Differential
Equations
& Applications
Volume 6, Number 3 (2014), 383–401
doi:10.7153/dea-06-21

ON ENTIRE SOLUTIONS FOR AN INDEFINITE QUASILINEAR SYSTEM OF MIXED POWER

CARLOS ALBERTO SANTOS AND MARIANA RAMOS REIS

(*Communicated by Norio Yoshida*)

Abstract. We prove non-existence and existence of entire positive solutions for a Schrödinger quasilinear elliptic system. To prove the non-existence, we combine a carefully-chosen test function with some results that we proved concerning the positivity of a kind of principal eigenvalue of a eigenvalue problem in $\mathbb{R}^N$ with indefinite weights. Contrary to the existence, the non-existence results for this class of problems have not been studied very much in recent years. For the existence we mainly used upper and lower solution methods combined with comparison principles.

1. Introduction

In this paper, our main purpose is to establish non-existence and existence of solutions for the problem

$$\begin{cases} -\Delta_p u + m_1(x)u^{p-1} = a(x)u^{\beta_1} + \lambda b(x)u^{\gamma_1}v^{\delta_1} + f(x) \text{ in } \mathbb{R}^N \\ -\Delta_q v + m_2(x)v^{q-1} = c(x)v^{\beta_2} + \mu d(x)v^{\gamma_2}u^{\delta_2} + g(x) \text{ in } \mathbb{R}^N \\ u,v > 0 \text{ in } \mathbb{R}^N; \quad u(x),v(x) \stackrel{|x|\to\infty}{\longrightarrow} 0, \end{cases} \tag{1.1}$$

where Δ_r is the usual r-Laplacian operator with $1 < r = p,q < N$; $a,b,c,d,f,g : \mathbb{R}^N \to [0,\infty)$ are continuous functions; $m_1,m_2 : \mathbb{R}^N \to \mathbb{R}$ are continuous functions which can change signs; $\beta_i,\gamma_i,\delta_i$ for $i = 1,2$ are appropriate real constants; $N \geqslant 1$ and $\lambda,\mu \geqslant 0$ are real parameters.

In particular, we note that in our results the coefficients a,b,c and d can vanish in open sets of $\mathbb{R}^N$ and $m_i < 0$ will be permitted. This possibility can lead the principal part of the operator to have a non-coercive behavior.

A solution of (1.1) is understood as a pair of positive functions in $C^1(\mathbb{R}^N)$ converging to zero at infinity which satisfy the equations in (1.1) in the distributional sense.

This problem belongs to the class of cooperative systems, because $b,d \geqslant 0$ and $\delta_1,\delta_2 \geqslant 0$ too. The study of this class of problems is motivated by various nonlinear

Mathematics subject classification (2010): 35B08, 35B09, 35B53, 35J10, 35J57, 35J92.
Keywords and phrases: quasilinear system, existence, non-existence, positive solutions, entire solutions.
The first author acknowledges the support of PROCAD/UFG/UnB and FAPDF under grant PRONEX 193.000.580/2009.

Paper DEA-06-21

phenomena, for instance, in the non-Newtonian fluid theory [10], in the generalized reaction-diffusion theory [13], nonlinear elasticity [4] and references therein.

There is at present a large number of papers and an extensive literature studying the existence of positive solutions for multi-parameters problems by using several and different techniques. While the study of nonexistence of positive solutions for this class of multi-parameters problems including Laplacian and r-Laplacian operators both in bounded and unbounded domains has not been much intensive.

Problems like (1.1) for smooth bounded domains have been intensively studied in recent years. We quote for instance Giacomoni et al [12], Zou [32], Zhen [31], Yan and Lu [27], Chen and Lu [5], Clement et al [6], Velin [25], Ahammou [1] and references therein.

In these works, many and different techniques have been used to deal with them. For existence of solutions, the main tools that have been used are variational and topological methods and for the nonexistence the moving planes, moving spheres techniques and the classical test functions methods have been considered.

In our result of existence, the direct application of these techniques does not seem to be so appropriate, because of the possibility of the powers either be very small (negative) or very large (positive). So, we have principally used the lower-upper solutions technique and comparison principle results. To do this, we constructed an upper solution by appropriated parameters and we solved a sublinear problem with the Schrödinger operator in R^N whose solution is already built less than or equal to upper solution and it is a lower solution of problem (1.1) too.

For the non-existence, first we proved an important result about of the positivity of a kind of eigenvalue of an eigenvalue problem with Schrödinger operator in R^N. So, combining this result (Lemma 2.4 which was proved by using the classical Picone's inequality) and a test function carefully chosen in $C_0^\infty(\Omega)$, we were able to construct a contradiction for large parameters by assuming the existence of solutions.

An overview about nonexistence. With regards to powers of problem (1.1), Mitidieri [16] (for $\delta_1, \delta_2 > 1$) and Serrin and Zou [22] (for $\delta_1, \delta_2 > 0$) showed that problem

$$\begin{cases} -\Delta u = v^{\delta_1} \text{ in } \mathbb{R}^N \\ -\Delta v = u^{\delta_2} \text{ in } \mathbb{R}^N \\ u, v > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

has no radial solution, if $1/(\delta_1+1)+1/(\delta_2+1) > (N-2)/N$. More, it was proved in [23] that (1.2) admits infinitely many radial solutions provided that $1/(\delta_1+1)+1/(\delta_2+1) \leqslant (N-2)/N$.

That is, the labeled Hardy-Littlewwod-Sobolev hyperbola given by

$$\frac{1}{\delta_1+1}+\frac{1}{\delta_2+1}=\frac{N-2}{N}$$

is critical for existence and nonexistence of classical radial solutions. It is conjectured that it is critical for non-radial solutions too, but this conjecture has not still been solved completely.

Still in [16], it was proved that the problem

$$\begin{cases} -\Delta u = au^{\beta_1} + bv^{\delta_1} \text{ in } \mathbb{R}^N \\ -\Delta v = cv^{\beta_2} + du^{\delta_2} \text{ in } \mathbb{R}^N \\ u, v > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

has no classical solutions, if a,b,c,d are positive constants, $\beta_1,\beta_2,\delta_1,\delta_2 > 1$ and

$$\max\left\{\frac{1}{\beta_1 - 1}, \frac{\delta_1 + 1}{\delta_1\delta_2 - 1}, \frac{1}{\beta_2 - 1}, \frac{\delta_1 + 1}{\delta_2\delta_2 - 1}\right\} \geqslant \frac{N-2}{N}.$$

Later, De Figueiredo and Siracov [11] proved that the system (1.3) has no classical bounded solution, if

$$\delta_1 = \beta_1 \frac{\beta_2 - 1}{\beta_1 - 1} \text{ and } \delta_2 = \beta_2 \frac{\beta_1 - 1}{\beta_2 - 1}, \text{ where } 1 < \beta_1, \beta_2 < \frac{N+2}{N-2}.$$

In this same sense, Zhang and Zhu [30] showed non-existence of solutions for a problem like (1.3), where a,b,c,d are positive constants, $0 < \beta_i, \gamma_i, \delta_i, \gamma_i + \delta_i < (N+2)/(N-2)$ for $i = 1,2$.

Recently, Wang and Hong [26] improved these results by proving that if $(u,v) \in (C^2(\mathbb{R}^N))^2$ is a solution of (1.3), under the hypotheses that a,b,c,d are positive constants, $\beta_1,\beta_2,\delta_1,\delta_2 > 0$ and there exist $\rho,\nu \geqslant N-2$ such that $\rho + 4 \geqslant \max\{\beta_1\rho, \delta_1\nu\}$ and $\nu + 4 \geqslant \max\{\beta_2\nu, \beta_1\rho\}$, then $\beta_1 = \beta_2 = \delta_1 = \delta_2 = (N+2)/(N-2)$ and

$$u(x) = \left(\frac{c_1}{d + |x - \overline{x}|^2}\right)^{(N-2)/2} \text{ and } v(x) = \left(\frac{c_2}{d + |x - \overline{x}|^2}\right)^{(N-2)/2}, \ x \in \mathbb{R}^N,$$

for some $d > 0$, $\overline{x} \in \mathbb{R}^N$ and c_1, c_2 appropriate positive constants.

Motivated principally by the above works, we have proved some results that improve or complement the prior results. In order to state them, we are going to define the non-negative and continuous function

$$m(x) = \min\{a(x), b(x), c(x), d(x)\}, \ x \in \mathbb{R}^N$$

and we shall assume

(H_1) $\quad -\infty < \beta_1 < p - 1 \leqslant \gamma_1 + \delta_1 \quad$ (H_2) $\quad -\infty < \gamma_1 + \delta_1 \leqslant p - 1 < \beta_1$,

(K_1) $\quad -\infty < \beta_2 < p - 1 \leqslant \gamma_2 + \delta_2 \quad$ (K_2) $\quad -\infty < \gamma_2 + \delta_2 \leqslant p - 1 < \beta_2$.

So, our first result is:

THEOREM 1. *[Non−existence]: Suppose (H_i) and (K_j) for some $i, j \in \{1,2\}$, $m \neq 0$, m_i continuous functions, $p = q$ and $\delta_i \geqslant 0$ for $i = 1,2$ hold. Then, there exist $0 \leqslant \lambda^*, \mu^* < \infty$ such that the system (1.1) has no solution for every $(\lambda,\mu) > (\lambda^*,\mu^*)$ given, where the size of $\lambda^*, \mu^* \geqslant 0$ depends on m_1, m_2 and m behavior.*

To highlight the last result and dependance of $\lambda^*,\mu^* \geqslant 0$ of m_1,m_2 and m behavior (see, for instance (3.1) and (3.2)), we restate it in a particular case.

COROLLARY 1. *Assume that (H_i) and (K_j) hold for some $i,j\in\{1,2\}$, $m_1,m_2\geqslant 0$, the terms a,b,c and d satisfy $|x|^p a(x),|x|^p b(x),|x|^p c(x),|x|^p d(x)\to\infty$ when $|x|\to\infty$ and and $\delta_i\geqslant 0$ for $i=1,2$. Then, the problem*

$$\begin{cases}-\Delta_p u = m_1(x)u^{p-1}+a(x)u^{\beta_1}+\lambda b(x)u^{\gamma_1}v^{\delta_1} \text{ in } \mathbb{R}^N\\ -\Delta_p v = m_2(x)v^{p-1}+c(x)v^{\beta_2}+\mu d(x)v^{\gamma_2}u^{\delta_2} \text{ in } \mathbb{R}^N\\ u,v>0 \text{ in } \mathbb{R}^n;\quad u(x),v(x)\overset{|x|\to\infty}{\longrightarrow} 0\end{cases}$$

has no solution for all $\lambda,\mu>0$.

REMARK 1. Theorem 1 and Corollary 1 hold for the more general system

$$\begin{cases}-\Delta_p u+m_1(x)u^{p-1} = a(x)u^{\beta_1}v^{\theta_1}+\lambda b(x)u^{\gamma_1}v^{\delta_1}+f(x) \text{ in } \mathbb{R}^N\\ -\Delta_p v+m_2(x)v^{q-1} = c(x)v^{\beta_2}u^{\theta_2}+\mu d(x)v^{\gamma_2}u^{\delta_2}+g(x) \text{ in } \mathbb{R}^N\\ u,v>0 \text{ in } \mathbb{R}^n;\quad u(x),v(x)\overset{|x|\to\infty}{\longrightarrow} 0,\end{cases}$$

with $\theta_i,\delta_i\geqslant 0$ for $i=1,2$, β_1 replaced by $\beta_1+\theta_1$ at (H_1) and (H_2) and β_2 substituted by $\beta_2+\theta_2$ at (K_1) and (K_2), respectively and the potentials in x like in Theorem 1 and Corollary 1.

About the existence of solution for (1.1), recently Moussaoui, Khodja and Tas in [17] applied Schauder's fixed point theorem in a regularized system to prove the existence of solution for

$$\begin{cases}-\Delta u+m_1(x)u = b(x)v^{\delta_1} \text{ in } \mathbb{R}^N\\ -\Delta v+m_2(x)v = d(x)v^{\gamma_2}u^{\delta_2} \text{ in } \mathbb{R}^N\\ u,v>0 \text{ in } \mathbb{R}^N;\quad u(x),v(x)\overset{|x|\to\infty}{\longrightarrow} 0\end{cases}$$

with $\delta_1<0$, $-1<\gamma_2<0$, $\delta_2>0$ such that $\delta_2+\gamma_2\leqslant 1$ and the functions $b,d,m_1,m_2\in L^\infty_{loc}(\mathbb{R}^N)$ are such that $m_1(x)\geqslant\alpha_0$, $m_2(x)\geqslant\beta_0$ for $|x|\geqslant R$ for some $\alpha_0,\beta_0,R>0$ with $b,d\geqslant 0$.

For a quasilinear problem, Manouni and Touzani in [9] used the Mountain Pass Theorem and local minimization for showing the existence of solutions of the problem

$$\begin{cases}-\Delta_p u+m_1(x)u^{p-1} = b(x)u^{\gamma}v^{\delta+1} \text{ in } \mathbb{R}^N\\ -\Delta_q v+m_2(x)v^{q-1} = b(x)v^{\delta}u^{\gamma+1} \text{ in } \mathbb{R}^N\\ u,v>0 \text{ in } \mathbb{R}^N;\quad u(x),v(x)\overset{|x|\to\infty}{\longrightarrow} 0,\end{cases}$$

where $1<p,q<N$, $0<\gamma\leqslant p-1$, $0<\delta\leqslant q-1$, $\max\{(N-p)/N,(N-q)/N\}<(\gamma+1)/p^*+(\delta+1)/q^*<1$, $b\geqslant 0$, the functions m_1,m_2 are continuous functions which satisfy $m_1(x) \geqslant \alpha_0 > 0$, $m_2(x) \geqslant \beta_0 > 0$, $x \in \mathbb{R}^N$ and $m_1(x), m_2(x) \to \infty$ when $|x| \to \infty$.

Yin and Yan in [28], by using sub and super solutions methods, showed existence of solution for small parameters and non-existence for large ones for the following problem with parameters

$$\begin{cases} -\Delta_p u = a(x)u^{\beta_1} + \lambda c(x)v^{\delta} \text{ in } \mathbb{R}^N \\ -\Delta_q v = b(x)v^{\beta_2} + \mu c(x)u^{\delta} \text{ in } \mathbb{R}^N \\ u, v > 0 \text{ in } \mathbb{R}^N; \quad u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0, \end{cases}$$

where $-\infty < \beta_1 < p-1 < \delta$; $-\infty < \beta_2 < q-1 < \delta$ and $a, b, c \geqslant 0$ are appropriate continuous terms.

Serag and Zahrani in [21] have obtained the existence of solution for the problem

$$\begin{cases} -\Delta_p u + m_1(x)u^{p-1} = b(x)u^{\gamma}v^{\delta+1} + f(x) \text{ in } \mathbb{R}^N \\ -\Delta_q v + m_2(x)v^{q-1} = d(x)v^{\delta}u^{\gamma+1} + g(x) \text{ in } \mathbb{R}^N \\ u, v > 0 \text{ in } \mathbb{R}^N; \quad u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0. \end{cases}$$

They used the lower-upper solution method and considered $m_1, m_2 < 0$; $b, d > 0$; f, g non-negative appropriate functions and $\gamma, \delta > 0$ such that $(\gamma+1)/p + (\delta+1)/q = 1$.

Using Schauder's fixed point theorem, Leadi and Marcos in [14], have showed that the following system has at least one solution,

$$\begin{cases} -\Delta_p u + a_1 m_1(x)u^{p-1} = \lambda b(x)v^{\delta_1} + f(x) \text{ in } \mathbb{R}^N \\ -\Delta_q v + a_2 m_2(x)v^{q-1} = \mu d(x)u^{\delta_2} + g(x) \text{ in } \mathbb{R}^N \\ u, v > 0 \text{ in } \mathbb{R}^N; \; u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0, \end{cases}$$

where $m_1, m_2, b, d > 0$; $f, g \geqslant 0$ are suitable functions, $\delta_1, \delta_2 \geqslant 1$, $\delta_2/p + 1/q = 1$, $\delta_1/q + 1/p = 1$ and appropriate $\lambda, \mu \geqslant 0$ depending on a_1 and a_2.

Inspired in these results, we proved the next result that improves or complements the last results in some sense. To enunciate it, we shall define $M_f, M_g \geqslant 0$ by

$$M_f(x) := \max\{a(x), b(x), f(x)\}, \; x \in \mathbb{R}^N,$$
$$M_g(x) := \max\{c(x), d(x), g(x)\}, \; x \in \mathbb{R}^N$$

and assume that the problems

$$\begin{cases} -\Delta_p u + m_1^+(x)u^{p-1} = M_f(x) \text{ in } \mathbb{R}^N \\ u > 0 \text{ in } \mathbb{R}^N; \quad u(x) \overset{|x|\to\infty}{\longrightarrow} 0 \end{cases} \tag{1.4}$$

and

$$\begin{cases} -\Delta_q v + m_2^+(x)v^{q-1} = M_g(x) \text{ in } \mathbb{R}^N \\ v > 0 \text{ in } \mathbb{R}^N; \quad v(x) \overset{|x|\to\infty}{\longrightarrow} 0 \end{cases} \tag{1.5}$$

have solutions in $C^1(\mathbb{R}^N)$, where we are denoting by $m_i^+(x) = \max\{m_i(x),0\}$ and $m_i^-(x) = \max\{-m_i(x),0\}$ for $x \in \mathbb{R}^N$.

Additionally, we are going to assume that $m_1^-/M_f \in L^\infty(\mathbb{R}^N)$, if $m_1^- \neq 0$; $m_2^-/M_g \in L^\infty(\mathbb{R}^N)$, if $m_2^- \neq 0$; to denote by $\|\cdot\|_\infty$ the $L^\infty(\mathbb{R}^N)$-norm and to consider

$$(H_3) \qquad 1 - \left\| \frac{m_1^-}{M_f} \right\|_\infty \|w_1\|_\infty^{p-1} > 0 \text{ and } 1 - \left\| \frac{m_2^-}{M_g} \right\|_\infty \|w_2\|_\infty^{q-1} > 0,$$

where $w_1, w_2 \in C^1(\mathbb{R}^N)$ are the solutions of problems (1.4) and (1.5), respectively.

THEOREM 2. [*Existence*]: *Assume that* (1.4) *and* (1.5) *have solutions in* $C^1(\mathbb{R}^N)$, (H_3), $a, c \neq 0$, $\delta_1, \delta_2 \geqslant 0$, $-\infty < \beta_1 < p-1$ *and* $-\infty < \beta_2 < q-1$ *holds. Then, there exist* $(\lambda_*, \mu_*) > (0,0)$ *such that the problem* (1.1) *has at least one solution for each* $(0,0) < (\lambda, \mu) < (\lambda_*, \mu_*)$ *given. Besides this,*

(i) $\lambda_* = \infty$, *if* $\gamma_1 + \delta_1 < p-1$ *and* $\gamma_2 + \delta_2 \leqslant q-1$ *and*

(ii) $\mu_* = \infty$, *if* $\gamma_2 + \delta_2 < q-1$ *and* $\gamma_1 + \delta_1 \leqslant p-1$.

REMARK 2. We proved in the Appendix that problem (1.4) admits a solution in $C^1(\mathbb{R}^N)$, if for instance

$$\int_0^\infty \left[s^{1-N} \int_0^s t^{N-1} \hat{M}_f(t)dt \right]^{\frac{1}{p-1}} ds < \infty,$$

where $\hat{M}_f(t) = \max_{|x|=t} M_f(x)$, $t \geqslant 0$; $m_1^+, M_f \in C(\mathbb{R}^N)$ and $1 < p < N$. (The same result holds for problem (1.5), when m_2^+ and M_g satisfy similar hypotheses.)

This paper is organized into four sections. In Section 2, we give some results concerning the first eigenvalue of a problem with an indefinite weight. In particular, we prove that the positivity of this eigenvalue is related to the existence of solution of an inhomogeneuos problem. In Sections 3 and 4, we prove our main results. Finally, an auxiliary result is proved in Appendix 5.

2. Auxiliary Results

Due to our approach, to prove theorems (1) and (2), we are going to use the solution of the problem

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = \rho(x) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a smooth and bounded domain and $\rho, V : \Omega \to \mathbb{R}$ are suitable functions. As a particular case of a result in [18], we have.

LEMMA 1. *Suppose that* $\rho, V \in L^\infty(\Omega)$ *are non-negative functions with* $\rho \neq 0$ *and* $1 < p < N$. *Then there exists a unique* $u \in W_0^{1,p}(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$ *solution of* (2.1).

Also, we are going to use some facts concerning the following eigenvalue problem,

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = \lambda\rho(x)u^{p-1} \text{ in } \Omega \\ u > 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

where ρ, V are such that

(H_4) $\rho, V \in L^r(\Omega)$, where $r > N/p$ if $1 < p \leqslant N$ and $r = 1$ if $p > N$.

We are going to denote by $\Omega_+(\rho) = \{x \in \Omega \ / \ \rho(x) > 0\}$ and to introduce

$$\alpha_\Omega(V,\rho) = \inf\left\{ \int_\Omega (|\nabla u|^p + V|u|^p)dx \ / \ u \in W_0^{1,p}(\Omega), \right.$$
$$\left. \int_\Omega |u|^p dx = 1, \int_\Omega \rho|u|^p dx = 0 \right\}.$$

So, for V, ρ satisfying (H_4), we have $-\infty < \alpha_\Omega(V,\rho) \leqslant \infty$ with $\alpha_\Omega(V,\rho) < \infty$ if and only if $|\Omega_+(\rho)| < |\Omega|$ (see [7]). Besides this, $\alpha_{\Omega_2}(V,\rho) \leqslant \alpha_{\Omega_1}(V,\rho)$ if $\Omega_1 \subseteq \Omega_2$ for $\Omega_1, \Omega_2 \subseteq \Omega$ smooth bounded domains.

It follows from Cuesta and Quoirin in [7, Theorem 7], that

LEMMA 2. *Assume* ρ, V *satisfies* (H_4) *and* $\rho \geqslant 0$ *with* $|\Omega_+(\rho)| > 0$. *Then there exists a principal eigenvalue of* (2.2) *if and only if* $\alpha_\Omega(V,\rho) > 0$. *In this case, the principal eigenvalue is unique and is characterized by*

$$\lambda_{1,\Omega}(V,\rho) := \inf\left\{ \int_\Omega (|\nabla u|^p + V|u|^p)dx \ / \ u \in W_0^{1,p}(\Omega); \int_\Omega \rho|u|^p dx = 1 \right\} \in (-\infty, \infty).$$

Now, we establish a version of the [19, Theorem 1.2, Lemma 2.3] for our class of problems.

LEMMA 3. *Assume* (H_4), $\rho \geqslant 0$ *with* $|\Omega_+(\rho)| > 0$, $\alpha_\Omega(V,\rho) > 0$ *and that given a* $\lambda \in \mathbb{R}$ *there exists a* $0 < v = v_\lambda \in W_{loc}^{1,p}(\Omega)$ *such that* $-\Delta_p v + V v^{p-1} \geqslant \lambda\rho v^{p-1}$ *in* Ω *in the distributional sense. Then* $\lambda \leqslant \lambda_{1,\Omega}(V,\rho)$. *In particular, if there exists a* $0 < w \in W_{loc}^{1,p}(\Omega) \cap L^\infty(\Omega)$ *satisfying* $-\Delta_p w + V w^{p-1} \geqslant \rho$ *in* Ω, *then* $\lambda_{1,\Omega}(V,\rho) \geqslant \|w\|_{L^\infty(\Omega)}^{1-p}$.

Proof. Pick $\{\varphi_n\}_{n\in\mathbf{N}} \subset C_0^\infty(\Omega)$ with $\varphi_n \geqslant 0$ and $\varphi_n \to \phi_1$ in $W_0^{1,p}(\Omega)$, where $\phi_1 = \phi_{1,\Omega}(V,\rho) > 0$ is the first eigenfunction of the problem (2.2) associated to its first eigenvalue $\lambda_{1,\Omega}(V,a)$. So, applying Picone's identity (see [2]) and density arguments, we have

$$0 \leqslant \int_\Omega |\nabla\varphi_n|^p dx - \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\left(\frac{\varphi_n^p}{v^{p-1}}\right)dx$$

$$\leqslant \int_\Omega |\nabla \varphi_n|^p dx + \int_\Omega V \varphi_n^p dx - \int_\Omega \lambda \rho(x) \varphi_n^p dx.$$

Now, making $n \to \infty$, we have

$$\lambda_{1,\Omega}(V,\rho) \int_\Omega \rho \phi_1^p dx = \int_\Omega |\nabla \phi_1|^p dx + \int_\Omega V \phi_1^p dx \geqslant \lambda \int_\Omega \rho \phi_1^p dx,$$

that is, $\lambda \leqslant \lambda_{1,\Omega}(V,\rho)$ because ρ is non-negative and not identically zero. To finish our proof, we define for each $\tau > 0$, $v(x) = v_\tau(x) = \tau \|w\|^{-1}_{L^\infty(\Omega)} w(x)$, $x \in \Omega$. So, we have that $0 < v \leqslant \tau$ and

$$\begin{aligned}
&\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi dx + \int_\Omega V(x) v^{p-1} \varphi dx \\
&\qquad = \frac{\tau^{p-1}}{\|w\|^{p-1}_{L^\infty(\Omega)}} \left[\int_\Omega |\nabla w|^{p-2} \nabla w \nabla \varphi dx + \int_\Omega V(x) w^{p-1} \varphi dx \right] \\
&\qquad \geqslant \frac{1}{\|w\|^{p-1}_{L^\infty(\Omega)}} \int_\Omega \rho(x) \tau^{p-1} \varphi dx \geqslant \frac{1}{\|w\|^{p-1}_{L^\infty(\Omega)}} \int_\Omega \rho(x) v^{p-1} \varphi dx,
\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$. So, from the first part, with $\lambda = \|w\|^{1-p}_{L^\infty(\Omega)}$, the proof ends.

Now, given functions $\rho, V : \mathbb{R}^N \to \mathbb{R}$ such that ρ, V satisfy

$(H_4)'$ $\rho, V \in L^r_{loc}(\mathbb{R}^N)$, where $r > N/p$ if $1 < p \leqslant N$ and $r = 1$ if $p > N$,

we can define

$$\alpha(V,\rho) = \lim_{k\to\infty} \alpha_{B_k}(V,\rho) \in [-\infty, \infty],$$

because of the monotonicity of $\alpha_{B_k}(V,\rho)$ in $k = 1, 2, \cdots$, where B_k is the ball centered at the origin of the $\mathbb{R}^N$ with radius k. Besides this, if $\rho \geqslant 0$, $|\{x \in \mathbb{R}^N \ / \ \rho(x) > 0\}| > 0$ and $\alpha_{B_k}(V,\rho) > 0$ for all $k = 1, 2, \cdots$ we can let

$$\lambda_1(V,\rho) := \lim_{k\to\infty} \lambda_{1,B_k}(V,\rho) \in [-\infty, \infty) \tag{2.3}$$

because $\lambda_{1,B_{k+1}}(V,\rho) \leqslant \lambda_{1,B_k}(V,\rho)$.

REMARK 3. We have $\alpha(V,\rho) > 0$ if, for instance, $\rho > 0$ a.e. in $\mathbb{R}^N$, because $|(B_k)_+(\rho)| = |B_k|$ for every $k \in \mathbb{N}$. In particular, we have well-defined $\lambda_1(V,\rho)$.

REMARK 4. It follows from Lemmas 1 – 3 that $\lambda_1(V,\rho) \geqslant 0$ for $\rho, V \in C(\mathbb{R}^N)$ with $\rho, V \geqslant 0$, $\rho \neq 0$ and $1 < p < N$. In fact, in this case, we note that $\alpha_{B_k}(V,\rho) \geqslant \lambda_{1,B_k}(0,1) > 0$ for all $k \in \mathbb{N}$. Hence, there exists $\lambda_{1,B_k}(V,\rho)$ and it satisfies

$$\lambda_{1,B_k}(V,\rho) \geqslant \|w_k\|_{L^\infty(B_k)},$$

where w_k is the solution of (2.1) given by Lemma 2.1.

So, as an application the of the last results, we have that

LEMMA 4. *Assume* $(H_4)'$ *with* $\rho \geqslant 0$ *and* $|\{x \in \mathbb{R}^N \,/\, \rho(x) > 0\}| > 0$, $\alpha_{B_k}(V,\rho) > 0$ *for* $k = 1, 2, \cdots$ *and that given* $\lambda \in \mathbb{R}$ *there exists a* $0 < v = v_\lambda \in W^{1,p}_{loc}(\mathbb{R}^N)$ *satisfying* $-\Delta_p v + V v^{p-1} \geqslant \lambda \rho v^{p-1}$ *in* $\mathbb{R}^N$ *in distributional sense. Then,* $\lambda \leqslant \lambda_1(V,\rho)$. *In particular, if there exists a function* $0 < w \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *such that* $-\Delta_p w + V w^{p-1} \geqslant \rho$ *in* $\mathbb{R}^N$, *then* $\lambda_1(V,\rho) \geqslant \|w\|^{1-p}_{L^\infty(\mathbb{R}^N)}$.

In the sequel, we are going to use some of the last results to ensure the existence of solution for the problem

$$\begin{cases} -\Delta_p u + V(x) u^{p-1} = \rho(x) u^\beta \text{ in } \Omega \\ u > 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

So, we have

THEOREM 3. *Assume* $\rho, V \in L^\infty(\Omega)$ *are non-negative functions with* $\rho \neq 0$ *and* $-\infty < \beta < p-1 < N-1$ *hold. Then, there exists a solution* $u \in C^1(\Omega) \cap C(\overline{\Omega})$ *of* (2.4) *such that*

$$0 < \|u\|_\infty \leqslant max\left\{1, \left[\left(2 - \frac{\beta}{p-1}\right)^2 \|w_\rho\|_\infty\right]^{\frac{p-1}{p-1-\beta}}\right\},$$

where $w_\rho \in C^1(\Omega) \cap C(\overline{\Omega})$ *is the solution of* (2.1) *given by Lemma* 1.

Proof. We are going to follow an idea as in [19] or [20]. Defining

$$\Phi(s) = \frac{s^{2-\frac{\beta}{p-1}}}{\left(2 - \frac{\beta}{p-1}\right)^2 \tau_\infty}, \; s > 0,$$

where

$$\tau_\infty = max\left\{1, \left[\left(2 - \frac{\beta}{p-1}\right)^2 \|w_\rho\|_\infty\right]^{\frac{p-1}{p-1-\beta}}\right\}$$

and $v(x) = \Phi^{-1}(w_\rho(x))$, $x \in \overline{\Omega}$, we have $v \in C^1(\Omega) \cap C(\overline{\Omega})$, $0 < v(x) = \Phi^{-1}(w_\rho(x)) \leqslant \Phi^{-1}(\|w_\rho\|_\infty) \leqslant \tau_\infty$, $x \in \Omega$ and $v(x) = \Phi^{-1}(w_\rho(x)) = 0$ on $\partial\Omega$.

Besides this, given $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$, it follows from $(\Phi^{-1}(s))' > 0$ and $(\Phi^{-1}(s))'' < 0$, $s > 0$ that

$$\begin{aligned} \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi dx + \int_\Omega V(x) v^{p-1} \varphi dx \\ &= \int_\Omega [(\Phi^{-1})'(w_\rho)]^{p-1} |\nabla w_\rho|^{p-2} \nabla w_\rho \nabla \varphi dx \\ &\quad + \int_\Omega V(x) [\Phi^{-1}(w_\rho)]^{p-1} \varphi dx \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega |\nabla w_\rho|^{p-2}\nabla w_\rho \nabla((\Phi^{-1})'(w_\rho)^{p-1}\varphi)dx \\
&\quad - \int_\Omega (p-1)[(\Phi^{-1})'(w_\rho)]^{p-2}[(\Phi^{-1})''(w_\rho)]|\nabla w_\rho|^p \varphi dx \\
&\quad + \int_\Omega V(x)[\Phi^{-1}(w_\rho)]^{p-1}\varphi dx \\
&\geqslant \int_\Omega |\nabla w_\rho|^{p-2}\nabla w_\rho \nabla((\Phi^{-1})'(w_\rho)^{p-1}\varphi)dx \\
&\quad + \int_\Omega v(x)[\Phi^{-1}(w_\rho)]^{p-1}\varphi dx.
\end{aligned}
$$

Since $w_\rho(x)[(\Phi^{-1})'(w_\rho(x))] \leqslant \Phi^{-1}(w_\rho(x))$, $x \in \Omega$ holds (to prove this inequality, first show that $\Phi(t)[(\Phi^{-1})'(\Phi(t))] \leqslant \Phi^{-1}(\Phi(t))$, $t > 0$), it follows

$$
\begin{aligned}
&\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi dx + \int_\Omega V(x)v^{p-1}\varphi dx \\
&\qquad \geqslant \int_\Omega |\nabla w_\rho|^{p-2}\nabla w_\rho \nabla((\Phi^{-1})'(w_\rho)^{p-1}\varphi)dx \\
&\qquad\quad + \int_\Omega V(x)w_\rho^{p-1}[(\Phi^{-1})'(w_\rho)]^{p-1}\varphi dx \\
&\qquad = \int_\Omega a(x)[(\Phi^{-1})'(w_\rho)]^{p-1}\varphi dx \\
&\qquad = \int_\Omega \rho(x)\left(2 - \frac{\beta}{p-1}\right)^{p-1} \tau_\infty^{p-1} v^{\beta-p+1}\varphi dx.
\end{aligned}
$$

So, as $\beta_1 < p-1$ and $a \geqslant 0$, we have

$$
\begin{aligned}
&\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi dx + \int_\Omega V(x)v^{p-1}\varphi dx \\
&\qquad \geqslant \int_\Omega \rho(x)\tau_\infty^{p-1}v^{\beta-p+1}\varphi dx \geqslant \int_\Omega \rho(x)v^\beta \varphi dx. \qquad (2.5)
\end{aligned}
$$

In particular, v is an upper solution of (2.4). Now, since $V \geqslant 0$, it follows that $\alpha_\Omega(V,\rho) \geqslant \lambda_{1,\Omega}(0,1) > 0$. So, from Lemmas 2 and 3, we have $\lambda_{1,\Omega}(V,\rho) \geqslant \|w_\rho\|_\infty^{1-p} > 0$. Now, define $w = \phi_{1,\Omega}$ with

$$
\|w\|_\infty^{p-1-\beta} \leqslant \min\left\{\tau_\infty^{p-1-\beta}, 1/\lambda_{1,\Omega}(V,\rho)\right\},
$$

where $\phi_{1,\Omega} > 0$ and $\lambda_{1,\Omega}(V,\rho)$ are the first eigenfunction and eigenvalue of eigenvalue problem (2.2), respectively.

Hence,

$$
\int_\Omega |\nabla w|^{p-2}\nabla w \nabla \psi dx + \int_\Omega V(x)w^{p-1}\psi dx \leqslant \int_\Omega w^{\beta-p+1}\rho(x)w^{p-1}\psi dx,
$$

for all $\psi \in C_0^\infty(\Omega), \psi \geqslant 0$. In particular, w is a lower solution of (2.4). Besides this, it follows from $\|w\|_\infty \leqslant \tau_\infty$, that

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla \psi dx + \int_\Omega V(x) w^{p-1}\psi dx \leqslant \int_\Omega w^{\beta-p+1}\tau_\infty^{p-1}\rho(x)\psi dx, \forall\ \psi\ \in C_0^\infty(\Omega). \quad (2.6)$$

Now, applying the comparison principle of Tolksdorf (see Lemma 3.1 in [24]), it follows from (2.5) and (2.6) that $0 < w(x) \leqslant v(x),\ x \in \Omega$. Therefore, by the lower and upper solution principle of Lee, Shivaji and Ye in [15, Lemma 1.8] there exists a $u \in C^1(\Omega) \cap C(\overline{\Omega})$ solution of (2.4). This completes the proof of Theorem 2.1.

Our main tool for the proof of the existence of solution will be a general method of lower and upper solution. Consider the system

$$\begin{cases} -\Delta_p u = f_1(x,u,v) \text{ in } \mathbb{R}^N \\ -\Delta_q v = f_2(x,u,v) \text{ in } \mathbb{R}^N, \end{cases} \quad (2.7)$$

where the functions $f_i : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

In order to enunciate a version of the lower and upper-solution method for our class of problems, we will introduce some definitions.

DEFINITION 1. A pair of functions $(\underline{u},\underline{v}) \in C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ is called a weak lower solution of the problem (2.7) if

$$\int_{\mathbb{R}^N} |\nabla \underline{u}|^{p-2}\underline{u}\nabla\varphi dx \leqslant \int_{\mathbb{R}^N} f_1(x,\underline{u},\underline{v})\varphi dx,$$

$$\int_{\mathbb{R}^N} |\nabla \underline{v}|^{q-2}\underline{v}\nabla\psi dx \leqslant \int_{\mathbb{R}^N} f_2(x,\underline{u},\underline{v})\psi dx,$$

for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi, \psi \geqslant 0$.

Similarly, an upper solution $(\overline{u},\overline{v})$ of (2.7) is defined by considering the converse inequality in the above definition.

The proof of the theorem below follows arguments as in [15].

THEOREM 4. *Assume that $f_i : \mathbb{R}^N \times (0,\infty) \times (0,\infty) \to \mathbb{R}$ for $i = 1,2$ are continuous functions satisfying:*

(F_1) *$f_1(x,s,t)$ is nondecreasing in $t > 0$ for all fixed $(x,s) \in \mathbb{R}^N \times (0,\infty)$,*
$f_2(x,s,t)$ is nondecreasing in $s > 0$ for all fixed $(x,t) \in \mathbb{R}^N \times (0,\infty)$,

(F_2) *given $a_i, b_i \in (0,\infty)$, $i = 1,2$, with $a_1 < a_2$, $b_1 < b_2$, then*

$$f_1(\cdot,s,t), f_2(\cdot,s,t) \in L^\infty_{loc}(\mathbb{R}^N)$$

for all $(s,t) \in [a_1,a_2] \times [b_1,b_2]$.

Suppose that $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ are a weak lower-solution and a weak upper-solution, respectively, of system (2.7) with

$$(\underline{u},\underline{v}) \leqslant (\overline{u},\overline{v}) \text{ a.e. } \mathbb{R}^N.$$

Then the problem (2.7) has at least one solution in $[\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$.

3. Proof of theorem 1

The proof of this theorem is based on an argument by contradiction which involves the results about the positivity of eigenvalues treated early and a special carefully-chosen test function.

Proof. First, let us define the continuous function $M(x) = \max\{m_1(x), m_2(x)\}$, $x \in \mathbb{R}^N$. So, we have by definition, that

$$\alpha_{B_k}(M^+, m) \geqslant \lambda_{1,B_k}(0,1) > 0,$$

for $k = 1, 2, \cdots$.

So, since $m, M^+ \in L^\infty(B_k)$ for all $k > 1$ and $m \neq 0$ in $\mathbb{R}^N$, it follows from Lemmas 1-3 that there exists $\lambda_{1,B_k}(M^+, m)$ for all $k \geqslant k_0$ for some $k_0 > 0$ and it satisfies

$$\lambda_{1,B_k}(M^+, m) \geqslant \|w_k\|^{1-p}_{L^\infty(B_k)} \text{ for all } k > k_0,$$

where $w_k \in C^1(\Omega) \cap C(\overline{\Omega})$ is the unique solution of (2.1) with $V = M^+$, $\rho = m$ and $\Omega = B_k$ given by Lemma 2.1.

So, from (2.3), we have that $\lambda_1(M^+, m) \geqslant 0$. Now we are going to define $\lambda^* \geqslant 0$ and $\mu^* \geqslant 0$ as $\lambda^* = \lambda_1(M^+, m)$, if $\gamma_1 + \delta_1 = p - 1$ in (H_i) for $i = 1, 2$; $\mu^* = \lambda_1(M^+, m)$, if $\gamma_2 + \delta_2 = p - 1$ in (K_j) for $j = 1, 2$ and for the other cases

$$\lambda^* = \lambda_1(M^+, m)^{\frac{\gamma_1+\delta_1-\beta_1}{p-1-\beta_1}} \left(\frac{p-1-\beta_1}{\gamma_1+\delta_1-\beta_1}\right) \left(\frac{\gamma_1+\delta_1-p+1}{\gamma_1+\delta_1-\beta_1}\right)^{\frac{\gamma_1+\delta_1-\beta_1}{p-1-\beta_1}} \geqslant 0, \tag{3.1}$$

and

$$\mu^* = \lambda_1(M^+, m)^{\frac{\gamma_2+\delta_2-\beta_2}{p-1-\beta_2}} \left(\frac{q-1-\beta_2}{\gamma_2+\delta_2-\beta_2}\right) \left(\frac{\gamma_2+\delta_2-q+1}{\gamma_2+\delta_2-\beta_2}\right)^{\frac{\gamma_2+\delta_2-\beta_2}{q-1-\beta_2}} \geqslant 0. \tag{3.2}$$

Now, given $(\lambda, \mu) > (\lambda^*, \mu^*)$, we are going to suppose by contradiction that problem (1.1) has a solution $(u, v) = (u_{\lambda,\mu}, v_{\lambda,\mu})$. So defining $h_\lambda, h_\mu : (0, \infty) \longrightarrow (0, \infty)$ by

$$h_\lambda(t) = t^{\beta_1-p+1} + \lambda t^{\gamma_1+\delta_1-p+1} \text{ and } h_\mu(t) = t^{\beta_2-p+1} + \mu t^{\gamma_2+\delta_2-p+1},$$

it follows from (H_i) and (K_j) for some $i, j \in \{1, 2\}$ that

$$h_\lambda(t) > \lambda_1(M^+, m) \text{ and } h_\mu(t) > \lambda_1(M^+, m) \text{ for all } t > 0. \tag{3.3}$$

Now, defining

$$\Omega_1 = \{x \in \mathbb{R}^N; u \leqslant v\}, \qquad \Omega_2 = \{x \in \mathbb{R}^N; u > v\}$$

and, for each n, $q_n : \mathbb{R}^N \to [0, 1]$ by $q_n(x) = \rho_n((u - v)(x))$, where $\rho_n : \mathbb{R} \longrightarrow [0, 1]$ is chosen satisfying $\rho_n \in C^1(\mathbb{R})$, $\rho_n' \leqslant 0$ on $(0, 1/n)$, $\rho_n(t) = 0$ if $t \geqslant 1/n$ and $\rho_n(t) = 1$, if $t \leqslant 0$, it follows that q_n satisfies

$$q_n(x) = \begin{cases} 0, & \text{if } \quad u(x) \geqslant v(x) + \frac{1}{n}, \\ 1, & \text{if } \quad u(x) \leqslant v(x). \end{cases}$$

Hence, $q_n \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\|q_n\|_\infty \leqslant 1$, $\nabla q_n = \rho_n'(u-v)\nabla(u-v)$ and

$$q_n(x) \xrightarrow{n\to\infty} \begin{cases} 0, & \text{if} \quad x \in \Omega_2, \\ 1, & \text{if} \quad x \in \Omega_1. \end{cases}$$

Now, define $U(x) = \min\{u(x), v(x)\}$, $x \in \mathbb{R}^N$. Given $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geqslant 0$, it follows after some calculations that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla U|^{p-2}\nabla U \nabla\varphi dx + \int_{\mathbb{R}^N} M^+(x)U^{p-1}\varphi dx \\
&\qquad \geqslant \int_{\Omega_1} |\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_{\Omega_1} m_1(x)u^{p-1}\varphi dx \\
&\qquad\qquad + \int_{\Omega_2} |\nabla v|^{p-2}\nabla v\nabla\varphi dx + \int_{\Omega_2} m_2(x)v^{p-1}\varphi dx \\
&\qquad \geqslant \limsup_{n\to\infty} \int_{\mathbb{R}^N} q_n(|\nabla u|^{p-2}\nabla u\nabla\varphi + m_1(x)u^{p-1}\varphi)dx \\
&\qquad\qquad + \limsup_{n\to\infty} \int_{\mathbb{R}^N} (1-q_n)(|\nabla v|^{p-2}\nabla v\nabla\varphi + m_2(x)v^{p-1}\varphi)dx. \qquad (3.4)
\end{aligned}$$

Since $\nabla(q_n\varphi) = q_n\nabla\varphi + \varphi\nabla q_n$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} q_n|\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_{\mathbb{R}^N} m_1(x)u^{p-1}(q_n\varphi)dx &= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla(q_n\varphi)dx \\
&\quad + \int_{\mathbb{R}^N} m_1(x)u^{p-1}(q_n\varphi)dx - \int_{\mathbb{R}^N} \big(|\nabla u|^{p-2}\nabla u\nabla q_n\big)\varphi dx \\
&= \int_{\mathbb{R}^N} \big(a(x)u^{\beta_1} + \lambda b(x)u^{\gamma_1}v^{\delta_1} + f(x)\big)(q_n\varphi)dx \qquad (3.5) \\
&\quad - \int_{\Omega_n} \big(|\nabla u|^{p-2}\nabla u\nabla q_n\big)\varphi dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} (1-q_n)|\nabla v|^{p-2}\nabla v\nabla\varphi dx + \int_{\mathbb{R}^N} m_2(x)v^{p-1}(1-q_n)\varphi dx \qquad (3.6) \\
&= \int_{\mathbb{R}^N} (c(x)v^{\beta_2} + \mu d(x)v^{\gamma_2}u^{\delta_2} + g(x))(1-q_n)\varphi dx + \int_{\Omega_n} \big(|\nabla v|^{p-2}\nabla v\nabla q_n\big)\varphi dx,
\end{aligned}$$

where $\Omega_n = \{x \in \mathbb{R}^N, v(x) \leqslant u(x) < v(x) + 1/n\}$.

So, from (3.4), (3.5), (3.6), $(|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u)\nabla(v-u) \geqslant 0$ and $\rho'(u-v) \leqslant 0$ in Ω_n, we obtain, passing to the limit with $n \to \infty$, that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla U|^{p-2}\nabla U\nabla\varphi dx + \int_{\mathbb{R}^N} M^+(x)U^{p-1}\varphi dx \\
&\qquad \geqslant \int_{\Omega_1} (a(x)u^{\beta_1} + \lambda b(x)u^{\gamma_1}v^{\delta_1})\varphi dx \\
&\qquad\quad + \int_{\Omega_2} (c(x)v^{\beta_2} + \mu d(x)v^{\gamma_2}u^{\delta_2})\varphi dx
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \int_{\Omega_1} m(x)(u^{\beta_1}+\lambda u^{\gamma_1+\delta_1})\varphi dx\\
&\quad + \int_{\Omega_2} m(x)(v^{\beta_2}+\mu v^{\gamma_2+\delta_2})\varphi dx\\
&= \int_{\Omega_1} m(x)h_\lambda(u)u^{p-1}\varphi dx + \int_{\Omega_2} m(x)h_\mu(v)v^{p-1}\varphi dx\\
&\geqslant \int_{\Omega_1} \min\{h_\lambda(t),h_\mu(t)\}m(x)U^{p-1}\varphi dx\\
&\quad + \int_{\Omega_2} \min\{h_\lambda(t),h_\mu(t)\}m(x)U^{p-1}\varphi dx\\
&= \int_{\mathbb{R}^N} \min\{h_\lambda(t),h_\mu(t)\}m(x)U^{p-1}\varphi dx.
\end{aligned}
$$

So, from Lemma 2.4, it follows that $\min\{h_\lambda(t),h_\mu(t)\} \leqslant \lambda_1(M^+,m)$. But it is impossible by (3.3). This ends the proof.

REMARK 5. In (H_i) for $i=1,2$, we can assume $\beta_1 = p-1$, if $\lambda_1(M^+,m) < 1$. In this case, we should define $\lambda^* = 0$. In a similar way, we can have in (K_j) for $j=1,2$ the equality $\beta_1 = p-1$, if once again $\lambda_1(M^+,m) < 1$. Now, we define $\mu^* = 0$.

REMARK 6. In the proof of Theorem 1.1, we have not used the condition

$$u(x), v(x) \to 0 \ \text{ as } |x| \to \infty.$$

Proof of Corollary 1.1. As a consequence of the proof of Theorem 1.1, to prove that $\lambda^* = \mu^* = 0$ it is enough to prove $\lambda_1(M^+,m) = 0$. First, it follows from $M \leqslant 0$ and properties of the first eigenvalue, that

$$0 < \lambda_{1,B_k}(M^+,m) = \lambda_{1,B_k}(0,m) \leqslant \lambda_{1,B_k}(0,\hat{m}), \ \text{ for all } \ k > 1,$$

where $\hat{m}(|x|) = \hat{m}(r) := \min_{|x|=r} m(x), r > 0$.

Hence, by definition, we have

$$0 \leqslant \lambda_1(M^+,m) = \lambda_1(0,m) \leqslant \lambda_1(0,\hat{m}). \tag{3.7}$$

Now, suppose, by contradiction, that $\lambda_1(0,m) > 0$. So, by (3.7), we would have $\lambda_1(0,\hat{m}) > 0$. Considering $\lambda := \lambda_1(0,\hat{m})$ and taking the unique solution $u \in C^2((0,\infty)) \cap C^1([0,\infty))$ of the initial value problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1}\hat{m}(r)u^{p-1} \ \text{ in } (0,\infty) \\ u(0) = 1; \ \ u'(0) = 0, \end{cases}$$

it follows that $v \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C^1(\mathbb{R}^N)$ defined by $v(x) = u(|x|)$, $x \in \mathbb{R}^N$ is a solution of

$$\begin{cases} -\Delta_p u = \lambda \hat{m}(x)u^{p-1} \ \text{ in } \mathbb{R}^N, \\ u(0) = 1. \end{cases} \tag{3.8}$$

On the other hand, we point out that $|x|^p a(x), |x|^p b(x), |x|^p c(x), |x|^p d(x) \to \infty$ when $|x| \to \infty$ implies $|x|^p m(x) \to \infty$ when $|x| \to \infty$. So, as a consequence of this, we have $|x|^p \hat{m}(|x|) \to \infty$ when $|x| \to \infty$. Indeed, suppose that there exists $r_n \to \infty$ such that $\liminf r_n^p \hat{m}(r_n) < \infty$. We can see that $\hat{m}(r_n) = m(x_n)$ with $|x_n| = r_n$. So, $\lim |x_n|^p m(x_n) = \lim r_n^p \hat{m}(r_n) < \infty$, when $n \to \infty$, what is an absurd, since $|x_n|^p m(x_n) \to \infty$.

So, since $|x|^p \hat{m}(x) \to \infty$ when $|x| \to \infty$, it follows by a result in [3] that the problem (3.8) does not have any positive entire solution. So, there exists a $R > 0$ such that $u(R) = 0$. With this, v satisfies

$$\begin{cases} -\Delta_p u = \lambda \hat{m}(x) u^{p-1} \text{ in } B_R(0) \\ u > 0 \text{ in } B_R(0); \;\; u = 0 \text{ on } \partial B_R(0). \end{cases}$$

Hence, by well-known properties concerning to the eigenvalue problems in bounded domain, we have

$$\lambda_1(0, \hat{m}) := \lambda = \lambda_{1, B_R(0)}(0, \hat{m}) > \lambda_1(0, \hat{m}).$$

This is impossible. So, we have showed that $\lambda_1(0, \hat{m}) = 0$. So, from (3.7), it follows that $\lambda_1(M^+, m) = \lambda_1(0, m) = 0$.

4. Proof of Theorem 2

Proof. First, we are going to construct an upper solution of (1.1) of the form $(\overline{u}, \overline{v}) = (tw_1, tw_2)$, for some $t = t_{\lambda,\mu} > 0$, where w_1, w_2 are solutions of (1.4) and (1.5), respectively. For this, it is enough to find a $t_{\lambda,\mu} > 0$ satisfying

$$t_{\lambda,\mu}^{p-1} \geqslant t_{\lambda,\mu}^{\beta_1} \|w_1\|_\infty^{\beta_1} + \lambda t_{\lambda,\mu}^{\gamma_1+\delta_1} \|w_1\|_\infty^{\gamma_1} \|w_2\|_\infty^{\delta_1} + 1 + \left\|m_1^-/M_f\right\|_\infty t_{\lambda,\mu}^{p-1} \|w_1\|_\infty^{p-1} \tag{4.1}$$

and

$$t_{\lambda,\mu}^{q-1} \geqslant t_{\lambda,\mu}^{\beta_2} \|w_2\|_\infty^{\beta_2} + \lambda t_{\lambda,\mu}^{\gamma_2+\delta_2} \|w_1\|_\infty^{\gamma_2} \|w_2\|_\infty^{\delta_2} + 1 + \left\|m_2^-/M_g\right\|_\infty t_{\lambda,\mu}^{q-1} \|w_2\|_\infty^{q-1}. \tag{4.2}$$

So defining $h, k : (0, \infty) \longrightarrow \mathbb{R}$ by

$$h(t) = \frac{1}{\|w_1\|_\infty^{\gamma_1} \|w_2\|_\infty^{\delta_1}} \left[A_1 t^{p-1-\gamma_1-\delta_1} - \|w_1\|_\infty^{\beta_1} t^{\beta_1-\gamma_1-\delta_1} - t^{-\gamma_1-\delta_1} \right]$$

and

$$k(t) = \frac{1}{\|w_2\|_\infty^{\gamma_2} \|w_1\|_\infty^{\delta_2}} \left[A_2 t^{q-1-\gamma_2-\delta_2} - \|w_2\|_\infty^{\beta_2} t^{\beta_2-\gamma_2-\delta_2} - t^{-\gamma_2-\delta_2} \right],$$

where

$$A_1 = \left(1 - \left\| \frac{m_1^-}{M_f} \right\|_\infty \|w_1\|_\infty^{p-1} \right) \text{ and } A_2 = \left(1 - \left\| \frac{m_2^-}{M_g} \right\|_\infty \|w_2\|_\infty^{q-1} \right),$$

we have that the existence of $t_{\lambda,\mu} > 0$ satisfying (4.1) and (4.2) is the same as showing that $h(t_{\lambda,\mu}) \geqslant \lambda$ and $k(t_{\lambda,\mu}) \geqslant \mu$ for $\lambda > 0$ and $\mu > 0$ given.

Now, denoting by

$$\mathrm{Im}^{+}(g) = \{t > 0 \ / \ g(t) > 0\} \text{ for } g = h,k$$

and

$$\mathrm{Im}^{+}(h,k) = \{t \in \mathrm{Im}^{+}(h) \cap \mathrm{Im}^{+}(k) \ / \ h(t) = k(t)\}$$

we are going to consider some cases.

Case 1. $p-1 < \gamma_1 + \delta_1$ and $q-1 < \gamma_2 + \delta_2$. We have that $h(t), k(t) \longrightarrow -\infty$ if $t \to 0$ and $h(t), k(t) \longrightarrow 0$ if $t \to \infty$. Since $\mathrm{Im}^{+}(h), \mathrm{Im}^{+}(k) \neq \emptyset$, we have just two possibilities:

If $\mathrm{Im}^{+}(h,k) \neq \emptyset$, then take a $t_0 \in \mathrm{Im}^{+}(h,k)$. So, it follows from the behavior of $h(t), k(t)$ that we can take $\lambda_* = \mu_* = h(t_0) = k(t_0) > 0$.

On the other hand, if $\mathrm{Im}^{+}(h,k) = \emptyset$, then we are going to define $\lambda_* = \mu_* = \min\{\max_{t>0} h(t), \max_{t>0} k(t)\} > 0$.

Case 2. $p-1 < \gamma_1 + \delta_1$ and $q-1 = \gamma_2 + \delta_2$. Now, we have that $h(t), k(t) \longrightarrow -\infty$ if $t \to 0$, $h(t) \longrightarrow 0$, if $t \to \infty$ and

$$k(t) \to \Big(\frac{1}{\|w_2\|_\infty^{\gamma_2} \|w_1\|_\infty^{\delta_2}}\Big)\Big(1 - \Big\|\frac{m_2^-}{M_g}\Big\|_\infty \|w_2\|_\infty^{q-1}\Big), \ t \to \infty.$$

Again, we have $\mathrm{Im}^{+}(h), \mathrm{Im}^{+}(k) \neq \emptyset$. So, if $\mathrm{Im}^{+}(h,k) \neq \emptyset$ we can consider $\lambda_* = \mu_* > 0$ as the last case. If $\mathrm{Im}^{+}(h,k) = \emptyset$, we will define $\lambda_* = \mu_* = \max_{t>0} h(t) > 0$.

Case 3. $p-1 < \gamma_1 + \delta_1$ and $q-1 > \gamma_2 + \delta_2$. Since $h(t) \to -\infty$ and $k(t) \to -c_2$ when $t \to 0$, for some $c_2 \in [0,\infty]$, $h(t) \to 0$ and $k(t) \to \infty$, if $t \to \infty$ and $\mathrm{Im}^{+}(h), \mathrm{Im}^{+}(k) \neq \emptyset$, it follows that if $\mathrm{Im}^{+}(h,k) \neq \emptyset$, then we can consider $\lambda_* = \mu_* > 0$ as in the case 1. If $\mathrm{Im}^{+}(h,k) = \emptyset$, we will define $\lambda_* = \mu_* = \max_{t>0} h(t) > 0$ as in the last case.

Case 4. $p-1 > \gamma_1 + \delta_1$ and $q-1 > \gamma_2 + \delta_2$. Since $h(t), k(t) \longrightarrow \infty$, if $t \to \infty$ and considering the different subcases $\beta_i < \gamma_i + \delta_i$, $\beta_i = \gamma_i + \delta_i$ and $\beta_i > \gamma_i + \delta_i$, for $i = 1,2$, we can prove that $h(t) \longrightarrow -c_1$ and $k(t) \longrightarrow -c_2$, if $t \to 0$, for some $c_1, c_2 \in [0,\infty]$. So, we have $\mathrm{Im}^{+}(h) = [h_1, \infty)$ and $\mathrm{Im}^{+}(k) = [k_1, \infty)$, for some $h_1, k_1 > 0$. That is, we can take $\lambda_* = \mu_* = \infty$.

Thus, in all the cases above, for each $(0,0) < (\lambda,\mu) \leqslant (\lambda_*, \mu_*)$ given, there exists a $t = t_{\lambda,\mu} > 0$ such that $k(t_{\lambda,\mu}) > \mu$ and $h(t_{\lambda,\mu}) > \lambda$. Now, computing, we show that $(\overline{u}, \overline{v}) = (tw_1, tw_2)$ is an upper solution of (1.1). The other five possible cases are treated in a similar way.

Below we are going to construct a lower solution of the problem (1.1). Since $a, c, m_1, m_2 \in L^\infty(B_k)$ for all $k > 1$ and $a, c \geqslant 0$, $a, c \neq 0$, it follows from theorem 3 that there exists (u_k, v_k) satisfying

$$\begin{cases} -\Delta_p u + m_1^+(x) u^{p-1} = a(x) u^{\beta_1} \text{ in } B_k(0) \\ -\Delta_q v + m_2^+(x) v^{q-1} = c(x) v^{\beta_2} \text{ in } B_k(0) \\ u, v > 0 \text{ in } B_k(0); \quad u = v = 0 \text{ on } \partial B_k(0) \end{cases}$$

for all $k > k_0$, for some $k_0 > 0$ such that $B_{k_0} \supseteq \{x \in \mathbb{R}^N \ / \ a(x), c(x) \neq 0\}$.

So, defining $u_k(x) = v_k(x) = 0$ for $|x| > k$, it follows that

$$\begin{aligned} u_1(x) \leqslant u_2(x) \leqslant \ldots \leqslant u_k(x) \leqslant u_{k+1}(x) \leqslant \ldots \leqslant \bar{u}(x), x \in \mathbb{R}^N, \\ v_1(x) \leqslant v_2(x) \leqslant \ldots \leqslant v_k(x) \leqslant v_{k+1}(x) \leqslant \ldots \leqslant \bar{v}(x), x \in \mathbb{R}^N. \end{aligned} \tag{4.3}$$

Thus, letting $\underline{u} = \lim_{k\to\infty} u_k(x)$, $\underline{v} = \lim_{k\to\infty} v_k(x)$ and following an idea similar to that of [29, Theorem 1.1], we show that $(\underline{u}, \underline{v})$ satisfies

$$\begin{cases} -\Delta_p u + m_1^+(x)u^{p-1} = a(x)u^{\beta_1} \text{ in } \mathbb{R}^N \\ -\Delta_q v + m_2^+(x)v^{q-1} = c(x)v^{\beta_2} \text{ in } \mathbb{R}^N \\ u, v > 0 \text{ in } \mathbb{R}^N; \quad u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0. \end{cases}$$

So, $(\underline{u}, \underline{v})$ is a lower solution of (1.1) and from (4.3), we have $(0,0) < (\underline{u}, \underline{v}) \leqslant (\bar{u}, \bar{v})$. Therefore, by Theorem 4, there exists a $(u,v) \in C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$ solution of (1.1) such that $(\underline{u}, \underline{v}) \leqslant (u,v) \leqslant (\bar{u}, \bar{v})$.

5. Appendix

In this Appendix, we are going to sketch the proof of Remark 1.2.

Proof. First, we note that by defining $w_1(x) = v_1(|x|)$, $x \in \mathbb{R}^N$, where

$$v_1(r) = \int_r^\infty \left[s^{1-N} \int_0^s t^{N-1} \widehat{M}_1(t)dt \right]^{\frac{1}{p-1}} ds, \ r \geqslant 0$$

it is easy to check that $w_1 \in C^1(\mathbb{R}^N)$ is a positive upper solution of (1.4) such that $w_1(x) \to 0$ when $|x| \to \infty$.

On the other hand, by Lemma 1, there exists a unique $u_k \in C^1(B_k) \cap C(\overline{B}_k)$ solution of

$$(P)_k \quad \begin{cases} -\Delta_p u + m_1^+(x)u^{p-1} = M_f(x) \text{ in } B_k, \\ u > 0 \text{ in } B_k; \ u(x) = 0 \text{ on } \partial B_k. \end{cases}$$

Now, considering $u_k = 0$ in $\mathbb{R}^N \setminus B_k$, we have by a comparison principle that

$$\underline{u}_1(x) \leqslant \underline{u}_2(x) \leqslant \ldots \leqslant \underline{u}_k(x) \leqslant \underline{u}_{k+1}(x) \leqslant \ldots \leqslant w_1(x), \ x \in \mathbb{R}^N.$$

So, after some standard calculations, we obtain the result claimed.

REFERENCES

[1] A. AHAMMOU, *Positive radial solutions of nonlinear elliptic systems*, New York J. Math., **7** (2001), 267–280.

[2] W. ALLEGRETTO AND Y. HUANG, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Anal., **32** 7 (1998), 819–830.

[3] S.N. ARMSTRONG AND B. SIRAKOV, *Nonexistence of positive supersolutions of elliptic equations via the maximum principle*, Comm. Partial Differential Equations, **36** (2011), 2011–2047.

[4] S.S. ANTMAN, *Nonlinear Problems of Elasticity*, Second edition, Springer, Applied Mathematical Sciences **107**, New York, 2005.

[5] S. CHEN AND G. LU, *Existence and nonexistence of positive radial solutions for a class of semilinear elliptic system*, Nonlinear Anal., **38** 7 (1999), 919–932.

[6] P. CLEMENT, J. FLECKINGER, E. MITIDIERI AND F. DE THELIN, *Existence of positive solutions for nonvariational quasilinear system*, J. Differential Equations, **166** (2000), 455–477.

[7] M. CUESTA AND H.R. QUOIRIN, *A weighted eigenvalue problem for the p-Laplacian plus a potential*, NoDEA Nonlinear Differential Equations Appl., **16** 4 (2009), 469–491.

[8] J.I. DÍAZ AND J.E. SAA, *Existence et unicité de solutions positives pour certains equations elliptiques quasilineaires*, C. R. Math. Acad. Sci. Paris, **305** 12 (1987), 521–524.

[9] S. EL MANOUNI AND A. TOUZANI, *On some nonlinear elliptic systems with coercive perturbations in* $\mathbb{R}^N$, Rev. Mat. Complut., **16** 2 (2003), 483–494.

[10] J.R. ESTEBAN AND J.L. VAZQUEZ, *On the equation of turbulent filtration in one-dimensional porous media*, Nonlinear Anal., **10** 11 (1986), 1303–1325.

[11] D. G. FIGUEIREDO AND B. SIRACOV, *Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems*, Math. Ann., **333** (2005), 231–260.

[12] J. GIACOMONI, J. HERNÁNDEZ AND A. MOUSSAOUI, *Quasilinear and singular systems: the cooperative case*, Contemp. Math., **540** (2011), 79–94.

[13] A. M. HERRERO AND J. L. VAZQUEZ, *On the propagation properties of a nonlinear degenerate parabolic equation*, Comm. Partial Differential Equations, **7** 12 (1982), 1381–1402.

[14] L. LEADI AND A. MARCOS, *Maximum principle and existence results for elliptic system on* $\mathbb{R}^N$, Electron. J. Differential Equations, **2010** 60 (2010), 1–13.

[15] E.K. LEE, R. SHIVAJI AND J. YE, *Classes of infinite semipositone systems*, Proc. Roy. Soc. Edinburgh Sect. A, **139** 4 (2009), 853–865.

[16] E. MITIDIERI, *Non-existence of positive solutions of semilinear systems in* $\mathbb{R}^N$, Differential Integral Equations, **9** (1996), 465–479.

[17] A. MOUSSAOUI, B. KHODJA AND S. TAS, *A singular Gierer Meinhardt system of elliptic equations in* $\mathbb{R}^N$, Nonlinear Anal., **71** 3-4 (2009), 708–716.

[18] L. PEZZO AND J. BONDER, *A optimization problem for the first eigenvalue of the p-Laplacian plus a potential*, Commun. Pure Appl. Anal., **5** 4 (2006), 675–690.

[19] C.A. SANTOS, *Non-existence and existence of entire solutions for a quasi-linear problem with singular and super-linear terms*, Nonlinear Anal., **72** (2010), 3813–3819.

[20] C.A. SANTOS, *Entire Solutions for a Quasilinear Problem in the Presence of Sublinear and Super-Linear Terms*, Bound. Value Probl., Article ID 845946 (2009), 16 pages.

[21] H.M. SERAG AND E.A. EL-ZAHRANI, *Maximum principle and existence of positive solutions for nonlinear systems on* $\mathbb{R}^N$, Electron. J. Differential Equations, **85** 12 (2005), 1–12.

[22] J. SERRIN AND H. ZOU, *Non-existence of positive solutions of Lane-Emden system*, Differential Integral Equations, **9** (1996), 635–653.

[23] J. SERRIN AND H. ZOU, *The existence of positive entire solutions of elliptic Hamiltonian system*, Comm. Partial Differential Equations, **23** (1998), 577–599.

[24] P. TOLKSDORF, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations, **8** 7 (1983), 773–817.

[25] J. VELIN, *Existence results for some nonlinear elliptic system with lack of compactness*, Nonlinear Anal., **52** 3 (2003), 1017–1034.

[26] W. WANG AND L. HONG, *Liouville-type theorems for semilinear elliptic systems*, Nonlinear Anal., **75** (2012), 5380–5391.

[27] Z. YANG AND Q. LU, *Non-existence of positive solution to a quasilinear elliptic system and blow-up estimates for a reaction-diffusion system*, Commun. Nonlinear Sci. Numer. Simul., **6** 4(2001), 222–226.

[28] H. YIN AND Z. YANG, *Existence and non-existence of entire positive solutions for quasilinear systems with singular and super-linear terms*, Differ. Equ. Appl., **2** 2 (2010), 241–249.

[29] Z. ZHANG, *A remark on the existence of positive entire solutions of a sublinear elliptic problem*, Nonlinear Anal., **67** 1 (2007), 147–153.

[30] Z. ZHANG AND L. ZHU, *Nonexistence and radial symmetry of positive solutions of semilinear elliptic systems*, Discrete Dyn. Nat. Soc., **2009** 8 (2009). Article ID 629749.
[31] S. ZHEN, *Nonexistence of positive solutions to a semilinear elliptic system and blow-up estimates for a reaction-diffusion system*, J. Math. Anal. Appl., **232** 2 (1999), 293–311.
[32] H. ZOU, *Existence and non-existence for strongly coupled quasi-linear cooperative elliptic systems*, J. Math. Soc. Japan, **59** 2 (2007), 393–421.

(Received February 13, 2014)
(Revised July 27, 2014)

Carlos Alberto Santos
Departamento de Matemática
Universidade de Brasília
70910-900 Brasília
DF - Brasil
e-mail: csantos@unb.br

Mariana Ramos Reis
Departamento de Matemática
Universidade Federal da Integração Latino-Americana
85867-970 Foz do Iguaçu
PR - Brasil
e-mail: mramosreis@gmail.com