta)\}.
\end{aligned}
$$

Since $F(t) = F(\theta)$ for all $t \geqslant \theta$, we have

$$
J_w^n[v] \leqslant J_w^n[u_w^n] - \frac{1}{2}\int_{\{u_w^n > \theta\}} \mathscr{A}_n(x,w)|\nabla u_w^n|^2 < J_w^n[u_w^n],
$$

which contradicts (2.5).

REMARK 2. One should note that if u is any nontrivial solution of (2.4) then $u > 0$ a.e. $x \in \Omega$ holds from maximum principle.

It is to be noted that if u is any solution of (2.4), then $u \in L^\infty(\Omega)$, follows from elliptic regularity theory [14]. Next lemma is a well known result, we refer to **[2]**. We will use this lemma without a proof.

LEMMA 5. *Let u_1 and u_2 be two distinct non trivial solutions of* (2.4)*, then $\frac{u_1}{u_2}$ and $\frac{u_2}{u_1}$ are in $L^\infty(\Omega)$.*

LEMMA 6. *There exists at most one nontrivial solution to* (2.4).

Proof. Let $u_1, u_2 \in H_0^1(\Omega)$ be two nontrivial solutions of (2.4). Fix $\varepsilon > 0$. Using $\phi_1 = (u_1^2 - u_2^2)/(u_1+\varepsilon) \in H_0^1(\Omega)$ in the Euler-Lagrange equation of u_1, we get

$$
\int_\Omega \mathscr{A}_n(x,w)\nabla u_1 \cdot \nabla\phi_1 = \lambda\int_\Omega f(u_1)\phi_1. \tag{2.10}
$$

Similarly, using $\phi_2 = (u_1^2 - u_2^2)/(u_2+\varepsilon) \in H_0^1(\Omega)$, in the Euler-Lagrange equation of u_2, we obtain

$$
\int_\Omega \mathscr{A}_n(x,w)\nabla u_2 \cdot \nabla\phi_2 = \lambda\int_\Omega f(u_2)\phi_2. \tag{2.11}
$$

Explicit calculations in (2.10) gives

$$
\int_\Omega \mathscr{A}_n(x,w)\nabla u_1 \cdot \left\{\frac{(u_1+\varepsilon)(2u_1\nabla u_1 - 2u_2\nabla u_2) - (u_1^2 - u_2^2)\nabla u_1}{(u_1+\varepsilon)^2}\right\}
$$

$$= \lambda \int_\Omega f(u_1) \frac{u_1^2 - u_2^2}{u_1 + \varepsilon}. \quad (2.12)$$

Similarly, from (2.11) it follows

$$\int_\Omega \mathscr{A}_n(x,w) \nabla u_2 \cdot \left\{ \frac{(u_2+\varepsilon)(2u_1\nabla u_1 - 2u_2 \nabla u_2) - (u_1^2 - u_2^2)\nabla u_2}{(u_2+\varepsilon)^2} \right\}$$
$$= \lambda \int_\Omega f(u_2) \frac{u_1^2 - u_2^2}{u_2 + \varepsilon}. \quad (2.13)$$

Subtracting the right hand side of (2.13) from the right hand side of (2.12), we get

$$\int_\Omega \mathscr{A}_n(x,w) \nabla u_1 \cdot \left\{ \frac{(u_1+\varepsilon)(2u_1\nabla u_1 - 2u_2 \nabla u_2) - (u_1^2 - u_2^2)\nabla u_1}{(u_1+\varepsilon)^2} \right\}$$
$$- \int_\Omega \mathscr{A}_n(x,w) \nabla u_2 \cdot \left\{ \frac{(u_2+\varepsilon)(2u_1\nabla u_1 - 2u_2 \nabla u_2) - (u_1^2 - u_2^2)\nabla u_2}{(u_2+\varepsilon)^2} \right\}$$
$$= \int_\Omega \mathscr{A}_n(x,w) |\nabla u_1|^2 \left\{ \frac{u_1^2 + u_2^2 + 2\varepsilon u_1}{(u_1+\varepsilon)^2} \right\} + \int_\Omega \mathscr{A}_n(x,w) |\nabla u_2|^2 \left\{ \frac{u_1^2 + u_2^2 + 2\varepsilon u_1}{(u_2+\varepsilon)^2} \right\}$$
$$- 2\int_\Omega \mathscr{A}_n(x,w) \nabla u_1 \cdot \nabla u_2 \left\{ \frac{u_1}{u_2+\varepsilon} + \frac{u_2}{u_1+\varepsilon} \right\}$$
$$= \int_\Omega \mathscr{A}_n(x,w) |\nabla u_1|^2 \left\{ 1 + \frac{u_2^2}{(u_1+\varepsilon)^2} - \frac{\varepsilon^2}{(u_1+\varepsilon)^2} \right\}$$
$$+ \int_\Omega \mathscr{A}_n(x,w) |\nabla u_2|^2 \left\{ 1 + \frac{u_1^2}{(u_2+\varepsilon)^2} - \frac{\varepsilon^2}{(u_2+\varepsilon)^2} \right\}$$
$$- 2\int_\Omega \mathscr{A}_n(x,w) \nabla u_1 \cdot \nabla u_2 \left\{ \frac{u_1}{u_2+\varepsilon} + \frac{u_2}{u_1+\varepsilon} \right\}$$
$$= \int_\Omega \mathscr{A}_n(x,w) \left\{ \left| \nabla u_1 - \frac{u_1}{u_2+\varepsilon} \nabla u_2 \right|^2 + \left| \nabla u_2 - \frac{u_2}{u_1+\varepsilon} \nabla u_1 \right|^2 \right\}$$
$$- \varepsilon^2 \int_\Omega \mathscr{A}_n(x,w) \left\{ \frac{|\nabla u_1|^2}{(u_1+\varepsilon)^2} + \frac{|\nabla u_2|^2}{(u_2+\varepsilon)^2} \right\} \quad (2.14)$$

Subtracting (2.13) from (2.12), we get

$$\int_\Omega \mathscr{A}_n(x,w) \left\{ \left| \nabla u_1 - \frac{u_1}{u_2+\varepsilon} \nabla u_2 \right|^2 + \left| \nabla u_2 - \frac{u_2}{u_1+\varepsilon} \nabla u_1 \right|^2 \right\}$$
$$- \varepsilon^2 \int_\Omega \mathscr{A}_n(x,w) \left\{ \frac{|\nabla u_1|^2}{(u_1+\varepsilon)^2} + \frac{|\nabla u_2|^2}{(u_2+\varepsilon)^2} \right\}$$
$$= \lambda \int_\Omega \left\{ \frac{f(u_1)}{u_1+\varepsilon} - \frac{f(u_2)}{u_2+\varepsilon} \right\} (u_1^2 - u_2^2). \quad (2.15)$$

Let us denote by $\mathscr{L}_\varepsilon$ and $\mathscr{R}_\varepsilon$ the left and the right hand side of (2.15) respectively. We want to show that,

$$\lim_{\varepsilon \to 0} \mathscr{L}_\varepsilon = \lim_{\varepsilon \to 0} \mathscr{R}_\varepsilon = 0.$$

First note that

$$\mathscr{L}_\varepsilon \geqslant -\int_\Omega \mathscr{A}_n(x,w)g_\varepsilon(x), \tag{2.16}$$

where

$$g_\varepsilon(x) = \varepsilon^2 \left\{ \frac{|\nabla u_1|^2}{(u_1+\varepsilon)^2} + \frac{|\nabla u_2|^2}{(u_2+\varepsilon)^2} \right\}.$$

Clearly $g_\varepsilon \geqslant 0$ in Ω and $g_\varepsilon \to 0$ point wise. For each fixed $x \in \Omega$ as $u_1(x),\ u_2(x) > 0$, we have

$$g_\varepsilon(x) \leqslant |\nabla u_1(x)|^2 + |\nabla u_2(x)|^2.$$

Since $|\nabla u_1(x)|^2 + |\nabla u_2(x)|^2 \in L^1(\Omega)$, we can apply dominated convergence theorem to get

$$\lim_{\varepsilon\to 0} \int_\Omega \mathscr{A}_n(x,\ w)g_\varepsilon(x) = 0.$$

Hence from (2.16),

$$\liminf_{\varepsilon\to 0} \mathscr{L}_\varepsilon \geqslant 0.$$

Set $\mathscr{R}_\varepsilon = \lambda I_\varepsilon + \lambda J_\varepsilon$, where

$$I_\varepsilon = \int_{\{u_1>u_2\}} \left\{ \frac{f(u_1)}{u_1+\varepsilon} - \frac{f(u_2)}{u_2+\varepsilon} \right\} (u_1^2 - u_2^2)$$

and

$$J_\varepsilon = \int_{\{u_1\leqslant u_2\}} \left\{ \frac{f(u_1)}{u_1+\varepsilon} - \frac{f(u_2)}{u_2+\varepsilon} \right\} (u_1^2 - u_2^2).$$

Using (1.5) we estimate I_ε from above,

$$\begin{aligned} I_\varepsilon &= \int_{\{u_1>u_2\}} \left\{ \frac{f(u_1)}{u_1+\varepsilon} - \frac{f(u_2)}{u_2+\varepsilon} \right\} (u_1^2 - u_2^2) \\ &\leqslant \varepsilon \int_{\{u_1>u_2\}} \frac{f(u_2)}{u_2} \left\{ \frac{(u_1-u_2)(u_1^2-u_2^2)}{(u_1+\varepsilon)(u_2+\varepsilon)} \right\} \\ &\leqslant \varepsilon L \int_{\{u_1>u_2\}} \frac{u_1^3+u_2^3}{(u_1+\varepsilon)(u_2+\varepsilon)} \\ &\leqslant \varepsilon L \int_{\{u_1>u_2\}} \frac{u_1^2}{u_2} + \varepsilon L \int_{\{u_1>u_2\}} \frac{u_2^2}{u_1} \\ &\leqslant \varepsilon L|\Omega|(||\frac{u_1}{u_2}||_\infty ||u_1||_\infty + ||\frac{u_2}{u_1}||_\infty ||u_2||_\infty). \end{aligned}$$

Since the right hand side goes to 0 as $\varepsilon \to 0$. We have

$$\limsup_{\varepsilon\to 0} I_\varepsilon \leqslant 0.$$

Using a similar argument for J_ε, it can be shown that

$$\limsup_{\varepsilon\to 0} J_\varepsilon \leqslant 0.$$

Combining the last two inequality, we have

$$\limsup_{\varepsilon \to 0} \mathscr{R}_\varepsilon \leqslant 0$$

and hence

$$\lim_{\varepsilon \to 0} \mathscr{L}_\varepsilon = \lim_{\varepsilon \to 0} \mathscr{R}_\varepsilon = 0.$$

This means that

$$\lim_{\varepsilon \to 0} \int_\Omega \left\{ \frac{f(u_1)}{u_1+\varepsilon} - \frac{f(u_2)}{u_2+\varepsilon} \right\} (u_1^2 - u_2^2) = 0. \tag{2.17}$$

Let us now consider the sequence $h_\varepsilon(x) = \left\{ \frac{f(u_1)}{u_1+\varepsilon} - \frac{f(u_2)}{u_2+\varepsilon} \right\} (u_1^2 - u_2^2)$. For any fixed $x \in \Omega$ one has

$$h_\varepsilon(x) \to \left\{ \frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2} \right\} (u_1^2 - u_2^2), \quad \text{as } \varepsilon \to 0. \tag{2.18}$$

Using $f(0) = 0$ and the Lipschitz continuity of f, we get

$$\begin{aligned} |h_\varepsilon(x)| \leqslant \left\{ \frac{f(u_1)}{u_1+\varepsilon} + \frac{f(u_2)}{u_2+\varepsilon} \right\} (u_1^2 + u_2^2) &\leqslant \left\{ \frac{f(u_1)}{u_1} + \frac{f(u_2)}{u_2} \right\} (u_1^2 + u_2^2) \\ &\leqslant 2L(||u_1||_\infty^2 + ||u_2||_\infty^2), \end{aligned}$$

where L denotes the Lipschitz constant of f. Hence from dominated convergence theorem, we obtain

$$\int_\Omega \left\{ \frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2} \right\} (u_1^2 - u_2^2) = 0$$

which is possible if and only if $u_1 = u_2$ *a.e.* $x \in \Omega$. This concludes the proof of the lemma.

REMARK 3. From the last theorem we know that (2.4) has an unique nontrivial solution. Also from Lemma 3 we have u_w^n is a nontrivial solution of (2.4), for $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \varepsilon}$. Hence u_w^n is the only nontrivial solution of the problem (2.4).

THEOREM 2. *For $\lambda > \frac{a_\infty \lambda_1 + \delta}{f'(0) - \varepsilon}$ the problem* (2.3) *admits a positive solution.*

Proof. Define the set

$$\mathscr{K} = \left\{ u \in L^2(\Omega) \;\middle|\; 0 \leqslant u \leqslant \theta \text{ } a.e. \text{ } x \in \Omega \right\}.$$

Clearly $\mathscr{K}$ is a closed convex subset of $L^2(\Omega)$. Fix $w \in \mathscr{K}$. Define the map $T : \mathscr{K} \to L^2(\Omega)$ as

$$T(w) = u_w^n$$

where u_w^n is as in the last remark.

1. 0 **does not belong to** $T(\mathscr{K})$**.**

The claim follows from the definition of T.

2. T **maps** $\mathscr{K}$ **to** $\mathscr{K}$**.**

This claim is a consequence of Lemma 4.

3. Continuity of T**.**

Let $\{w_k\}_k \subset \mathscr{K}$ be such that

$$w_k \to w \quad \text{in } L^2(\Omega). \tag{2.19}$$

The Euler-Lagrange equation associated to (2.4) satisfied by w_k is given by

$$\int_\Omega \mathscr{A}_n(x,w_k)\nabla T(w_k)\cdot\nabla v = \lambda\int_\Omega f(T(w_k))v, \quad \forall v \in H_0^1(\Omega). \tag{2.20}$$

Taking $v = T(w_k)$ in (2.20), we obtain using Hölder and Poincaré inequality

$$\left(\int_\Omega |\nabla T(w_k)|^2\right)^{\frac{1}{2}} \leqslant \frac{\lambda||f||_\infty}{a_0}\sqrt{\frac{|\Omega|}{\lambda_1}}$$

where λ_1 is as in (3.2). Thus the sequence $\{T(w_k)\}_k$ is bounded in $H_0^1(\Omega)$, hence there exists a function $p \in H_0^1(\Omega)$ such that up to a subsequence $\{w_{k_m}\}_m$ of $\{w_k\}_k$, we have

$$\begin{aligned} T(w_{k_m}) &\to p && \text{in } L^2(\Omega), \\ T(w_{k_m}) &\rightharpoonup p && \text{in } H_0^1(\Omega), \\ \nabla T(w_{k_m}) &\rightharpoonup \nabla p && \text{in } L^2(\Omega). \end{aligned} \tag{2.21}$$

First we show that p is nontrivial. From Lemma 3 we have

$$\frac{1}{2}\int_\Omega \mathscr{A}_n(x,w_{m_k})|\nabla T(w_{m_k})|^2 - \lambda\int_\Omega F(T(w_{m_k})) \leqslant -\frac{t_\varepsilon^2\delta}{2}.$$

Using $\mathscr{A}_n \geqslant a_0$, we have

$$\frac{a_0}{2}\int_\Omega |\nabla T(w_{m_k})|^2 - \lambda\int_\Omega F(T(w_{m_k})) \leqslant -\frac{t_\varepsilon^2\delta}{2}.$$

Using the lower semi continuity for the weak convergence of H_0^1 norm and the continuity of F, we have

$$\frac{a_0}{2}\int_\Omega |\nabla p|^2 - \lambda\int_\Omega F(p) \leqslant -\frac{t_\varepsilon^2\delta}{2} < 0.$$

This proves that p cannot be trivial. Now considering the left hand side of (2.20), we have

$$\int_\Omega \mathscr{A}_n(x,w_{k_m})\nabla T(w_{k_m})\cdot\nabla v$$

$$= \int_\Omega \{\mathscr{A}_n(x,w_{k_m}) - \mathscr{A}_n(x,w)\}\nabla T(w_{k_m})\cdot\nabla v + \int_\Omega \mathscr{A}_n(x,w)\nabla T(w_{k_m})\cdot\nabla v$$
$$:= I_1^m + I_2^m.$$

We first estimate the term I_1^m.

$$|I_1^m| \leqslant \int_\Omega |\mathscr{A}_n(x,w_{k_m}) - \mathscr{A}_n(x,w)||\nabla T(w_{k_m})||\nabla v|$$
$$\leqslant \left(\int_\Omega |\mathscr{A}_n(x,w_{k_m}) - \mathscr{A}_n(x,w)|^2|\nabla v|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla T(w_{k_m})|^2\right)^{\frac{1}{2}}.$$

Now using $\int_\Omega |\nabla T(w_{k_m})|^2 \leqslant C^2$, where $C = \frac{\lambda||f||_\infty}{a_0}\sqrt{\frac{|\Omega|}{\lambda_1}}$, we get

$$|I_1^m| \leqslant C\left(\int_\Omega |\mathscr{A}_n(x,w_{k_m}) - \mathscr{A}_n(x,w)|^2|\nabla v|^2\right)^{\frac{1}{2}}. \tag{2.22}$$

From (2.19) we have

$$w_n \to w \text{ a.e. } x\in\Omega.$$

Since $w_n \leqslant \theta$ forall n, this implies from Lebesgue theorem that

$$w_n \to w \quad \text{in } L^p(\Omega),\ \forall p.$$

Therefore from (3.1) we have

$$\mathscr{A}_n(x,w_{k_m}) \to \mathscr{A}_n(x,w) \quad \text{a.e. } x\in\Omega.$$

Also

$$|\mathscr{A}_n(x,w_{k_m}) - \mathscr{A}_n(x,w)|^2|\nabla v|^2 \leqslant 4a_\infty^2|\nabla v|^2, \quad \forall v\in H_0^1(\Omega).$$

Now since $4a_\infty^2|\nabla v|^2 \in L^1(\Omega)$, we can pass trough the limit in (2.22) using dominated convergence theorem to get

$$I_1^m \to 0.$$

Also by (2.21),

$$I_2^m \to \int_\Omega \mathscr{A}_n(x,w)\nabla p\cdot\nabla v.$$

Therefore

$$\int_\Omega \mathscr{A}_n(x,w_{k_m})\nabla T(w_{k_m})\cdot\nabla v \to \int_\Omega \mathscr{A}_n(x,w)\nabla p\cdot\nabla v.$$

Using Lipschitz continuity of f and (2.20), we have

$$\int_\Omega f(T(w_{k_m}))v \to \int_\Omega f(p)v.$$

Therefore we have

$$\int_\Omega \mathscr{A}_n(x,w)\nabla p\cdot\nabla v = \lambda\int_\Omega f(p)v, \qquad \forall v\in H_0^1(\Omega).$$

Hence from the above equation we get $T(w)=p$ and since the possible limit is unique we have,

$$T(w_k)\to T(w),\quad \text{in } L^2(\Omega).$$

This completes the proof of continuity of T.

4. Compactness of T

Let $w_n\to w$ in $L^2(\Omega)$. We want to show that

$$T(w_k)\to T(w)\ \text{ in } H_0^1(\Omega).$$

Compactness of the mapping T then follows from the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$. The Euler-Lagrange equation satisfied by $T(w_k)$ is

$$\int_\Omega \mathscr{A}_n(x,w_k)\nabla T(w_k)\cdot\nabla v=\lambda\int_\Omega f(T(w_k))v,\quad \forall v\in H_0^1(\Omega). \tag{2.23}$$

That is

$$\begin{aligned}\int_\Omega \mathscr{A}_n(x,w_k)\nabla(T(w_k)-T(w))\cdot\nabla v+\int_\Omega(\mathscr{A}_n(x,w_k)-\mathscr{A}_n(x,w))\nabla T(w)\cdot\nabla v\\ =\lambda\int_\Omega f(T(w_k))v-\int_\Omega \mathscr{A}_n(x,w)\nabla T(w)\cdot\nabla v=\lambda\int_\Omega\{f(T(w_k))-f(T(w))\}v.\end{aligned}$$

Using $v=T(w_k)-T(w)$, (1.2) and Lipschitz continuity of f, we have

$$\begin{aligned}a_0\int_\Omega|\nabla(T(w_k)-T(w))|^2\leqslant \lambda L\int_\Omega|T(w_k)-T(w)|^2\\ +\int_\Omega|\mathscr{A}_n(x,w_k)-\mathscr{A}_n(x,w)||\nabla(T(w_k)-T(w))||\nabla T(w)|.\end{aligned} \tag{2.24}$$

Application of Young's inequality gives

$$\begin{aligned}a_0\int_\Omega|\nabla(T(w_k)-T(w))|^2\leqslant \lambda L\int_\Omega|T(w_k)-T(w)|^2\\ +\frac{a_0}{2}\int_\Omega|\nabla(T(w_k)-T(w))|^2+\frac{2}{a_0}\int_\Omega|\mathscr{A}_n(x,w_k)-\mathscr{A}_n(x,w)|^2|\nabla T(w)|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{a_0}{2}\int_\Omega|\nabla(T(w_k)-T(w))|^2\leqslant \lambda L\int_\Omega|T(w_k)-T(w)|^2\\ +\frac{2}{a_0}\int_\Omega|\mathscr{A}_n(x,w_k)-\mathscr{A}_n(x,w)|^2|\nabla T(w)|^2.\end{aligned}$$

The first integral on the RHS of the above inequality tends to 0 from the last part and the second integral converges to 0, following a similar argument, that shows the convergence of the I_m^1 in (2.22).

Schauder fixed point theorem.

The map $T:\mathscr{K}\to\mathscr{K}$ is compact and $\mathscr{K}$ is closed, convex set in $L^2(\Omega)$. By Schauder fixed point theorem T has a fixed point. Since the function 0 doesn't belongs to $T(\mathscr{K})$, the above obtained fixed point is nontrivial. This finishes the proof of the theorem.

Let u_n denotes the nontrivial solution obtained for the problem (2.3) for large λ. In the above theorem it should be noted that the choice of λ doesn't depends on n. Now the goal is to pass through the limit in (2.3) and obtain a nontrivial solution for the problem (1.4).

Proof of Theorem 1

First of all it is clear that

$$\mathscr{A}_n(x,u)\to\mathscr{A}(x,u)$$

for each fixed $x\in\Omega$ and $u\in L^2(\Omega)$. This follows from the property of mollification. The equation satisfied by u_n is written as, for fixed $\phi\in H_0^1(\Omega)$,

$$\int_\Omega \mathscr{A}_n(x,u_n)\nabla u_n\cdot\nabla\phi=\lambda\int_\Omega f(u_n)\phi. \tag{2.25}$$

Using $\phi=u_n$ in (2.25), we get

$$\int_\Omega \mathscr{A}_n(x,u_n)|\nabla u_n|^2=\lambda\int_\Omega f(u_n)u_n.$$

Since $\mathscr{A}_n\geqslant a_0$, we have

$$a_0\int_\Omega|\nabla u_n|^2\leqslant\lambda||f||_\infty\int_\Omega u_n\leqslant\lambda||f||_\infty|\Omega|^{\frac12}\left(\int_\Omega|u_n|^2\right)^{\frac12}.$$

Now using Poincaré's inequality, we get

$$\left(\int_\Omega|\nabla u_n|^2\right)^{\frac12}\leqslant\frac{\lambda||f||_\infty|\Omega|^{\frac12}}{a_0\sqrt{\lambda_1}}.$$

Thus for a subsequence, which we again denote by $\{n\}$, there exist $u_0\in H_0^1(\Omega)$ such that

$$u_n\rightharpoonup u_0\quad\text{in } H_0^1(\Omega)$$

and strongly in $L^2(\Omega)$. The theorem will be proved if we show that $u_0\in L^p(\Omega)$, $\forall p\geqslant 1$, nontrivial and for fixed ϕ, the following holds

$$\int_\Omega\mathscr{A}(x,u_0)\nabla u_0\cdot\nabla\phi=\lambda\int_\Omega f(u_0)\phi.$$

For all n, one has from Lemma 4 that

$$0<u_n\leqslant\theta,\ \text{a.e. } x\in\Omega.$$

This implies that $0 \leqslant u_0 \leqslant \theta$ a.e. $x \in \Omega$ from almost every where convergence of u_n to u_0. Hence $u_0 \in L^p(\Omega),\ \forall p \geqslant 1$.

Now let us start from the left hand side of (2.25).

$$\int_\Omega \mathscr{A}_n(x,u_n)\nabla u_n \cdot \nabla\phi = \int_\Omega \big\{\mathscr{A}_n(x,u_n) - \mathscr{A}_n(x,u_0)\big\}\nabla u_n \cdot \nabla\phi \\ + \int_\Omega \big\{\mathscr{A}_n(x,u_0) - \mathscr{A}(x,u_0)\big\}\nabla u_n \cdot \nabla\phi + \int_\Omega \mathscr{A}(x,u_0)\nabla u_n \cdot \nabla\phi \\ := I_n^1 + I_n^2 + I_n^3.$$

Clearly from the weak convergence of u_n to u_0, we have

$$I_n^3 \to \int_\Omega \mathscr{A}(x,u_0)\nabla u_0 \cdot \nabla\phi.$$

We claim that both I_n^1 and I_n^2 converges to 0. First we will estimate the term I_n^1.

$$|I_n^1| \leqslant \int_\Omega |\mathscr{A}_n(x,u_n) - \mathscr{A}_n(x,u_0)||\nabla u_n||\nabla\phi| \\ \leqslant \int_\Omega \left(\int_{B(0,\frac{1}{n})} \big|\mathscr{A}(x-y,u_n) - \mathscr{A}(x-y,u_0)\big| \psi_{\frac{1}{n}}\, dy \right) |\nabla u_n|\ |\nabla\phi|.$$

Using $|\mathscr{A}(x-y,u_n) - \mathscr{A}(x-y,u_0)| \leqslant ||\mathscr{A}(x,u_n) - \mathscr{A}(x,u_0)||_\infty$ and $\int_{B(0,\frac{1}{n})} \psi_{\frac{1}{n}} = 1$, we get

$$|I_n^1| \leqslant ||\mathscr{A}(x,u_n) - \mathscr{A}(x,u_0)||_\infty \int_\Omega |\nabla u_n||\nabla\phi| \\ \leqslant ||\mathscr{A}(x,u_n) - \mathscr{A}(x,u_0)||_\infty\ ||\nabla u_n||_{L^2}||\nabla\phi||_{L^2} \\ \leqslant C||\nabla\phi||_{L^2}||\mathscr{A}(x,u_n) - \mathscr{A}(x,u_0)||_\infty,$$

where $C = \frac{\lambda ||f||_\infty |\Omega|^{\frac{1}{2}}}{a_0\sqrt{\lambda_1}}$.

Now as $u_n \to u_0$ in $L^p(\Omega)$, this implies from (1.3) that

$$||\mathscr{A}(x,u_n) - \mathscr{A}(x,u_0)||_\infty \to 0$$

and hence

$$I_n^1 \to 0.$$

Let us now estimate the term I_n^2.

$$|I_n^2| \leqslant \int_\Omega |\mathscr{A}_n(x,u_0) - \mathscr{A}(x,u_0)||\nabla\phi||\nabla u_n| \\ \leqslant ||\{\mathscr{A}_n(x,u_0) - \mathscr{A}(x,u_0)\}\,|\nabla\phi|||_{L^2}||\nabla u_n||_{L^2} \\ \leqslant C||\{\mathscr{A}_n(x,u_0) - \mathscr{A}(x,u_0)\}\,|\nabla\phi|||_{L^2}.$$

As mentioned above in the beginning of the proof, we have

$$\mathscr{A}_n(x,u_0) \to \mathscr{A}(x,u_0) \quad \text{a.e. } x \in \Omega.$$

Also

$$|\mathscr{A}_n(x,u_0) - \mathscr{A}(x,u_0)|^2 |\nabla\phi|^2 \leqslant 4a_\infty^2 |\nabla\phi|^2$$

where $|\nabla\phi|^2 \in L^1(\Omega)$. Therefore by dominated convergence theorem, we have

$$||\{\mathscr{A}_n(x,u_0) - \mathscr{A}(x,u_0)\}|\nabla\phi|||_{L^2} \to 0.$$

Thus we have proved that

$$\int_\Omega \mathscr{A}_n(x,u_n)\nabla u_n \cdot \nabla\phi \to \int_\Omega \mathscr{A}(x,u_0)\nabla u_0 \cdot \nabla\phi.$$

The right hand side of (2.25) can be written as

$$\int_\Omega f(u_n)\phi = \int_\Omega \{f(u_n) - f(u_0)\}\phi + \int_\Omega f(u_0)\phi.$$

Now as $n \to \infty$, we have

$$|\int_\Omega \{f(u_n) - f(u_0)\}\phi| \leqslant L\int_\Omega |u_n - u_0||\phi| \leqslant L||u_n - u_0||_{L^2}||\phi||_{L^2} \to 0.$$

Hence we have

$$\int_\Omega f(u_n)\phi \to \int_\Omega f(u_0)\phi.$$

The proof will be completed once we show u_0 is not identically equals to 0. For proving that we use the weak lower semi continuity of the H_0^1 norm, the Lipschitz continuity of F and the energy estimates done in Lemma 3. We have

$$a_0\int_\Omega |\nabla u_n|^2 - \lambda\int_\Omega F(u_n) \leqslant \int_\Omega \mathscr{A}_n(x,u_n)|\nabla u_n|^2 - \lambda\int_\Omega F(u_n) \leqslant -\frac{t_\varepsilon^2\delta}{2}.$$

Again since $u_n \rightharpoonup u_0$, we have

$$a_0\int_\Omega |\nabla u_0|^2 - \lambda\int_\Omega F(u_0) \leqslant \liminf_{n\to\infty} a_0\int_\Omega |\nabla u_n|^2 - \lambda\int_\Omega F(u_n).$$

Combining the last two equations we get

$$a_0\int_\Omega |\nabla u_0|^2 - \lambda\int_\Omega F(u_0) \leqslant -\frac{t_\varepsilon^2\delta}{2} < 0$$

which is impossible if u_0 identically vanishes. In particular, if u_0 is not trivial then it has to be strictly positive in Ω. This again follows from the maximum principle.

Since the choice of $\varepsilon,\ \delta > 0$ is kept arbitrary, this proves Theorem 1.

SOME APPLICATIONS

Now we turn to examine the kinds of $\mathscr{A}$ that is suitable to fulfill our assumptions. For now let $\mathscr{B}(x,u)$ denote a Carathéodory function, that is $\mathscr{B}$ is defined from $\Omega\times\mathbb{R}$ into $\mathbb{R}$ such that

$$\begin{aligned} x\mapsto\mathscr{B}(x,u) \quad &\text{is measurable} \quad \forall u\in\mathbb{R},\\ u\mapsto\mathscr{B}(x,u) \quad &\text{is Lipschitz continuous} \quad \text{a.e. } \forall x\in\Omega, \end{aligned}$$

with the Lipschitz constant independent of x, and satisfying for some positive constants

$$0<a_0\leqslant\mathscr{B}(x,u)\leqslant a_\infty \qquad \text{a.e. } x\in\Omega,\ \forall u\in\mathbb{R}.$$

At first we look at the population distribution model. Let

$$\mathscr{A}(x,u)=\mathscr{B}(x,\int_\Omega u). \tag{2.26}$$

If u denotes the density of population, then the total population is denoted by $\int_\Omega u$. One can also look at the total population of a sub region, by replacing $\int_\Omega u$ by

$$\int_{\Omega'}u \qquad \text{where } \Omega'\subset\Omega.$$

Then it is quite obvious that $\mathscr{A}(x,u)$ defined by (2.26) satisfies our assumptions.

One can also consider non locality of the type

$$\mathscr{A}(u)=a\left(\int_\Omega gu\right)$$

with Lipschitz continuous function a. From the point of view of application when

$$g=\frac{1}{|\Omega|}$$

with $|\Omega|$ denoting the Lebesgue measure of Ω and if $f(u)$ is replaced by a force term f in (1.4), then the global minimization of the appropriate energy functional corresponds to the displacement of an elastic membrane spanned along the boundary of Ω, and submitted to a force f. Uniqueness, non-uniqueness issues for such kind of operator is studied in [7].

Another important class of nonlocal operator that suits our criterion is as follows. If Ω is a domain of $A-type$, that is for fixed $0<r<diam(\Omega)$, there exists a constant $A>0$ such that $|\Omega(x,r)|\geqslant Ar^d$ where $\Omega(x,r)=\Omega\cap B(x,r)$. If a is a Lipschitz continuous function then the nonlocal operator defined by

$$\mathscr{A}(x,u)=a\left(\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)}u(y)dy\right)$$

also satisfies our criterion.

3. An eigenvalue problem.

In this section we will work with an weaker condition than (1.3). We assume the mapping

$$u \mapsto \mathscr{A}(x,u) \text{ is continuous from } L^p(\Omega) \text{ into } \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{3.1}$$

It is easy to check that (1.3) implies the above condition. We will restrict ourself to the case of $p = 2$.

Let λ_1 and u_1 denotes the first eigenvalue and first eigenfunction of the problem

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \\ u_1 > 0, \ \int_\Omega u_1^2 = 1. \end{cases} \tag{3.2}$$

The main result of this section is the following.

THEOREM 3. *Under the assumption* (1.1)-(1.2)*, the problem*

$$\begin{cases} -\operatorname{div}(\mathscr{A}(x,u)\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \int_\Omega u^2 = 1, \end{cases} \tag{3.3}$$

admits a nontrivial solution for some $\lambda = \lambda^*$. *Further* $\lambda^* \in [a_0\lambda_1, a_\infty\lambda_1]$, *where* λ_1 *is as in* (3.2).

Proof. Fix $w \in L^2(\Omega)$ and consider the following problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}(x,w)\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \int_\Omega u^2 = 1, \\ u > 0 \quad \text{a.e. } x \in \Omega. \end{cases} \tag{3.4}$$

The above problem is an eigenvalue problem for an elliptic operator in divergence form. From the standard results of elliptic theory, we can conclude that there exists unique $\lambda = \lambda_w^1$ and $u = u_w$ that solves (3.4), where λ_w^1 denotes the first eigenvalue and u_w is the corresponding first eigenfunction. It is also well known that λ_w^1 has the following characterisation,

$$\lambda_w^1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \mathscr{A}(x,w)|\nabla u|^2}{\int_\Omega u^2} \tag{3.5}$$

and $u_w > 0$ a.e. $x \in \Omega$.

1. Forall $w \in L^2(\Omega)$, **we have** $\lambda_w^1 \in [a_0\lambda_1, a_\infty\lambda_1]$.

From (1.2) we have

$$a_0 \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \leqslant \frac{\int_\Omega \mathscr{A}(x,w)|\nabla u|^2}{\int_\Omega u^2} \leqslant a_\infty \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

The claim then follows by taking infimum over the set $H_0^1(\Omega) \setminus \{0\}$.

Define the set

$$\mathscr{K} = \left\{ u \in L^2(\Omega) \middle| \ \int_\Omega |\nabla u|^2 \leqslant \frac{a_\infty \lambda_1}{a_0} \right\}.$$

The set $\mathscr{K}$ is a compact, convex subset of $L^2(\Omega)$. Define the map $T : \mathscr{K} \to L^2(\Omega)$ as

$$T(w) = u_w,$$

where u_w solves (3.4). Clearly any fixed point of T is a solution of the problem (3.3).

2. T maps $\mathscr{K}$ to $\mathscr{K}$.

Fix $w \in \mathscr{K}$. From (3.4) and (3.5) we have

$$\lambda_w^1 = \int_\Omega \mathscr{A}(x,w)|\nabla u_w|^2.$$

Now using the last claim and (1.2), we get $u_w \in \mathscr{K}$.

3. $T : \mathscr{K} \to \mathscr{K}$ is continuous.

Let $\{w_k\}_k \subset \mathscr{K}$ be such that

$$w_k \to w \quad \text{in } L^2(\Omega). \tag{3.6}$$

Since $T(w_k) \in \mathscr{K}$, the sequence $\{T(w_k)\}_k$ is bounded in $H_0^1(\Omega)$. Hence there exists a function $p \in H_0^1(\Omega)$ such that up to a subsequence $\{w_{k_m}\}_m$ of $\{w_k\}_k$, we can have

$$\begin{aligned} T(w_{k_m}) &\to p \quad &&\text{in } L^2(\Omega), \\ T(w_{k_m}) &\rightharpoonup p \quad &&\text{in } H_0^1(\Omega), \\ T(w_{k_m}) &\to p \quad &&\text{a.e. } x \in \Omega. \end{aligned} \tag{3.7}$$

Since $T(w_{k_m}) > 0$ a.e. $x \in \Omega$, it follows from the convergence above that

$$p \geqslant 0 \quad \text{a.e. } x \in \Omega$$

and $\int_\Omega p^2 = 1$. This implies that p cannot be a trivial function. Since $\lambda_{w_{k_m}}^1 \in [a_0\lambda_1, a_\infty\lambda_1]$, there exists a further subsequence $\left\{k_{m_j}\right\}_j$ of $\{k_m\}_m$, such that

$$\lambda_{w_{k_{m_j}}}^1 \to \lambda_w^*$$

where $\lambda_w^* \in [a_0\lambda_1, a_\infty\lambda_1]$. The Euler-Lagrange equation satisfied by $T(w_{k_{m_j}})$ is given by

$$\int_\Omega \mathscr{A}(x, w_{k_{m_j}}) \nabla T(w_{k_{m_j}}) \nabla v = \lambda_{w_{k_{m_j}}}^1 \int_\Omega T(w_{k_{m_j}}) v, \quad \forall v \in H_0^1(\Omega). \tag{3.8}$$

Consider the left hand side of (3.8),

$$\begin{aligned}\int_\Omega &\mathscr{A}(x,w_{k_{m_j}})\nabla T(w_{k_{m_j}})\nabla v\\ &= \int_\Omega \{\mathscr{A}(x,w_{k_{m_j}}) - \mathscr{A}(x,w)\}\nabla T(w_{k_{m_j}})\nabla v + \int_\Omega \mathscr{A}(x,w)\nabla T(w_{k_{m_j}})\nabla v\\ &:= I_1^j + I_2^j.\end{aligned}$$

We first estimate the term I_1^j.

$$\begin{aligned}|I_1^j| &\leqslant \int_\Omega |\mathscr{A}(x,w_{k_{m_j}}) - \mathscr{A}(x,w)||\nabla T(w_{k_{m_j}})||\nabla v|\\ &\leqslant \left(\int_\Omega |\mathscr{A}(x,w_{k_{m_j}}) - \mathscr{A}(x,w)|^2|\nabla v|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla T(w_{k_{m_j}})|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now using $\int_\Omega |\nabla T(w_{k_{m_j}})|^2 \leqslant \frac{a_\infty \lambda_1}{a_0}$ we get

$$|I_1^j| \leqslant \left(\frac{a_\infty\lambda_1}{a_0}\right)^{\frac{1}{2}}\left(\int_\Omega |\mathscr{A}(x,w_{k_{m_j}}) - \mathscr{A}(x,w)|^2|\nabla v|^2\right)^{\frac{1}{2}}. \tag{3.9}$$

From (3.1) and (3.6) we have

$$\mathscr{A}(x,w_{k_{m_j}}) \to \mathscr{A}(x,w) \quad \text{a.e. } x \in \Omega$$

and

$$|\mathscr{A}(x,w_{k_{m_j}}) - \mathscr{A}(x,w)|^2|\nabla v|^2 \leqslant 4a_\infty^2|\nabla v|^2, \quad \forall v \in H_0^1(\Omega).$$

Now since $4a_\infty^2|\nabla v|^2 \in L^1(\Omega)$, we can pass trough the limit in (3.9) using dominated convergence theorem to get

$$I_1^j \to 0.$$

From (3.7),

$$I_2^j \to \int_\Omega \mathscr{A}(x,w)\nabla p\nabla v.$$

Therefore

$$\int_\Omega \mathscr{A}(x,w_{k_{m_j}})\nabla T(w_{k_{m_j}})\nabla v \to \int_\Omega \mathscr{A}(x,w)\nabla p\nabla v.$$

From (3.7) it also follows that

$$\int_\Omega T(w_{k_{m_j}})v \to \int_\Omega pv.$$

Therefore we have

$$\int_\Omega \mathscr{A}(x,w)\nabla p\nabla v = \lambda_w^*\int_\Omega pv, \qquad \forall v \in H_0^1(\Omega).$$

It is well known [14] that the first eigenfunction of the problem (3.4) is its only solution, that has a strict sign almost everywhere. Since p is nontrivial and $p \geqslant 0$, it has to be the first eigenfunction and λ_w^* has to be the first eigenvalue (λ_w^1). Therefore

$$\int_\Omega \mathscr{A}(x,w)\nabla p \nabla v = \lambda_w^1 \int_\Omega pv, \qquad \forall v \in H_0^1(\Omega).$$

Hence $T(w) = p$ holds. Since the possible limit is unique, we have

$$T(w_k) \to T(w) \quad \text{in } L^2(\Omega).$$

This proves continuity of the map T.

4. Schauder fixed point theorem.

The map $T : \mathscr{K} \to \mathscr{K}$ is continuous where $\mathscr{K}$ is compact and convex subset of $L^2(\Omega)$. Therefore from Schauder fixed point theorem the map T has a fixed point, that is $T(z) = z$ for some $z \in \mathscr{K}$.

Non triviality of z follows since $\int_\Omega |T(w)|^2 = 1, \ \forall w \in \mathscr{K}$. It is also clear from the definition of T that $\lambda^* = \lambda_z^1$ and hence $\lambda^* \in [a_0\lambda_1, a_\infty\lambda_1]$. This finishes the proof of the theorem.

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