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# STABILITY RESULTS OF SOME ABSTRACT EVOLUTION EQUATIONS

N.S. HOANG

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*Abstract.* The stability of the solution to the equation $\dot{u} = A(t)u + G(t,u) + f(t)$, $t \geqslant 0$, $u(0) = u_0$ is studied. Here $A(t)$ is a linear operator in a Hilbert space $H$ and $G(t,u)$ is a nonlinear operator in $H$ for any fixed $t \geqslant 0$. We assume that $\|G(t,u)\| \leqslant \alpha(t)\|u\|^p$, $p > 1$, and the spectrum of $A(t)$ lies in the half-plane $\mathrm{Re}\,\lambda \leqslant \gamma(t)$ where $\gamma(t)$ can take positive and negative values. We proved that the equilibrium solution $u(t) \equiv 0$ to the equation is Lyapunov stable under persistantly acting perturbations $f(t)$ if $\sup_{t\geqslant 0}\int_0^t \gamma(\xi)\,d\xi < \infty$ and $\int_0^\infty \alpha(\xi)\,d\xi < \infty$. In addition, if $\int_0^t \gamma(\xi)\,d\xi \to -\infty$ as $t \to \infty$, then we proved that the equilibrium solution $u(t) \equiv 0$ is asymptotically stable under persistantly acting perturbations $f(t)$. Sufficient conditions for the solution $u(t)$ to be bounded and for $\lim_{t\to\infty} u(t) = 0$ are proposed and justified.

## 1. Introduction

Consider the equation

$$\dot{u} = A(t)u + G(t,u) + f(t), \quad t \geqslant 0, \quad u(0) = u_0, \quad \dot{u} := \frac{du}{dt}. \tag{1.1}$$

Here, $u(t)$ is a function of $t \geqslant 0$ with values in a Hilbert space $H$, $A(t) : H \to H$ is a linear, closed, and densely defined operator in $H$,

$$\mathrm{Re}\langle u, A(t)u\rangle \leqslant \gamma(t)\|u\|^2, \qquad t \geqslant 0, \qquad \forall u \in D(A(t)), \tag{1.2}$$

$G(t,u)$ is a nonlinear operator in $H$ for any fixed $t \geqslant 0$,

$$\|G(t,u)\| \leqslant \alpha(t)\|u\|^p, \qquad p > 1, \quad t \geqslant 0, \qquad \forall u \in H, \tag{1.3}$$

and $f(t)$ is a function on $\mathbb{R}_+ = [0,\infty)$ with values in $H$,

$$\|f(t)\| \leqslant \beta(t), \qquad t \geqslant 0. \tag{1.4}$$

Note that inequality (1.3) implies that $G(t,0) = 0$. Thus, $u(t) \equiv 0$ is an equilibrium solution to the equation

$$\dot{u} = A(t)u + G(t,u), \qquad t \geqslant 0.$$

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It is assumed that $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ in inequalities (1.2)–(1.4) are in $L_{1,loc}([0,\infty))$ and that $\alpha(t)$ and $\beta(t)$ are nonnegative on $[0,\infty)$. Also, we assume that equation (1.1) has a unique local solution. A stronger assumption on the local existence of equation (1.1) is made in Assumption A below. By a solution to problem (1.1) we mean a classical solution. Specifically, a global solution to (1.1) is a continuous differentiable function $u:[0,\infty)\to H$ which satisfies equation (1.1). A local solution to equation (1.1) is a continuous differentiable function $u:[0,T)\to H$, for some $T>0$, which solves equation (1.1). Thus, the solution space for global existence is $C^1([0,\infty);H)$ and for local existence is $C^1([0,T);H)$. Recall that a local solution to problem (1.1) exists and is unique if $A(t)$ is a generator of a $C_0$-semigroup.

Take inner product of both sides of equation (1.1) with $u$ to get

$$\langle u,\dot{u}\rangle = \langle u,A(t)u\rangle + \langle u,G(t,u)\rangle + \langle u,f(t)\rangle, \qquad t\geqslant 0.$$

Denote $g(t):=\|u(t)\|$, take the real part of the equation above, and use the triangle inequality to get

$$\begin{aligned}\dot{g}(t)g(t) &\leqslant \operatorname{Re}\langle u,A(t)u\rangle + |\langle u,G(t,u)\rangle| + \beta(t)g(t),\\ &\leqslant \gamma(t)g^2(t)+\alpha(t)g^{p+1}(t)+\beta(t)g(t), \qquad t\geqslant 0.\end{aligned}$$

This implies

$$\dot{g}\leqslant \gamma(t)g(t)+\alpha(t)g^p(t)+\beta(t), \qquad t\geqslant 0, \quad g(0)=\|u_0\|. \tag{1.5}$$

Note that in inequality (1.5) the functions $\alpha(t)$ and $\beta(t)$ are non negative on $\mathbb{R}_+$.

The stability of solutions to equation (1.1) has been studied in the literature (see, e.g., [1], [2], [4], and [6]). Stability of solutions of abstract equations in Banach and Hilbert spaces was studied in [3], [5], and [13]. In [7] stability of solutions of abstract equations in Hilbert spaces was studied using nonlinear inequalities. In [8]–[11] stability of the solution to equation (1.1) was studied using nonlinear inequalities under the assumption that the spectrum of $A(t)$ lie in the half-plane $\operatorname{Re}\lambda\leqslant\gamma(t)$ where $0>\gamma(t)\to 0$ as $t\to\infty$ (see [8] and [9]) or $0<\gamma(t)\to 0$ as $t\to\infty$ (see [10]). In [12] stability of solutions to abstract evolution equations with delay was studied.

The classical stability result of equation (1.1) states that if $A(t)\equiv A$ a constant matrix whose eigenvalues lie in the half-plane $\operatorname{Re}\lambda<\sigma_0<0$, and $\alpha(t)$ and $f(t)$ are identically equal to zero, then the solution to problem (1.1) exists globally, is unique, and is asymptotically stable. If the matrix $A$ has an eigenvalue in the half-plane $\operatorname{Re}(\lambda)>0$, then, in general, $\lim_{t\to\infty}u(t)=\infty$.

In this paper we study the stability of the solution to equation (1.1) under a more relaxed condition on the spectrum of $A(t)$ than those used in the literature. Namely, we allow the spectrum of $A(t)$ to lie in the half-plane $\operatorname{Re}(\lambda)\leqslant\gamma(t)$, where $\gamma(t)$ can take positive and negative values. In [8]–[10] it was assumed either $\gamma(t)>0$ or $\gamma(t)<0$ on $\mathbb{R}_+$. We give sufficient conditions on the functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ which yield stability properties of the solution to equation (1.1).

The novelty of the stability results in this paper compared to those in [8]–[11] is: Our results do not require to find a function $\mu(t)>0$ which solves a nonlinear inequality as those in [8]–[10]. In particular, our results are applicable for the case when $\gamma(t) = \sin t$ (or $\gamma(t) = \frac{\sin t}{(t+1)^a}, 0 < a < 1$) and $\alpha(t)$ and $\beta(t)$ are measure, positive, and locally integrable functions on $\mathbb{R}_+$. These cases are not easy to treat by the methods in [8]–[10] as it is not easy to find the functions $\mu(t)$ which solve the nonlinear inequalities in [8]–[10] for general $\gamma(t)$, $\alpha(t)$, and $\beta(t)$. The conditions on $\alpha(t)$ and $\gamma(t)$ in Theorem 1 in this paper are also more relaxed than those in Theorem 2 in [10]. Specifically, in Theorem 1 we have proved that if $\sup_{t\geqslant 0}\int_0^t \gamma(\xi)\,d\xi < \infty$ and $\int_0^\infty \alpha(\xi)\,d\xi < \infty$, then the equilibrium solution to problem (1.1) is Lyapunov stable under persistently acting perturbations. In Theorem 2 in [10] it is required that $\gamma(t) > 0$ and that $\int_0^\infty [\gamma(\xi) + \alpha(\xi)]\,d\xi$ is not 'large' to get the same stability. Other results in this paper are Theorem 2 and Theorem 4 in which we give sufficient conditions for the solution to problem (1.1) to be asymptotically stable. The rate of decay of the solution to problem (1.1) of exponential type is given in Theorem 2 and Corrollary 1.

Throughout the paper, we assume that the following assumption holds.

**Assumption A.** The equation

$$\dot{u} = A(t)u + G(t,u) + f(t), \quad t \geqslant t_0, \quad u(t_0) = \tilde{u}_0, \quad \dot{u} := \frac{du}{dt}$$

where $A(t)$, $G(t,u)$, and $f(t)$ are defined as earlier has a unique local solution for any $t_0 \geqslant 0$ and $\tilde{u}_0 \in H$.

## 2. Main results

THEOREM 1. *Assume that*

$$M := \sup_{t\geqslant 0}\int_0^t \gamma(\xi)\,d\xi < \infty, \qquad \int_0^\infty \alpha(t)\,dt < \infty. \tag{2.1}$$

*Then the equilibrium solution $u = 0$ to problem (1.1) is Lyapunov stable under persistently acting perturbations $f(t)$.*

REMARK 1. The term $f(t)$ in equation (1.1) is called *persistently acting perturbations*. *'Stable under persistently acting perturbations $f(t)$'* means that given any $\varepsilon > 0$ arbitrarily small, if $\|f(t)\|$ is sufficiently small, then there exists $\delta > 0$ such that if $\|u(0)\| < \delta$ then $\|u(t)\| < \varepsilon$ for all $t \geqslant 0$.

The first condition in (2.1) is necessary for the solution to equation (1.1) to be bounded, in general. Indeed, if the first condition in (2.1) does not hold, then the function $v(t) := u_0 e^{\int_0^t \gamma(\xi)\,d\xi}$, $u_0 \neq 0$, is unbounded. This unbounded $v(t)$ solves the equation $\dot{u} = \gamma(t)u$, $t \geqslant 0$, $u(0) = u_0$ which is a special case of equation (1.1) when $A(t)u = \gamma(t)u$, $G(t,u) \equiv 0$, and $f(t) \equiv 0$.

*Proof.* [Proof of Theorem 1] Let $\varepsilon > 0$ be arbitrarily small. Define

$$\mu(t) := e^{-\int_0^t [\gamma(\xi) + \varepsilon^{p-1}\alpha(\xi)]\,d\xi}, \qquad t \geqslant 0. \tag{2.2}$$

Then

$$\mu(t) \geqslant e^{-M_1}, \quad t \geqslant 0, \qquad M_1 := M + \varepsilon^{p-1}\int_0^\infty \alpha(\xi)\,d\xi. \tag{2.3}$$

Choose $\delta \in (0,\varepsilon)$ sufficiently small such that

$$\delta e^{M_1} < \frac{\varepsilon}{3}. \tag{2.4}$$

*Let us prove that if* $0 \leqslant g(0) = \|u_0\| < \delta$ *and* $\beta(t) = \|f(t)\|$ *is sufficiently small, then* $\|u(t)\| < \varepsilon$ *for all* $t \geqslant 0$.

Since $g(0) < \delta < \varepsilon$ and $g(t)$ is continuous, there exists $\theta > 0$ such that $g(t) < \varepsilon$, $\forall t \in [0,\theta)$. Let $T > 0$ be the largest real value such that

$$g(t) = \|u(t)\| < \varepsilon, \qquad \forall t \in [0,T). \tag{2.5}$$

We claim that $T = \infty$. Assume the contrary. Thus, $T$ is finite and, by the continuity of $g(t)$,

$$g(T) = \|u(T)\| = \varepsilon. \tag{2.6}$$

Choose $f(t)$ such that the function $\beta(t) = \|f(t)\|$ satisfies the inequality

$$\frac{\int_0^t \beta(\xi)\mu(\xi)\,d\xi}{\mu(t)} < \frac{\varepsilon}{3}, \qquad \mu(t) = e^{-\int_0^t(\gamma(\xi)+\varepsilon^{p-1}\alpha(\xi))\,d\xi}. \tag{2.7}$$

Inequality (2.7) holds true if $\|f(t)\| = \beta(t)$ is sufficiently small. It follows from inequalities (1.5) and (2.5) that

$$\dot{g} \leqslant \gamma(t)g(t) + \alpha(t)\varepsilon^{p-1}g(t) + \beta(t), \qquad 0 \leqslant t < T.$$

This implies

$$\frac{d}{dt}\big(g(t)\mu(t)\big) \leqslant \beta(t)\mu(t), \qquad \mu(t) = e^{-\int_0^t(\gamma(\xi)+\varepsilon^{p-1}\alpha(\xi))\,d\xi}, \qquad 0 \leqslant t < T. \tag{2.8}$$

Integrate inequality (2.8) from $0$ to $t$ to get

$$g(t)\mu(t) - g(0)\mu(0) \leqslant \int_0^t \beta(\xi)\mu(\xi)\,d\xi, \qquad 0 \leqslant t < T.$$

This, inequality (2.3), and inequality (2.7) imply

$$g(t) \leqslant \frac{g(0)}{\mu(t)} + \frac{\int_0^t \beta(\xi)\mu(\xi)\,d\xi}{\mu(t)} \leqslant g(0)e^{M_1} + \frac{\varepsilon}{3}, \qquad \forall t \in [0,T). \tag{2.9}$$

It follows from inequalities (2.4) and (2.9) and the inequality $g(0) < \delta$ that

$$g(t) \leqslant \delta e^{M_1} + \frac{\varepsilon}{3} \leqslant \frac{2\varepsilon}{3}, \qquad \forall t \in [0,T). \tag{2.10}$$

This and the continuity of $g(t)$ imply $g(T) \leqslant \frac{2\varepsilon}{3}$ which contradicts to relation (2.6). This contradiction implies that $T = \infty$, i.e.,

$$\|u(t)\| = g(t) \leqslant \varepsilon, \qquad \forall t \geqslant 0.$$

Thus, the equilibrium solution $u = 0$ is Lyapunov stable under persistently acting perturbations $f(t)$. Theorem 1 is proved. □

THEOREM 2. *Assume that*

$$M := \sup_{t\geqslant 0} \int_0^t \gamma(\xi)\, d\xi < \infty, \tag{2.11}$$

$$\frac{1}{(g(0)+\omega)^{p-1}} > (p-1)\int_0^\infty \frac{\alpha(\xi)}{\nu^{p-1}(\xi)}\, d\xi, \qquad \omega = const > 0, \quad \nu(t) = e^{-\int_0^t \gamma(\xi)\, d\xi}. \tag{2.12}$$

*If* $\beta(t) = \|f(t)\|$ *satisfies the inequality*

$$\frac{\beta(t)\nu^p(t)}{\alpha(t)} \leqslant \omega^p, \qquad t \geqslant 0, \tag{2.13}$$

*then the solution* $u(t)$ *to problem* (1.1) *exists globally, is bounded, and satisfies*

$$\|u(t)\| \leqslant C_2 e^{\int_0^t \gamma(\xi)\, d\xi}, \qquad t \geqslant 0, \quad C_2 = const > 0. \tag{2.14}$$

*Moreover, if*

$$\lim_{t\to\infty} \int_0^t \gamma(\xi)\, d\xi = -\infty, \tag{2.15}$$

*then*

$$\lim_{t\to\infty} u(t) = 0. \tag{2.16}$$

REMARK 2. Inequality (2.12) is a natural assumption. If inequality (2.12) does not hold for any $\omega \geqslant 0$, then the solution $g(t)$ to inequality (1.5) may blow up at a finite time even for the case when $\beta(t) = \|f(t)\| \equiv 0$. For example, one can verify that the solution to the equation

$$\dot{g} = \gamma(t)g(t) + \alpha(t)g^p(t), \qquad t \geqslant 0, \qquad g(0) = g_0,$$

is

$$g(t) = \tilde{g}(t) := \frac{1}{\nu(t)}\left(\frac{1}{g_0^{1-p} - (p-1)\int_0^t \frac{\alpha(\xi)}{\nu^{p-1}(\xi)}\, d\xi}\right)^{\frac{1}{p-1}}.$$

The function $\tilde{g}(t)$ blows up at a finite time $t = t_0$ if $t_0$ is the solution to the equation

$$0 = \frac{1}{g_0^{p-1}} - (p-1)\int_0^t \frac{\alpha(\xi)}{\nu^{p-1}(\xi)}\, d\xi.$$

This equation has a solution $t_0 > 0$ if

$$\frac{1}{g_0^{p-1}} < (p-1)\int_0^\infty \frac{\alpha(\xi)}{\nu^{p-1}(\xi)}\,d\xi.$$

If $g(t)$ blows up at a finite time, then the solution $u(t)$ to equation (1.1) blows up at a finite time as well due to the relation $g(t) = \|u(t)\|$.

Inequality (2.13) holds if $\beta(t) = \|f(t)\|$ is sufficiently small. Relation (2.16) implies, under assumptions (2.12) and (2.15), that the equilibrium solution $u = 0$ to problem (1.1) is asymptotically stable under persistantly acting perturbations $f(t)$.

*Proof.* [Proof of Theorem 2] *Let us first show that the solution $u(t)$ to problem (1.1) exists globally.* Assume the contrary. Thus, there exists a finite number $T > 0$ such that the maximal interval of existence of $u(t)$ is $[0,T)$. Inequality (1.5) is equivalent to

$$\frac{d}{dt}\big(g(t)\nu(t)\big) \leqslant \alpha(t)g^p(t)\nu(t) + \beta(t)\nu(t), \qquad 0 \leqslant t < T, \quad \nu(t) := e^{-\int_0^t \gamma(\xi)\,d\xi}. \tag{2.17}$$

Inequalities (2.17) and (2.13) imply

$$\begin{aligned} \frac{d}{dt}\big(g(t)\nu(t)\big) &\leqslant \frac{\alpha(t)}{\nu^{p-1}(t)}(g(t)\nu(t))^p + \beta(t)\nu(t) \\ &= \frac{\alpha(t)}{\nu^{p-1}(t)}\left[(g(t)\nu(t))^p + \frac{\beta(t)\nu^p(t)}{\alpha(t)}\right] \\ &\leqslant \frac{\alpha(t)}{\nu^{p-1}(t)}\left[(g(t)\nu(t))^p + \omega^p\right] \\ &\leqslant \frac{\alpha(t)}{\nu^{p-1}(t)}\Big(g(t)\nu(t) + \omega\Big)^p, \qquad 0 \leqslant t < T, \qquad p > 1. \end{aligned} \tag{2.18}$$

Here we have used the inequality $a^p + b^p \leqslant (a+b)^p$, $a,b \geqslant 0$, $p > 1$. Inequality (2.18) can be rewritten as

$$\frac{d}{dt}\left(\frac{\big[g(t)\nu(t)+\omega\big]^{1-p}}{1-p}\right) \leqslant \frac{\alpha(t)}{\nu^{p-1}(t)}, \qquad 0 \leqslant t < T.$$

Integrate this inequality from 0 to $t$ to get

$$\frac{\big[g(t)\nu(t)+\omega\big]^{1-p} - \big[g(0)+\omega\big]^{1-p}}{1-p} \leqslant \int_0^t \frac{\alpha(\xi)}{\nu^{p-1}(\xi)}\,d\xi, \qquad 0 \leqslant t < T. \tag{2.19}$$

Therefore,

$$\big[g(t)\nu(t)+\omega\big]^{p-1} \leqslant \frac{1}{(g(0)+\omega)^{1-p} - (p-1)\int_0^t \frac{\alpha(\xi)}{\nu^{p-1}(\xi)}\,d\xi}, \qquad 0 \leqslant t < T. \tag{2.20}$$

Inequality (2.12) implies that the right-hand side of (2.20) is well-defined for all $t \geqslant 0$. Thus, from (2.20) one gets

$$\big[g(t)v(t)+\omega\big]^{p-1} \leqslant \frac{1}{(g(0)+\omega)^{1-p}-(p-1)\int_0^\infty \frac{\alpha(\xi)}{v^{p-1}(\xi)}\,d\xi} := M_3, \qquad 0 \leqslant t < T. \tag{2.21}$$

It follows from relation (2.11) that

$$v(t) = e^{-\int_0^t \gamma(\xi)\,d\xi} \geqslant e^{-M}, \qquad 0 \leqslant t < T.$$

This and inequality (2.21) imply that

$$g(t) \leqslant \frac{M_3^{\frac{1}{p-1}}-\omega}{v(t)} \leqslant e^M\big(M_3^{\frac{1}{p-1}}-\omega\big), \qquad 0 \leqslant t < T. \tag{2.22}$$

This and the continuity of $u(t)$ imply that $\|u(T)\|$ is finite and $u(t)$ exists on $[0,T]$. This and Assumption A imply that the existence of the solution $u(t)$ to equation (1.1) can be extended to a larger interval, namely, $[0,T+\delta)$ for some $\delta > 0$. This contradicts the definition of $T$. The contradiction implies that $T=\infty$, i.e., $u(t)$ exists globally. The boundedness of $u(t)$ follows directly from inequality (2.22) with $T=\infty$.

*Let us prove* (2.16) *assuming that* (2.15) *holds.* Let $C_2 := M_3^{\frac{1}{p-1}}-\omega$. Then inequality (2.14) follows from the first inequality in (2.22) and the relations $g(t)=\|u(t)\|$ and $v(t)=e^{-\int_0^t \gamma(\xi)\,d\xi}$. If relation (2.15) holds, then

$$v(t) = e^{-\int_0^t \gamma(\xi)\,d\xi} \to \infty \text{ as } t \to \infty.$$

This and inequality (2.14) imply (2.16). This completes the proof of Theorem 2. □

Consider the following inequality:

$$\frac{\beta(t)v^p(t)}{\alpha(t)} \leqslant C < \left(\frac{1}{(p-1)\int_0^\infty \frac{\alpha(\xi)}{v^{p-1}(\xi)}d\xi}\right)^{\frac{p}{p-1}}, \qquad t \geqslant 0, \quad C>0, \quad p>1. \tag{2.23}$$

Let $\omega_0 := C^{\frac{1}{p}}$, i.e., $\omega_0^p = C$. Then it follows from the first inequality in (2.23) that inequality (2.13) holds for $\omega=\omega_0$. From the second inequality in (2.23) and the relation $C=\omega_0^p$, one gets

$$\omega_0^p < \left(\frac{1}{(p-1)\int_0^\infty \frac{\alpha(\xi)}{v^{p-1}(\xi)}d\xi}\right)^{\frac{p}{p-1}}.$$

This implies

$$\omega_0^{1-p} > (p-1)\int_0^\infty \frac{\alpha(\xi)}{v^{p-1}(\xi)}d\xi.$$

Thus, if $g(0)>0$ is sufficiently small, then we have

$$\big[g(0)+\omega_0\big]^{1-p} > (p-1)\int_0^\infty \frac{\alpha(\xi)}{v^{p-1}(\xi)}d\xi, \qquad p>1.$$

Therefore, the function

$$\frac{1}{(g(0)+\omega)^{1-p}-(p-1)\int_0^t \frac{\alpha(\xi)}{v^{p-1}(\xi)}\,d\xi}$$

which appears in the right-hand side of (2.20) is well-defined for all $t \geqslant 0$ when $\omega = \omega_0$ and $g(0) > 0$ is sufficiently small. From the remarks above and the proof of Theorem 2 we have the following corollary:

COROLLARY 1. *Assume that*

$$\sup_{t\to\infty}\int_0^t \gamma(\xi)\,d\xi < \infty, \tag{2.24}$$

$$\frac{\beta(t)v^p(t)}{\alpha(t)} \leqslant C < C_1 := \left(\frac{1}{(p-1)\int_0^\infty \frac{\alpha(\xi)}{v^{p-1}(\xi)}d\xi}\right)^{\frac{p}{p-1}}, \qquad v(t) = e^{-\int_0^t \gamma(\xi)\,d\xi}, \tag{2.25}$$

*for all $t \geqslant 0$. If $\|u_0\|$ is sufficiently small so that*

$$\left(\|u_0\| + C^{\frac{1}{p}}\right)^p < C_1,$$

*then the solution $u(t)$ to problem (1.1) exists globally, is bounded, and satisfies*

$$\|u(t)\| \leqslant C_2 e^{\int_0^t \gamma(\xi)\,d\xi}, \qquad t \geqslant 0, \qquad C_2 = const > 0. \tag{2.26}$$

*In addition, if*

$$\lim_{t\to\infty}\int_0^t \gamma(\xi)\,d\xi = -\infty,$$

*then*

$$\lim_{t\to\infty} u(t) = 0. \tag{2.27}$$

THEOREM 3. *Assume that $g(0) = \|u(0)\| \neq 0$ and that $\alpha(t) \geqslant 0$ satisfies the inequality*

$$\alpha(t) \leqslant \frac{(q-1)\beta(t)}{(q\zeta(t))^p}, \qquad t \geqslant 0, \quad q > 1, \tag{2.28}$$

*where*

$$\zeta(t) := \frac{g(0)}{v(t)} + \frac{\int_0^t \beta(\xi)v(\xi)\,d\xi}{v(t)}, \qquad v(t) = e^{-\int_0^t \gamma(\xi)\,d\xi}. \tag{2.29}$$

*Then the solution $u(t)$ to problem (1.1) exists globally and*

$$\|u(t)\| < q\zeta(t), \qquad \forall t \geqslant 0. \tag{2.30}$$

*In addition;*

(a) *If the function* $\zeta(t)$ *is bounded on* $[0,\infty)$*, then the solution* $u(t)$ *to problem* (1.1) *is bounded.*

(b) *If* $\lim_{t\to\infty}\zeta(t)=0$*, then*

$$\lim_{t\to\infty} u(t) = 0. \tag{2.31}$$

*Proof.* Recall from our earlier assumptions that the functions $\alpha(t), \beta(t)$, and $\gamma(t)$ are in $L_{1,loc}([0,\infty))$ and $\alpha(t)\geqslant 0$, $\beta(t)\geqslant 0$, $\forall t\geqslant 0$. Thus, the integrals

$$\int_0^t \gamma(\xi)\,d\xi \quad \text{and} \quad \int_0^t \beta(\xi)\nu(\xi)\,d\xi$$

are well-defined for all $t\geqslant 0$ and $\nu(t)>0$, $\forall t\geqslant 0$. Therefore, the function $\zeta(t)$ in (2.29) is well-defined on $[0,\infty)$.

*Let us prove that the solution* $u(t)$ *to problem* (1.1) *exists globally.* Assume the contrary that the maximal interval of existence of $u(t)$ is $[0,T)$ where $0<T<\infty$. Let us first prove that

$$g(t) = \|u(t)\| < q\zeta(t), \qquad 0\leqslant t<T. \tag{2.32}$$

Since $\nu(0)=1$, it follows from (2.29) with $t=0$ that $g(0)=\zeta(0)<q\zeta(0)$. This and the continuity of $g(t)$ and $\zeta(t)$ imply that there exists $\theta>0$ so that $g(t)<q\zeta(t)$, $\forall t\in[0,\theta)$. Let $T_1\in(0,T]$ be the largest real number such that

$$g(t) < q\zeta(t), \qquad \forall t\in[0,T_1). \tag{2.33}$$

Let us prove that $T_1=T$. Assume the contrary. Then $0<T_1<T$. From the continuity of $g(t)$ and the definition of $T_1$, one has

$$g(T_1) = q\zeta(T_1), \qquad g(t)<q\zeta(t), \qquad 0\leqslant t<T_1. \tag{2.34}$$

Inequalities (1.5), (2.28), and (2.33) imply

$$\begin{aligned} \dot g &\leqslant \gamma(t)g(t) + \alpha(t)(q\zeta(t))^p + \beta(t) \\ &\leqslant \gamma(t)g(t) + (q-1)\beta(t) + \beta(t) = \gamma(t)g(t) + q\beta(t), \qquad \forall t\in[0,T_1]. \end{aligned} \tag{2.35}$$

This implies

$$\frac{d}{dt}(g(t)\nu(t)) \leqslant q\beta(t)\nu(t), \quad t\in[0,T_1], \qquad \nu(t) = e^{-\int_0^t \gamma(\xi)\,d\xi}. \tag{2.36}$$

Integrate this inequality from 0 to $t$ to get

$$g(t)\nu(t) - g(0)\nu(0) \leqslant q\int_0^t \beta(\xi)\nu(\xi)\,d\xi, \qquad t\in(0,T_1]. \tag{2.37}$$

Thus,

$$g(t) \leqslant \frac{g(0)}{\nu(t)} + \frac{q\int_0^t \beta(\xi)\nu(\xi)\,d\xi}{\nu(t)} < q\left(\frac{g(0)}{\nu(t)} + \frac{\int_0^t \beta(\xi)\nu(\xi)\,d\xi}{\nu(t)}\right) = q\zeta(t), \tag{2.38}$$

for all $t \in (0,T_1]$. Inequality (2.38) for $t = T_1$ is $g(T_1) < q\zeta(T_1)$ which contradicts to the first equality in (2.34). This contradiction implies that $T_1 = T$, i.e., inequality (2.32) holds.

Inequality (2.32) and the continuity of $\|u(t)\|$ imply that $\|u(t)\|$ is finite on the interval $[0,T]$. Thus, by using Assumption A with $t_0 = T$ one can extend the solution $u(t)$ to a large interval. In other words, there exists $\delta > 0$ so that the solution $u(t)$ to equation (1.1) exists on $[0,T+\delta]$. This contradicts the definition of $T$. The contradiction implies that $T = \infty$, i.e., the solution $u(t)$ to equation (1.1) exists globally.

Inequality (2.30) follows from inequality (2.32) when $T = \infty$. It follows directly from inequality (2.30) that if $\zeta(t)$ is bounded on $[0,\infty)$, then the solution $u(t)$ to equation (1.1) is bounded and that if $\lim_{t\to\infty}\zeta(t) = 0$, then $\lim_{t\to\infty} u(t) = 0$. Theorem 3 is proved. □

A consequence of Theorem 3 is the following result.

THEOREM 4. *Assume that* $g(0) = \|u(0)\| \neq 0$ *and that* $\alpha(t) \geqslant 0$ *satisfies the inequality*

$$\alpha(t) \leqslant \frac{(q-1)\beta(t)}{(q\zeta(t))^p}, \qquad t \geqslant 0, \quad q > 1, \tag{2.39}$$

*where*

$$\zeta(t) = \frac{g(0)}{\nu(t)} + \frac{\int_0^t \beta(\xi)\nu(\xi)\,d\xi}{\nu(t)}, \qquad \nu(t) = e^{-\int_0^t \gamma(\xi)\,d\xi}. \tag{2.40}$$

*Then the solution* $u(t)$ *to problem* (1.1) *exists globally.*

*In addition;*

*(a) If*

$$L := \sup_{t\geqslant 0}\left|\int_0^t \gamma(\xi)\,d\xi\right| < \infty, \qquad \int_0^\infty \beta(t)\,dt < \infty, \tag{2.41}$$

*then the solution* $u(t)$ *to problem* (1.1) *is bounded.*

*(b) If*

$$\lim_{t\to\infty}\int_0^t \gamma(\xi)\,d\xi = -\infty, \qquad \lim_{t\to\infty}\frac{\beta(t)}{\gamma(t)} = 0, \tag{2.42}$$

*then*

$$\lim_{t\to\infty} u(t) = 0. \tag{2.43}$$

*Proof.* The global existence of $u(t)$ follows from Theorem 3.

*Let us proved that* $u(t)$ *is bounded given that inequality* (2.41) *holds.* From (2.30) it suffices to show that the function $\zeta(t)$ is bounded. From the first inequality in (2.41), one gets

$$e^{-L} \leqslant \nu(t) = e^{-\int_0^t \gamma(\xi)\,d\xi} \leqslant e^L, \qquad \forall t \geqslant 0. \tag{2.44}$$

Thus,

$$0 \leqslant \int_0^\infty \beta(\xi)\nu(\xi)\,d\xi \leqslant e^L \int_0^\infty \beta(\xi)\,d\xi < \infty. \tag{2.45}$$

It follows from (2.44) and (2.45) that

$$\frac{g(0)}{\nu(t)} + \frac{\int_0^t \beta(\xi)\nu(\xi)\,d\xi}{\nu(t)} \leqslant g(0)e^L + e^{2L}\int_0^\infty \beta(\xi)\,d\xi < \infty, \qquad \forall t \geqslant 0.$$

Therefore, the function $\zeta(t)$ defined in (2.29) is bounded. Thus, $u(t)$ is bounded as a consequence of inequality (2.30).

*Let us prove* (2.43) *given that the relations in* (2.42) *hold.* It follows from the first relation in (2.42) that

$$\lim_{t\to\infty} \nu(t) = \lim_{t\to\infty} e^{-\int_0^t \gamma(\xi)\,d\xi} = \infty.$$

We claim that

$$\lim_{t\to\infty} \frac{\int_0^t \beta(\xi)\nu(\xi)\,d\xi}{\nu(t)} = 0. \tag{2.46}$$

Indeed, if $\int_0^\infty \beta(\xi)\nu(\xi)\,d\xi < \infty$, then $\int_0^t \beta(\xi)\nu(\xi)\,d\xi$ is bounded on $\mathbb{R}_+$ and relation (2.46) follows from the relation $\lim_{t\to\infty} \nu(t) = \infty$. If $\int_0^\infty \beta(\xi)\nu(\xi)\,d\xi = \infty$, then relation (2.46) follows from L'Hospital's rule and the relation $\lim_{t\to\infty} \frac{\beta(t)}{\gamma(t)} = 0$ (cf. (2.42)).

From (2.38) one gets

$$0 \leqslant g(t) \leqslant \frac{g(0)}{\nu(t)} + \frac{q\int_0^t \beta(\xi)\nu(\xi)\,d\xi}{\nu(t)}, \qquad \forall t \geqslant 0. \tag{2.47}$$

It follows from relation (2.46), the relation $\lim_{t\to\infty} \nu(t) = \infty$, and inequality (2.47) that $\lim_{t\to\infty} g(t) = 0$. Thus, relation (2.43) holds. Theorem 4 is proved. □

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## REFERENCES


[1] L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Springer–Verlag, Berlin, 1963.

[2] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGrawHill, NewYork, 1955.

[3] Y. Daleckii and M. Krein, *Stability of solutions of differential equations in Banach spaces*, Amer. Math. Soc., Providence, RI, 1974.

[4] B. Demidovich, *Lectures on Mathematical Theory of Stability*, Nauka: Moscow, 1967. (in Russian)

[5] P. Hartman, *Ordinary Differential Equations*, Wiley, NewYork, 1964.

[6] A. Lyapunov, *Collected Works*, II. Acad. Sci., Moscow, 1954. (in Russian)

[7] A. G. Ramm and N. S. Hoang, *Dynamical Systems Method and Applications, Theoretical Developments and Numerical examples*, Wiley, Hoboken, 2012.

[8] A. G. Ramm, Asymptotic stability of solutions to abstract differential equations, *J. Abstr. Differ. Equ. Appl.*, 1 (2010), N1, 27–34.

[9] A. G. Ramm, Stability of solutions to some evolution problems, *Chaotic Modeling and Simulation (CMSIM)*, 1 (2011), 17–27.

[10] A. G. Ramm, A stability result for abstract evolution problems, *Math. Meth. Appl. Sci.*, 36 (2012), N4, 422–426.

[11] A. G. Ramm, Stability of the solutions to evolution problems, *Mathematics*, 1 (2013), 46–64.

[12] A. G. RAMM, Stability of solutions to abstract evolution equations with delay, *J. Math. Anal. Appl.*, 396 (2012), 523–527.

[13] R. TEMAM, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, NewYork, 1997.

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*N.S. Hoang*
*Department of Mathematics*
*University of West Georgia*
*Carrollton, GA 30116*
*USA*
*e-mail:* nhoang@westga.edu