Differential Equations & Applications
Volume 6, Number 4 (2014), 455–466
doi:10.7153/dea-06-26

EXISTENCE THEORY FOR NONLINEAR STURM–LIOUVILLE PROBLEMS WITH UNBOUNDED NONLINEARITIES

DANIEL MARONCELLI AND JESÚS RODRÍGUEZ

(*Communicated by Christopher C. Tisdell*)

Abstract. In this work we provide conditions for the existence of solutions to nonlinear Sturm-Liouville problems of the form,

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(\varepsilon, x(t))$$

subject to

$$ax(0) + bx'(0) = 0 \text{ and } cx(1) + dx'(1) = 0.$$

Our approach will be topological, utilizing both degree theory and the Lyapunov-Schmidt procedure.

1. Introduction

In this paper we provide criteria for the solvability of nonlinear Sturm-Liouville problems of the form,

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(\varepsilon, x(t)) \tag{1}$$

subject to

$$ax(0) + bx'(0) = 0 \text{ and } cx(1) + dx'(1) = 0. \tag{2}$$

Throughout, we assume that $f : \mathbb{R}^2 \to \mathbb{R}, p : [0,1] \to \mathbb{R}$ and $q : [0,1] \to \mathbb{R}$ are continuous, $p(t) > 0$ for all $t \in [0,1]$, $a^2 + b^2, c^2 + d^2 > 0$, and λ is an eigenvalue of the associated linear Sturm-Liouville problem.

Section 2 contains preliminary material. We give a brief introduction to Sturm-Liouville theory and the Lyapunov-Schmidt procedure. In section 3 we consider the nonlinear boundary value problem (1)-(2) when λ is an arbitrary eigenvalue of the corresponding linear Sturm-Liouville problem. We prove that if $f(0,\cdot)$ exibits sublinear growth and is bounded away from 0 in an appropriate way, then (1)-(2) has a solution.

The case where λ is the first eigenvalue of the linear Sturm-Liouville problem is discussed in section 4. In this case, the solvability of (1)-(2) is obtained under less restrictive conditions on the nonlinearty.

Mathematics subject classification (2010): 34B15.
Keywords and phrases: boundary value problems,Sturm-Liouville, Lyapunov-Schmidt procedure, topological degree theory.

The literature devoted to nonlinear Sturm-Liouville boundary value problem is extensive. Pertinent references from the point of view of this paper are [6, 7, 8, 10, 11, 13].

The approach we have used in this paper, that is, the Lyapunov-Schmidt procedure in combination with topological degree theory, has also been successfully applied to the study of periodic behavior in both discrete and continuous systems. For readers interested in this topic we suggest [1, 2, 3, 4, 12, 14].

2. Preliminaries

The nonlinear boundary value problem (1)-(2) will be viewed as an operator equation. We let $X = C[0,1]$ represent the continuous real-valued functions defined on $[0,1]$ and the norm on X will be the supremum norm. The subspace of X consisting of the continuously differentiable functions will be denoted by $C^1[0,1]$.

Operators B and D will be defined as follows:

$B : X \to \mathbb{R}$ is given by

$$B\phi = a\phi(0) + b\phi'(0)$$

and $D : X \to \mathbb{R}$ is given by

$$D\phi = c\phi(1) + d\phi'(1).$$

We define a linear operator $\mathscr{L} : dom(\mathscr{L}) \subset X \to X$ by

$$\mathscr{L}x(t) = (p(t)x'(t))' + q(t)x(t),$$

where

$$dom(\mathscr{L}) = \{\phi \in C^1[0,1] \mid p\phi' \in C^1[0,1] \text{ and } B\phi = D\phi = 0\}.$$

For each ε in $\mathbb{R}$, we let

$$\mathscr{F}_\varepsilon(x)(t) = f(\varepsilon, x(t)).$$

Solving the nonlinear boundary value problem (1)-(2) is now equivalent to solving

$$\mathscr{L}_\lambda x = \mathscr{F}_\varepsilon(x), \tag{3}$$

where $\mathscr{L}_\lambda = \mathscr{L} + \lambda I$.

We begin our study of the nonlinear boundary value problem by recalling some well known facts regarding the linear Sturm-Liouville problem,

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = 0 \tag{4}$$

subject to the boundary conditions (2).

For those readers interested in a more detailed introduction to linear Sturm-Liouville problems, we suggest [5].

REMARK 2.1. Using our notation from above, the linear Sturm-Liouville problem is equivalent to $\mathscr{L}_\lambda = 0$.

It is well known that $\mathscr{L}_\lambda$ is self-adjoint, that λ is a simple eigenvalue, and that

$$X = Ker(\mathscr{L}_\lambda) \oplus Im(\mathscr{L}_\lambda).$$

We may therefore choose a vector, ψ, as a basis for $Ker(\mathscr{L}_\lambda)$. Without loss of generality we assume

$$\int_0^1 \psi^2(t)\,dt = 1.$$

With this in mind, we make the following definition which will play a crucial role in our ability to analyze the nonlinear Sturm-Liouville problem, (1)-(2), using a projection scheme.

DEFINITION 2.2. Define $P\colon X \leftarrow X$ by

$$Px(t) = \psi(t)\int_0^1 \psi(s)x(s)\,ds.$$

It is clear that P is a projection with $Im(P) = Ker(\mathscr{L}_\lambda)$. Since $\mathscr{L}_\lambda$ is self-adjoint, we have that $I-P$ is a projection onto the $Im(\mathscr{L}_\lambda)$.

In our analysis of the nonlinear Sturm-Liouville problem we will use a projection scheme ofter referred to as the Lyapunov-Schmidt procedure. The result of the Lyapunov-Schmidt reduction will allow us to write the operator equation (3) as an equivalent system in which a degree theoretic argument may be applied to prove the existence of solutions. The following is the standard formulation of the Lyapunov-Schmidt procedure. Interested readers may consult [1, 9]. We include the proof for the benefit of the reader.

PROPOSITION 2.3. *Solving $\mathscr{L}_\lambda x = \mathscr{F}_\varepsilon(x)$ is equivalent to solving the system*

$$\begin{cases} x = Px + M_p(I-P)\mathscr{F}_\varepsilon(x) \\ \quad\textit{and} \\ P\mathscr{F}_\varepsilon(x) = 0 \end{cases}$$

where M_p is $(\mathscr{L}_{\lambda|Ker(P)\cap dom(\mathscr{L})})^{-1}$.

Proof.

$$\mathscr{L}_\lambda x = \mathscr{F}_\varepsilon(x) \iff \begin{cases} (I-P)(\mathscr{L}_\lambda x - \mathscr{F}_\varepsilon(x)) = 0 \\ \quad\text{and} \\ P(\mathscr{L}_\lambda x - \mathscr{F}_\varepsilon(x)) = 0 \end{cases}$$

$$\iff \begin{cases} \mathscr{L}_\lambda x - (I-P)\mathscr{F}_\varepsilon(x) = 0 \\ \quad\text{and} \\ P\mathscr{F}_\varepsilon(x) = 0 \end{cases}$$

$$\iff \begin{cases} M_p\mathscr{L}_\lambda x - M_p(I-P)\mathscr{F}_\varepsilon(x) = 0 \\ \text{and} \\ P\mathscr{F}_\varepsilon(x) = 0 \end{cases}$$

$$\iff \begin{cases} (I-P)x - M_pE\mathscr{F}_\varepsilon(x) = 0 \\ \text{and} \\ P\mathscr{F}_\varepsilon(x) = 0 \end{cases}$$

$$g \iff \begin{cases} (I-P)x - M_pE\mathscr{F}_\varepsilon(x) = 0 \\ \text{and} \\ \psi(\cdot)\int_0^1 \psi(s)f(\varepsilon,x(s))\,ds = 0 \end{cases}$$

$$\iff \begin{cases} (I-P)x - M_pE\mathscr{F}_\varepsilon(x) = 0 \\ \text{and} \\ \int_0^1 \psi(s)f(\varepsilon,x(t))\,ds = 0. \end{cases}$$

The following proposition will play a significant role in the proof of our main results.

PROPOSITION 2.4. *Let A_η denote $\{t \in [0,1] \mid |\psi(t)| \geqslant \eta\}$. Suppose $f(0,\cdot)$ is sublinear, i.e. $|f(0,x)| \leqslant M_1|x|^\beta + M_2$ where $0 \leqslant \beta < 1$. We then have that there exists positive constants, C_1 and C_2, such that for each pair $(\alpha,x) \in \mathbb{R} \times Im(\mathscr{L}_\lambda)$ satisfying $x = M_p(I-P)\mathscr{F}_0(\alpha\psi + x)$ and $|\alpha| \geqslant \dfrac{C_1}{\eta}$,*

$$|x(t)| \leqslant C_2|\alpha\psi(t)|^\beta$$

provided $t \in A_\eta$.

Proof.

$$\begin{aligned} |x(t)| &\leqslant \big\|M_p(I-P)\big\|\big(M_1(|\alpha\psi(t)| + |x(t)|)^\beta + M_2\big) \\ &= \big\|M_p(I-P)\big\|M_1|\alpha\psi(t)|^\beta\big(1 + \frac{|x(t)|}{|\alpha\psi(t)|}\big)^\beta + \big\|M_p(I-P)\big\|M_2 \\ &\leqslant \big\|M_p(I-P)\big\|M_1|\alpha\psi(t)|^\beta\big(1 + \frac{\beta|x(t)|}{|\alpha\psi(t)|}\big) + \big\|M_p(I-P)\big\|M_2. \end{aligned}$$

Thus,

$$\frac{|x(t)|}{|\alpha\psi(t)|^\beta} - \big\|M_p(I-P)\big\|M_1\frac{\beta|x(t)|}{|\alpha\psi(t)|} \leqslant \big\|M_p(I-P)\big\|M_1 + \frac{\big\|M_p(I-P)\big\|M_2}{|\alpha\psi(t)|^\beta}$$

or

$$\frac{|x(t)|}{|\alpha\psi(t)|^\beta}\left(1-\frac{\|M_p(I-P)\|M_1\beta}{|\alpha\psi(t)|^{1-\beta}}\right)\leqslant \|M_p(I-P)\|M_1+\frac{\|M_p(I-P)\|M_2}{|\alpha\psi(t)|^\beta}.$$

Now fix γ, with $0<\gamma<1$. If

$$|\alpha|\geqslant\frac{1}{\eta}\left(\frac{\|M_p(I-P)\|M_1\beta}{\gamma}\right)^{\frac{1}{1-\beta}},$$

then

$$\frac{|x(t)|}{|\alpha\psi(t)|^\beta}\leqslant\frac{\|M_p(I-P)\|M_1}{1-\gamma}+\frac{\|M_p(I-P)\|M_2}{(1-\gamma)\left|\frac{\|M_p(I-P)\|M_1\beta}{\gamma}\right|^{\frac{\beta}{1-\beta}}}.$$

The result now follows by taking

$$C_1=\left(\frac{\|M_p(I-P)\|M_1\beta}{\gamma}\right)^{\frac{1}{1-\beta}}$$

and

$$C_2=\frac{\|M_p(I-P)\|M_1}{1-\gamma}+\frac{\|M_p(I-P)\|M_2}{(1-\gamma)\left|\frac{\|M_p(I-P)\|M_1\beta}{\gamma}\right|^{\frac{\beta}{1-\beta}}}.$$

3. Existence at an arbitrary Eigenvalue

We first consider the solvabilty of (1)-(2) for a general eigenvalue. For simplicity, we denote $f(0,x)$ by $f(x)$ and $\mathscr{F}_0$ by $\mathscr{F}$.

THEOREM 3.1. *Suppose the following conditions hold:*

C1. The function f is "sublinear"; that is, there exist real numbers M_1,M_2 and β, with $0\leqslant\beta<1$, such that $|f(x)|\leqslant M_1|x|^\beta+M_2$.

C2. There exists $J>0$ and $\hat{z}>0$ such that for all $z\geqslant\hat{z}$, $f(-z)\leqslant -J<0<J\leqslant f(z)$.

Then, for sufficiently small ε, there exists a solution to (1)-(2).

Proof. In order to apply a degree theoretic argument, we make $Im(\mathscr{L}_\lambda)\times\mathbb{R}$ a Banach space using the following norm

$$\|(x,\alpha)\|=\max\{\|x\|,|\alpha|\}.$$

We then define an operator $H : Im(\mathscr{L}_\lambda) \times \mathbb{R} \leftarrow Im(\mathscr{L}_\lambda) \times \mathbb{R}$ by

$$H(x,\alpha) = \begin{bmatrix} M_p(I-P)\mathscr{F}(\alpha\psi(\cdot)+x) \\ \alpha - \int_0^1 \psi(t) f(\alpha\psi(t)+x(t))\,dt \end{bmatrix}.$$

We know, from Proposition 2.3, that the fixed points of H are precisely the solutions of the nonlinear boundary value problem (1)-(2). Now, M_p is an integral operator and thus a compact linear map. Combining this with the fact that $\mathscr{F}$ is sublinear, we have that H is a nonlinear compact map. We will show that H has a fixed point by showing that for an appropriately chosen set, the Leray-Schauder degree of $I-H$ is nonzero.

Let C_1 and C_2 be as in Proposition 2.4. The following is trivial, but will be useful in what follows: there exists r^* such that for all $r \geqslant r^*$, we have

$$r - C_2 r^\beta \geqslant \hat{z}$$

and

$$r - \|M_p(I-P)\|\big(M_1(2\|\psi\|)^\beta r^\beta + M_2\big) > 0.$$

Without loss of generality we will assume that $C_1 > r^* > 1$. For $\eta < 1$, we define

$$\Omega_\eta = B\big(0, \frac{C_1}{\eta}\big) \text{ in } Im(\mathscr{L}_\lambda) \times \mathbb{R}.$$

We now analyze H on $\partial(\Omega_\eta)$. It is clear that if $(x,\alpha) \in \partial(\Omega_\eta)$, then one of two following condtitions is true:

(i) $\|x\| = \dfrac{C_1}{\eta}$ and $|\alpha| \leqslant \dfrac{C_1}{\eta}$.

(ii) $\|x\| \leqslant \dfrac{C_1}{\eta}$ and $|\alpha| = \dfrac{C_1}{\eta}$.

With this in mind, we assume $(x,\alpha) \in \partial(\Omega_\eta)$ and that i. holds.

Since $\dfrac{C_1}{\eta} > r^*$, we have

$$\frac{C_1}{\eta} - \|M_p(I-P)\|\big(M_1(2\|\psi\|)^\beta \big(\frac{C_1}{\eta}\big)^\beta + M_2\big) > 0.$$

However,

$$\|\alpha\psi + x\| \leqslant \|\psi\|\frac{C_1}{\eta} + \frac{C_1}{\eta} \leqslant 2\|\psi\|\frac{C_1}{\eta},$$

so that for $s \in (0,1)$,

$$\begin{aligned}\|sM_p(I-P)\mathscr{F}(\alpha\psi + x)\| &\leqslant \|M_p(I-P)\|\big(M_1\big(2\|\psi\|\frac{C_1}{\eta}\big)^\beta + M_2\big) \\ &< \frac{C_1}{\eta} = \|x\|.\end{aligned}$$

It follows that

$$x \neq sM_p(I-P)\mathscr{F}(\alpha\psi(\cdot)+x). \tag{5}$$

We now assume $(x,\alpha) \in \partial(\Omega_\eta)$ and that ii. holds. We may assume without loss of generality that $x = sM_p(I-P)\mathscr{F}(\alpha\psi + x)$ for some $s \in (0,1)$. For the moment we assume $\alpha = C_1/\eta$. We introduce the following sets which will be useful in what follows:

$$O^+ = \{t \in [0,1] \mid \psi(t) > 0\} \quad \text{and} \quad O^- = \{t \in [0,1] \mid \psi(t) < 0\}.$$

Fix $t \in O^+ \cap A_\eta$, where A_η is as in Proposition 2.4. Since $|\alpha| = C_1/\eta$, we conclude that

$$|x(t)| \leqslant C_2|\alpha\psi(t)|^\beta,$$

from which is follows that

$$\begin{aligned}\alpha\psi(t)+x(t) &\geqslant \alpha\psi(t) - |x(t)|\\ &\geqslant \alpha\psi(t) - C_2(\alpha\psi(t))^\beta\\ &\geqslant \hat{z}. \quad (\text{since } \alpha\psi(t) \geqslant \frac{C_1}{\eta}\eta = C_1 > r^*.)\end{aligned}$$

We then have that

$$\psi(t)f(\alpha\psi(t)+x(t)) > \psi(t)J = |\psi(t)|J,$$

whenever $t \in O^+ \cap A_\eta$.

The same argument shows that for $t \in O^- \cap A_\eta$,

$$\alpha\psi(t)+x(t) < -\hat{z}$$

and

$$\psi(t)f(\alpha\psi(t)+x(t)) > \psi(t)(-J) = |\psi(t)|J.$$

Now,

$$\begin{aligned}\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt &= \int_{O^+\cap A_\eta} \psi(t)f(\alpha\psi(t)+x(t))\,dt\\ &\qquad + \int_{O^-\cap A_\eta} \psi(t)f(\alpha\psi(t)+x(t))\,dt\\ &\qquad\quad + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))\,dt\\ &\geqslant \int_{A_\eta} |\psi(t)|J\,dt + \int_{A_\eta^c} \psi(t)f(\alpha\psi(t)+x(t))\,dt\\ &\geqslant \int_{A_\eta} |\psi(t)|J\,dt - m(A_\eta^c)\eta\big(M_1(2|\alpha|\|\psi\|)^\beta + M_2\big)\\ &= \int_{A_\eta} |\psi(t)|J\,dt - m(A_\eta^c)\eta\big(M_1\big(2\frac{C_1}{\eta}\|\psi\|\big)^\beta + M_2\big),\end{aligned}$$

where m denotes Lebesgue measure on $[0,1]$.

Since the Lebesgue measure of $\{t \in [0,1] \mid \psi(t) = 0\}$ is 0 and $\beta < 1$, we have that

$$\lim_{\eta \to 0} m(A_\eta^c)\eta\big(M_1\big(2\frac{C_1}{\eta}\|\psi\|\big)^\beta + M_2\big) = 0.$$

Thus, there exists η^* such that for all $(x,\alpha) \in \partial(\Omega_{\eta^*})$ satisfying $\alpha = \dfrac{C_1}{\eta^*}$, we have

$$\int_0^1 \psi(t) f(\alpha\psi(t) + x(t)) > 0.$$

Now,

$$\begin{aligned}\big|(1-s)\alpha + s\int_0^1 \psi(t) f(\alpha\psi(t)+x(t))dt\big|^2 \\ = (1-s)^2\alpha^2 + s^2\big(\int_0^1 \psi(t) f(\alpha\psi(t)+x(t))dt\big)^2 \\ + 2(1-s)s\alpha\int_0^1 \psi(t) f(\alpha\psi(t)+x(t))dt.\end{aligned}$$

Since $\int_0^1 \psi(t) f(\alpha\psi(t)+x(t))\,dt \neq 0$, and α and $\int_0^1 \psi(t) f(\alpha\psi(t)+x(t))\,dt$ have the same sign, we conclude

$$\big|(1-s)\alpha + s\int_0^1 \psi(t) f(\alpha\psi(t)+x(t))\,dt\big|^2 > 0. \tag{6}$$

The same argument shows that if $(x,\alpha) \in \partial(\Omega_{\eta^*})$ and $\alpha = -\dfrac{C_1}{\eta^*}$, then

$$\int_0^1 \psi(t) f(\alpha\psi(t) + x(t)) < 0.$$

Thus, (6) holds for all $(x,\alpha) \in \partial(\Omega_{\eta^*})$ satisfying ii..

If we define $Q : [0,1] \times \overline{\Omega_{\eta^*}} \leftarrow Im(I-P) \times \mathbb{R}$ by

$$Q(s,(x,\alpha)) = \begin{bmatrix} x - sM_p(I-P)F(\alpha\psi(\cdot)+x) \\ (1-s)\alpha + s\int_0^1 \psi(t) f(\alpha\psi(t)+x(t))\,dt \end{bmatrix},$$

then it is clear that Q is a homotopy between I and $I-H$. Further, (5) and (6) show that $Q(s,(x,\alpha)) \neq 0$ for all $(x,\alpha) \in \partial(\Omega_{\eta^*})$. This establishes the result for $\varepsilon = 0$, by the invariance of the Leray-Schauder degree under homotopy.

To finish the proof we define

$H_\varepsilon : Im(\mathscr{L}_\lambda)\times\mathbb{R} \leftarrow Im(\mathscr{L}_\lambda)\times\mathbb{R}$ by

$$H_\varepsilon(x,\alpha) = \begin{bmatrix} M_p(I-P)\mathscr{F}_\varepsilon(\alpha\psi(\cdot)+x) \\ \alpha - \int_0^1 \psi(t)f(\varepsilon,\alpha\psi(t)+x(t))\,dt \end{bmatrix}.$$

Our previous work shows that the Leray-Schauder degree, $deg(H_0,\Omega_{\eta^*},0)$, is nonzero. It then follows that $deg(H_\varepsilon,\Omega_{\eta^*},0)$ is nonzero for sufficiently small ε by the continuity of the Leray-Schauder degree.

4. Existence at the First Eigenvalue

In the case where λ is the first eigenvalue of the linear Sturm-Liouville problem, the following result establishes the existence of solutions under weaker assumptions than that of Theorem 3.1. We remind the reader of our notation when $\varepsilon = 0$, where we denote $f(0,x)$ by $f(x)$ and $\mathscr{F}_0$ by $\mathscr{F}$.

THEOREM 4.1. *Suppose the following conditions hold:*

C1. λ *is the first eigenvalue of the linear Sturm-Liouville problem (1)-(2).*

C2. The function f *is "sublinear"; that is, there exists real numbers* M_1,M_2 *and* β*, with* $0\leqslant\beta<1$*, such that* $|f(x)|\leqslant M_1|x|^\beta+M_2$.

C3. There exists a $\hat{z}>0$ *such that for all* $z>\hat{z}$*,* $f(-z)<0<f(z)$.

Then, for sufficiently small ε*, there exists a solution to (1)-(2).*

Proof.
It is well known that the eigenfunction corresponding to the first eigenvalue has no zeros in the interval $[0,1]$. We assume, without loss of generality, that $\psi>0$ and let

$$\eta = \inf_{t\in[0,1]}\{\psi(t)\} > 0.$$

We let C_1, C_2 and A_η be as in Proposition 2.4. We note that by the definition of η, we have $A_\eta = [0,1]$. We now follow the line of reasoning used in Theorem 3.1.

Let $H : Im(\mathscr{L}_\lambda)\times\mathbb{R} \leftarrow Im(\mathscr{L}_\lambda)\times\mathbb{R}$ be defined by

$$H(x,\alpha) = \begin{bmatrix} M_p(I-P)\mathscr{F}(\alpha\psi(\cdot)+x) \\ \alpha - \int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt \end{bmatrix}.$$

We choose r^* such that for all $r \geqslant r^*$, we have

$$r - C_2 r^\beta \geqslant \hat{z}$$

and

$$r - \|M_p(I-P)\|\big(M_1(2\|\psi\|)^\beta r^\beta + M_2\big) > 0.$$

Again, we assume that $C_1 > r^* > 1$ and define

$$\Omega_\eta = B\big(0, \frac{C_1}{\eta}\big) \text{ in } Im(\mathscr{L}_\lambda) \times \mathbb{R}.$$

We now analyze H on $\partial(\Omega_\eta)$. Recall that if $(x,\alpha) \in \partial(\Omega_\eta)$, then one of two following condtitions is true:

(i) $\|x\| = \dfrac{C_1}{\eta}$ and $|\alpha| \leqslant \dfrac{C_1}{\eta}$.

(ii) $\|x\| \leqslant \dfrac{C_1}{\eta}$ and $|\alpha| = \dfrac{C_1}{\eta}$.

We assume first that $(x,\alpha) \in \partial(\Omega_\eta)$ and that i. holds. For this case, the proof is as in Theorem 3.1.

Since $\dfrac{C_1}{\eta} > r^*$, we have

$$\frac{C_1}{\eta} - \|M_p(I-P)\|\big(M_1(2\|\psi\|)^\beta \big(\frac{C_1}{\eta}\big)^\beta + M_2\big) > 0.$$

However,

$$\|\alpha\psi + x\| \leqslant \|\psi\|\frac{C_1}{\eta} + \frac{C_1}{\eta} \leqslant 2\|\psi\|\frac{C_1}{\eta},$$

so that for $s \in (0,1)$

$$\begin{aligned}\|sM_p(I-P)\mathscr{F}(\alpha\psi + x)\| &\leqslant \|M_p(I-P)\|\big(M_1\big(2\|\psi\|\frac{C_1}{\eta}\big)^\beta + M_2\big) \\ &< \frac{C_1}{\eta} = \|x\|.\end{aligned}$$

It follows that

$$x \neq sM_p(I-P)\mathscr{F}(\alpha\psi + x). \tag{7}$$

We now assume $(x,\alpha) \in \partial(\Omega_\eta)$ and that ii. holds. We will again assume that $x = sM_p(I-P)\mathscr{F}(\alpha\psi + x)$ for some $s \in (0,1)$ and restrict our attention to the case where $\alpha = \dfrac{C_1}{\eta}$.

Since $|\alpha| = \dfrac{C_1}{\eta}$, we conclude that

$$|x(t)| \leqslant C_2|\alpha\psi(t)|^\beta,$$

It follows that

$$\begin{aligned}\alpha\psi(t)+x(t) &\geqslant \alpha\psi(t)-|x(t)|\\ &\geqslant \alpha\psi(t)-C_2(\alpha\psi(t))^{\beta}\\ &\geqslant \hat{z}. \quad \big(\text{since } \alpha\psi(t)\geqslant \frac{C_1}{\eta}\eta = C_1 > r^*.\big)\end{aligned}$$

We then have that for all $t \in A_\eta = [0,1]$,

$$\psi(t)f(\alpha\psi(t)+x(t)) > 0.$$

Thus,

$$\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt > 0.$$

Now,

$$\begin{aligned}&\left|(1-s)\alpha+s\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt\right|^2\\ &\qquad = (1-s)^2\alpha^2+s^2\left(\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))dt\right)^2\\ &\qquad\qquad + 2(1-s)s\alpha\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))dt.\end{aligned}$$

Since $\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt \neq 0$, and α and $\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt$ have the same sign, we conclude

$$\left|(1-s)\alpha+s\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt\right|^2 > 0. \tag{8}$$

The same argument shows that if $(x,\alpha)\in\partial(\Omega_\eta)$ and $\alpha = -\frac{C_1}{\eta}$, then

$$\int_0^1 \psi(t)f(\alpha\psi(t)+x(t)) < 0.$$

Thus, (6) holds for all $(x,\alpha)\in\partial(\Omega_\eta)$ satisfying ii..

If we again define $Q:[0,1]\times\overline{\Omega_\eta}\leftarrow Im(I-P)\times\mathbb{R}$ by

$$Q(s,(x,\alpha)) = \begin{bmatrix} x - sM_p(I-P)F(\alpha\psi(\cdot)+x) \\ \\ (1-s)\alpha+s\int_0^1 \psi(t)f(\alpha\psi(t)+x(t))\,dt \end{bmatrix},$$

and $H_\varepsilon : Im(\mathscr{L}_\lambda)\times\mathbb{R}\leftarrow Im(\mathscr{L}_\lambda)\times\mathbb{R}$ by

$$H_\varepsilon(x,\alpha) = \begin{bmatrix} M_p(I-P)\mathscr{F}_\varepsilon(\alpha\psi(\cdot)+x) \\ \\ \alpha - \int_0^1 \psi(t)f(\varepsilon,\alpha\psi(t)+x(t))\,dt \end{bmatrix}.$$

then the result follows just as in Theorem 3.1.

REFERENCES

[1] S. CHOW, J.K. HALE, *Methods of Bifurcation Theory*, Springer, Berlin, 1982.

[2] D.L. ETHERIDGE, J. RODRÍGUEZ, *Periodic solutions of nonlinear discrete-time systems*, Appl. Anal., **62** (1996), 119–137.

[3] J.K. HALE, *Applications of Alternative Problems*, Lecture Notes, Brown University , Providence, RI.

[4] J.K. HALE, *Ordinary Differential Equations*, Robert E. Kreiger Publishing Company, Malabar FL, 1980.

[5] W.G. KELLEY, A.C. PETERSON, *The theory of differential Equations*, Springer 2010.

[6] E.M. LANDESMAN, A.C. LAZER, *Nonlinear Perturbations of Linear Elliptic Boundary Value Problems at Resonances*, J. Math. Mech., **19** (1969), 609–623.

[7] A.C. LAZER, D.F. LEACH, *Bounded perturbations of forced harmonic oscillators at resonance*, Ann. Mat. Pure Appl., **82** (1969), 49–68.

[8] R. MA, *Nonlinear discrete Sturm-Liouville problems at resonance*, Nonlinear Anal., **67** (2007), 3050–3057.

[9] D. MARONCELLI, J. RODRÍGUEZ, *On the solvability of nonlinear impulsive boundary value problems*, Topol. Methods Nonlinear Anal., accepted.

[10] J. RODRÍGUEZ, *Nonlinear discrete Sturm-Liouville problems*, J. Math. Anal. Appl., **308**, Issue 1 (2005), 380–391.

[11] J. RODRÍGUEZ, Z. ABERNATHY, *On the solvability of nonlinear Sturm-Liouville problems*, J. Math. Anal. Appl., **387**, Issue 1 (2012), 310–319.

[12] J. RODRÍGUEZ, D.L. ETHERIDGE, *Periodic solutions of nonlinear second-order difference equations*, Adv. Difference Eqn., **2** (2005), 119–137.

[13] J.F. RODRÍGUEZ, *Existence theory for nonlinear eigenvalue problems*, Appl. Anal., **87** (2008), 293–301.

[14] N. ROUCHE, J. MAWHIN, *Ordinary differential equations*, Pitman, London, 1980.

(Received March 14, 2014)

D. Maroncelli
Department of Mathematics
Concordia University St. Paul
1282 Concordia Avenue St. Paul, MN 55104-5494
e-mail: maroncelli@csp.edu

J. Rodríguez
Department of Mathematics
Box 8205, North Carolina State University
Raleigh, NC, 27695-8205
U.S.A
e-mail: rodrigu@ncsu.edu