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ASYMPTOTIC PROPERTIES OF SOLUTIONS TO A NONLINEAR SYSTEM OF NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this article we study the behavior of solutions to the system of delay differential equations

$$\begin{aligned} \big[y_1(t)+a(t)y_1(g(t))\big]' &= p_1(t)f_1\big(y_2(h_2(t))\big)\\ y_2'(t) &= p_2(t)f_2\big(y_3(h_3(t))\big)\\ &\dots\\ y_{n-1}'(t) &= p_{n-1}(t)f_{n-1}\big(y_n(h_n(t))\big)\\ y_n'(t) &= f_n\big(t,y_1(h_1(t))\big), \end{aligned}$$

where the coefficients p_i may have zeros, and the components of the solution may change signs. We prove properties to the components of the solutions as t approaches infinity.

1. Introduction

We consider the system of nonlinear differential equations

$$\begin{aligned} \big[y_1(t)+a(t)y_1(g(t))\big]' &= p_1(t)f_1\big(y_2(h_2(t))\big)\\ y_2'(t) &= p_2(t)f_2\big(y_3(h_3(t))\big)\\ &\dots\\ y_{n-1}'(t) &= p_{n-1}(t)f_{n-1}\big(y_n(h_n(t))\big)\\ y_n'(t) &= f_n\big(t,y_1(h_1(t))\big), \end{aligned} \tag{1.1}$$

where a, f_i, g, h_i, and p_i are given functions that satisfy the conditions stated below. Since the deviating arguments $h_i(t)$ and $g(t)$ may or may not be larger than t, we can have advanced, or retarded, or neutral differential equations. Our goal is to study the limit of the components y_i as t approaches infinity, by considering the possible limits of the quotient $[y_1+ay_1]/\int p_1\int p_2\dots$.

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Systems of functional differential equations with deviating arguments have been studied by many authors; see for example [1]–[15] and their references. When the coefficients $p_i(t)$ do not have zeros, (1.1) can be rewritten as an n-th order differential equation, for which there are several publications available; see for example [4, 5]. However, most of these publications study only consider non-oscillatory solutions.

The main two features in this article are allowing the coefficients p_i to have zeros, and allowing the functions y_i to oscillate. This work is also motivated by the work of Kitamura [8] for the system

$$\begin{aligned} y_1'(t) &= p_1(t)y_2(t) \\ y_2'(t) &= f_n\big(t, y_1(h_1(t))\big). \end{aligned}$$

Several techniques from [8] are used in our proofs, as indicated at the appropriate places.

By a solution we mean a continuous function $y = (y_1, \dots, y_n)$ that satisfies (1.1) on an interval $[t_0, \infty)$. Obviously, this requires the components to be defined for $t \geqslant \min\{t_0, \inf_{t \geqslant t_0} h_i(t), \inf_{t \geqslant t_0} g(t)\}$.

We will use the following assumptions:

(H1) For $i = 1, \dots, n$, the delay/advance functions h_i belong to $C([0,\infty),[0,\infty))$ and $\lim_{t\to\infty} h_i(t) = \infty$.

(H2) For $i = 1, \dots, n-1$, the coefficients p_i belong to $C([0,\infty),[0,\infty))$ and satisfy $\int_0^\infty p_i(s)\,ds = \infty$.

(H3) For $i = 1, \dots, n-1$: the functions f_i belong to $C(\mathbb{R},\mathbb{R})$; there exist constants β_i such that $|f_i(x)| \leqslant \beta_i |x|$ for all $x \in \mathbb{R}$; and $f_i(x) = 0$ if and only if $x = 0$.

(H4) The function f_n belongs to $C([t_0,\infty)\times\mathbb{R},\mathbb{R})$; there exists a non-negative function ω such that $|f_n(t,x)| \leqslant \omega(t,|x|)$; and $\omega(t,x)$ is non-decreasing with respect to x for each $t \geqslant t_0$.

Initially, we set $a(t) = 0$ and study the system

$$\begin{aligned} y_1'(t) &= p_1(t) f_1\big(y_2(h_2(t))\big) \\ &\dots \\ y_{n-1}'(t) &= p_{n-1}(t) f_{n-1}\big(y_n(h_n(t))\big) \\ y_n'(t) &= f_n\big(t, y_1(h_1(t))\big). \end{aligned} \tag{1.2}$$

Later, we add some assumptions and obtain results for (1.1).

2. Results for $a(t) \equiv 0$

Let

$$P_{i,\dots,j}(t) = \int_{t_0}^{t} p_i(x_i) \int_{t_0}^{x_i} p_{i+1}(x_{i+1}) \dots \int_{t_0}^{x_{j-1}} p_j(x_j)\,dx_j \dots dx_{i+1}\,dx_i,$$

$$P_i(t) = \int_{t_0}^{t} p_i(s)\,ds\,.$$

Using (H2) we can show that the functions $P_i(t)$ and $P_{i,\dots,j}(t)$ are non-decreasing and both approach infinity as $t \to \infty$. It is easy to show that $P_i(t)P_{i+1}(t) \geqslant P_{i,i+1}(t)$, and $\frac{d}{dt}P_{i,\dots,j}(t) = p_i(t)P_{i+1,\dots,j}(t)$. Using L'Hôpital's Rule, for $i < j$, we have

$$\lim_{t\to\infty} \frac{P_{i,i+1}(t)}{P_{i,\dots,j}(t)} = \lim_{t\to\infty} \frac{P_{i+1}(t)}{P_{i+1,\dots,j}(t)}, \quad \lim_{t\to\infty} \frac{P_i(t)}{P_{i,\dots,j}(t)} = 0\,.$$

LEMMA 2.1. *Assume* (H1)–(H3), *and let* $(y_1,\dots,y_n)$ *be a solution of* (1.2). *If*

$$\limsup_{t\to\infty} \frac{|y_1(t)|}{P_{1,\dots,n-1}(t)} = \infty,$$

then

$$\limsup_{t\to\infty} \frac{|y_2(t)|}{P_{2,\dots,n-1}(t)} = \infty,\ \dots,\ \limsup_{t\to\infty} \frac{|y_{n-1}(t)|}{P_{n-1}(t)} = \infty, \quad \limsup_{t\to\infty} |y_n(t)| = \infty$$

and $\limsup_{t\to\infty} y_1(t) = \infty$.

Proof. Using a contrapositive argument, we assume that there are constants M and $t^* > t_0$ such that $|y_2(t)|/P_{2,\dots,n-1}(t) \leqslant M$ for all $t \geqslant t*$, and show that $|y_1(t)|/P_{1,\dots,n-1}(t)$ is bounded for $t \geqslant t^*$. By (1.2) and (H3), for $t \geqslant t^*$,

$$|y_1(t)| \leqslant |y_1(t_0)| + \beta_2 M \int_{t_0}^{t} p_1(s)P_{2,\dots,n-1}(s)\,ds = |y_1(t_0)| + \beta_2 M P_{1,\dots,n-1}(t)\,,$$

which completes the contrapositive argument for the first limit.

By the same process if $|y_3(t)|/P_{3,\dots,n-1}(t)$ is bounded, then $|y_2(t)|/P_{2,\dots,n-1}(t)$ is bounded. Then recursively we obtain the desired results.

That $\limsup_{t\to\infty} y_1 = \infty$ follows from the fact that there is a sequence $\{t_m\}$ approaching infinity such that the fraction $|y_1(t_m)|/P_{1,\dots,n-1}(t_m)$ approaches infinity. Since the denominator approaches infinity, $|y_1(t_m)|$ must approach infinity, as $m \to \infty$. This completes the proof.

We remark that the converse of Lemma 2.1 is not necessarily true. For example, if y_2 is unbounded and integrable on $[0,\infty)$, and $p_1 = 1$, then $\limsup |y_1|/P_1 = 0$.

LEMMA 2.2. *Let* $(y_1,\dots,y_n)$ *be a solution of* (1.2), *and assume that* (H1)–(H4) *hold, and that*

$$\int_0^\infty \omega\big(s, MP_{1,\dots,n-1}(h_1(s))\big)\,ds < \infty \quad \textit{for each positive constant } M\,. \tag{2.1}$$

If

$$\limsup_{t\to\infty} \frac{|y_1(t)|}{P_{1,\dots,n-1}(t)} < \infty, \tag{2.2}$$

then y_n has finite limit at infinity, with

$$\lim_{t\to\infty} y_n(t) = \alpha_n, \quad \textit{and} \quad y_n(t) = \alpha_n - \int_t^\infty f_n\big(s, y_1(h_1(s))\big)\, ds.$$

Proof. By (2.2), there exist constants M and $t^* > t_0$ such that $|y_1(t)| \leqslant MP_{1,\dots,n-1}(t)$ for all $t \geqslant t^*$. From (H4) and (2.1),

$$\int_{t^*}^t f_n\big(s, y_1(h_1(s))\big)\, ds \leqslant \int_{t^*}^\infty \omega\big(s, MP_{1,\dots,n-1}(h_1(s))\big)\, ds < \infty.$$

This allows as to define $\alpha_n = y_n(t_0) + \int_{t_0}^\infty f_n\big(s, y_1(h_1(s))\big)\, ds$. Then the conclusion follows.

LEMMA 2.3. *Let* $(y_1, \dots, y_n)$ *be a solution of* (1.2)*, and assume* (H1)–(H4)*,* (2.1) *and* (2.2)*. If* $\lim_{t\to\infty} y_n(t) = \alpha_n \neq 0$*, then*

$$\lim_{t\to\infty} \frac{y_{n-1}(t)}{P_{n-1}(t)} = f_{n-1}(\alpha_n) \neq 0, \quad \lim_{t\to\infty} \frac{|y_{n-2}(t)|}{P_{n-2,n-1}(t)} = f_{n-2}\big(f_{n-1}(\alpha_n)\big) \neq 0,$$
$$\dots, \lim_{t\to\infty} \frac{y_1(t)}{P_{1,\dots,n-1}(t)} = f_1\big(\dots f_{n-2}\big(f_{n-1}(\alpha_n)\big)\big) \neq 0.$$

Proof. Since $\alpha_n \neq 0$, there exists t^* such that $|y_n(t)| \geqslant |\alpha_n/2| > 0$ for all $t \geqslant t^*$. By (H1) and (H3), there exist positive constants M and t^{**} such that $|f_{n-1}(y_n(h_n(t)))| \geqslant M$ for all $t \geqslant t^{**}$. Then by (H2),

$$y_{n-1}(t_0) + \lim_{t\to\infty} \int_{t_0}^t p_{n-1}(s) f_{n-1}(y_n(h_n(t)))\, ds = \pm\infty.$$

Thus we can apply L'Hôpital Rule to obtain

$$\lim_{t\to\infty} \frac{y_{n-1}(t)}{P_{n-1}(t)} = \lim_{t\to\infty} \frac{y_{n-1}(t_0)}{P_{n-1}(t)} + \lim_{t\to\infty} \frac{\int_{t_0}^t p_{n-1}(s) f_{n-1}\big(y_n(h_n(s))\big)\, ds}{\int_{t_0}^t p_{n-1}(s)\, ds}$$
$$= 0 + \lim_{t\to\infty} f_{n-1}\big(y_n(h_n(t))\big) = f_{n-1}(\alpha_n) \neq 0.$$

Here we used (H2) and the continuity of f_{n-1} and y_n.

Since the fraction $y_{n-1}(t)/P_{n-1}(t)$ approaches a non-zero number and the denominator approaches infinity, the numerator must approach $\pm\infty$. By (H1) and (H3), there exist positive constants M and t^* such that $|f_{n-2}(y_{n-1}(h_{n-1}(t)))| \geqslant M$ for all $t \geqslant t^*$. Then by (H2),

$$y_{n-2}(t_0) + \lim_{t\to\infty} \int_{t_0}^t p_{n-2}(s) f_{n-2}(y_n(h_n(t)))\, ds = \pm\infty.$$

Thus we can apply L'Hôpital Rule to obtain

$$\lim_{t\to\infty} \frac{y_{n-2}(t)}{P_{n-2,n-1}(t)} = f_{n-2}\big(f_{n-1}(\alpha_n)\big) \neq 0.$$

Reaping this process, we obtain the desired conclusions.

For the next proof we use repeatedly the following equality which is obtained by integration by parts:

$$\int_{t_0}^{t} p(\xi) \int_{\xi}^{\infty} f(s)\,ds\,d\xi = \int_{t_0}^{t} P(s) f(s)\,ds + P(t) \int_{t}^{\infty} f(s)\,ds, \tag{2.3}$$

where $P(t) = \int_{t_0}^{t} p(\xi)\,d\xi$. Also in the next theorem we use a condition more restrictive that (2.1); in fact condition (2.5) implies (2.1).

THEOREM 2.1. *Let* $(y_1,\ldots,y_n)$ *be a solution of* (1.2) *with* $h_i(t) = t$ *for* $i = 2,\ldots,n-1$ *and* $h_n(t) \geqslant t$. *Assume* (H1)–(H4), (2.2), *that*

$$\frac{\omega(t,x)}{x} \text{ is non-decreasing with respect to } x, \tag{2.4}$$

and that

$$\int_{t_0}^{\infty} \max\Big\{1, \frac{P_{n-1}(s)}{P_{n-1}(h_1(s))}\Big\}\, \omega\big(s, MP_1(h_1(s)) \cdots P_{n-1}(h_1(s))\big)\,ds < \infty \tag{2.5}$$

for each positive constant M. *If* $\lim_{t\to\infty} y_n(t) = 0$, *then:*

(1) $\limsup_{t\to\infty} \frac{|y_1(t)|}{P_1(t),\ldots,P_{n-2}(t)} < \infty$,

(2) y_{n-1} *has finite limit at infinity, with*

$$\lim_{t\to\infty} y_{n-1}(t) = \alpha_{n-1}, \quad y_{n-1}(t) = \alpha_{n-1} - \int_{t}^{\infty} p_{n-1}(s) f_n\big(y_n(h_n(s))\big)\,ds,$$

(3) y_n *converges to zero at least at the rate given by* $\lim_{t\to\infty} P_{n-1}(t) y_n(t) = 0$.

Proof. Since all limits are at infinity, through out this proof we restrict t to be larger than a value $t^* > t_0$ such that $h_1(s) > t_0$ for all $s \geqslant t^*$. This way we ensure that $P_i(h_1(t)$ and $P_i(t)$ are never zero and can be used in denominators.

Our first step is to find a bound for y_1. By (1.2), (H3), $h_{n-1}(t) = t$, $h_n(t) \geqslant t$, and (2.3), we have

$$\begin{aligned}
|y_{n-1}(t)| &\leqslant |y_{n-1}(t_0)| + \int_{t_0}^{t} p_{n-1}(\xi) |f_{n-1}\big(y_n(h_n(\xi))\big)\,d\xi \\
&\leqslant |y_{n-1}(t_0)| + \beta_{n-1} \int_{t_0}^{t} p_{n-1}(\xi) \int_{\xi}^{\infty} |f_n|\,ds\,d\xi \\
&\leqslant |y_{n-1}(t_0)| + \beta_{n-1} \Big[\int_{t_0}^{t} P_{n-1}(s)|f_n|\,ds + P_{n-1}(t) \int_{t}^{\infty} |f_n|\,ds \Big].
\end{aligned}$$

Then

$$|y_{n-2}(t)| \leqslant |y_{n-2}(t_0)| + \beta_{n-2} \int_{t_0}^{t} p_{n-2}(s) |y_{n-1}(s)|\,ds$$

$$
\begin{aligned}
&\leqslant |y_{n-2}(t_0)| + \beta_{n-2}P_{n-2}(t)|y_{n-1}(t_0)| \\
&\quad + \beta_{n-2}\beta_{n-1}P_{n-2}(t)\Big[\int_{t_0}^{t} P_{n-1}(s)|f_n|\,ds + P_{n-1}(t)\int_t^\infty |f_n|\,ds\Big].
\end{aligned}
$$

Recursively, we obtain an inequality for $|y_1(t)|$. Then dividing by $\big(P_1(t)\cdots P_{n-1}(t)\big)$, we obtain

$$
\begin{aligned}
\frac{|y_1(t)|}{P_1(t)\cdots P_{n-1}(t)} &\leqslant \frac{|y_1(t_0)|}{P_1(t)\cdots P_{n-1}(t)} + \frac{\beta_1|y_2(t_0)|}{P_2(t)\cdots P_{n-1}(t)} + \frac{\beta_1\beta_2|y_3(t_0)|}{P_3(t)\cdots P_{n-1}(t)} + \dots \\
&\quad + \frac{(\beta_1\cdots\beta_{n-2})|y_{n-1}(t_0)|}{P_{n-1}(t)} + \frac{(\beta_1\cdots\beta_{n-1})}{P_{n-1}(t)}\int_{t_0}^{t} P_{n-1}(s)|f_n|\,ds \qquad (2.6)\\
&\quad + (\beta_1\cdots\beta_{n-1})\int_t^\infty |f_n|\,ds\,.
\end{aligned}
$$

Taking the derivative with respect to t, we observe that the right-hand side is non-increasing; thus the right-hand side is an upper bound for the function

$$
u(t) = \sup_{s\geqslant t}\frac{|y_1(s)|}{P_1(s)\cdots P_{n-1}(s)}\,.
$$

Note that $u(t)$ is a non-increasing function.

By (2.4), for values $0 < a \leqslant 1$ and $b > 0$, we have

$$
\omega(s,ab) \leqslant a\omega(s,b)\,. \qquad (2.7)
$$

By (2.2) and $P_{1,\dots,n-1}(t) \leqslant P_1(t)\cdots P_{n-1}(t)$, there exists a positive constant M such that

$$
\frac{|y_1(t)|}{P_1(t)\cdots P_{n-1}(t)} \leqslant M \quad \text{for all } t \geqslant t^*\,.
$$

Then by (H4) and (2.7),

$$
\begin{aligned}
|f_n(s,y_1(h_1(s)))| &\leqslant \omega(s,y_1(h_1(s))) \\
&\leqslant \frac{|y_1(h_1)|}{MP_1(h_1)\cdots P_{n-1}(h_1)}\omega(s,MP_1(h_1(s))\cdots P_{n-1}(h_1(s))) \\
&= \frac{1}{M}u(h_1(s))\omega\big(s,MP_1(h_1(s))\cdots P_{n-1}(h_1(s))\big)\,.
\end{aligned}
$$

Using that the right-hand side of (2.6) is an upper bound for $u(t)$, and multiplying by $P_{n-1}(t)$, we have

$$
\begin{aligned}
&P_{n-1}(t)u(t) \\
&\leqslant k + \frac{(\beta_1\cdots\beta_{n-1})}{M}\int_{t_0}^{t} P_{n-1}(s)u(h_1(s))\omega(s,MP_1(h_1(s))\cdots P_{n-1}(h_1(s)))\,ds \qquad (2.8)\\
&\quad + \frac{(\beta_1\cdots\beta_{n-1})}{M}P_{n-1}(t)\int_t^\infty u(h_1(s))\omega\big(s,MP_1(h_1(s))\cdots P_{n-1}(h_1(s))\big)\,ds.
\end{aligned}
$$

Where k is a constant such that for $t \geqslant t^*$,

$$k \geqslant \frac{|y_1(t_0)|}{P_1(t)\cdots P_{n-2}(t)} + \frac{\beta_1 |y_2(t_0)|}{P_2(t)\cdots P_{n-2}(t)} + \ldots + (\beta_1 \cdots \beta_{n-2})|y_{n-1}(t_0)|.$$

By (2.5) there exists a value $t_1 \geqslant t^* > t_0$ such that

$$\frac{(\beta_1\cdots\beta_{n-1})}{M}\int_{t_1}^{\infty} \max\{1, \frac{P_{n-1}(s)}{P_{n-1}(h_1(s))}\}\omega\big(s, MP_1(h_1(s))\cdots P_{n-1}(h_1(s))\big)\,ds < \frac{1}{3}. \tag{2.9}$$

To estimate the integrals in (2.8), as in [8], we split the interval $[t_1,\infty)$ into two sets:

$$I = \{s \geqslant t_1 : h_1(s) \leqslant t\}, \quad J = \{s \geqslant t_1 : h_1(s) > t\}.$$

On the set $I_t = [t_1,t]\cap I$, we have $[P_{n-1}(h(s))u(h(s))] \leqslant \sup_{t_0\leqslant s\leqslant t}[P_{n-1}(s)u(s)]$ and

$$\int_{I_t} P_{n-1}(s)u(h_1(s))\omega(s,\ldots)\,ds \leqslant \sup_{t_0\leqslant s\leqslant t}[P_{n-1}(s)u(s)]\int_{I_t}\frac{P_{n-1}(s)}{P_{n-1}(h_1(s))}\omega(s,\ldots)\,ds. \tag{2.10}$$

On the set $J_t = [t_1,t]\cap J$, using that $u(h_1(s)) \leqslant u(t)$ and that $P_{n-1}(s) \leqslant P_{n-1}(t)$, we have

$$\begin{aligned}\int_{J_t} P_{n-1}(s)u(h_1(s))\omega(s,\ldots)\,ds &\leqslant P_{n-1}(t)u(t)\int_{J_t}\omega(s,\ldots)\,ds\\ &\leqslant \sup_{t_0\leqslant s\leqslant t}[P_{n-1}(s)u(s)]\int_{J_t}\omega(s,\ldots)\,ds.\end{aligned} \tag{2.11}$$

On the set $I^t = (t,\infty)\cap I$, using that $P_{n-1}(t) \leqslant P_{n-1}(s)$, we have

$$\int_{I^t} P_{n-1}(s)u(h_1(s))\omega(s,\ldots)\,ds \leqslant \sup_{t_0\leqslant s\leqslant t}[P_{n-1}(s)u(s)]\int_{I^t}\frac{P_{n-1}(s)}{P_{n-1}(h_1(s))}\omega(s,\ldots)\,ds. \tag{2.12}$$

On the set $J^t = (t,\infty)\cap J$, using that $u(h_1(s)) \leqslant u(t)$, we have

$$P_{n-1}(t)\int_{J^t} u(h_1(s))\omega(s,\ldots)\,ds \leqslant P_{n-1}(t)u(t)\int_{J^t}\omega(s,\ldots)\,ds. \tag{2.13}$$

Since I_t, J_t and I^t are disjoint subsets of $[t_1,\infty)$, (2.10)–(2.12) can be combined into single integral inequality. From (2.8), (2.10)–(2.13), for $t \geqslant t_1$, we have

$$\begin{aligned}&P_{n-1}(t)u(t)\\ &\leqslant k_1 + \sup_{t_0\leqslant s\leqslant t}[P_{n-1}(s)u(s)]\frac{(\beta_1\cdots\beta_{n-1})}{M}\int_{t_1}^{\infty}\max\{1,\frac{P_{n-1}(s)}{P_{n-1}(h_1 s)}\}\omega(s,\ldots)\,ds\\ &\quad + P_{n-1}(t)u(t)\frac{(\beta_1\cdots\beta_{n-1})}{M}\int_{t_1}^{\infty}\omega(s,\ldots)\,ds\end{aligned}$$

where

$$k_1 = k + \frac{(\beta_1 \cdots \beta_{n-1})}{M} \int_{t_0}^{t_1} P_{n-1}(s) u(h_1(s)) \omega(s, \dots)\, ds.$$

Using (2.9), we have

$$\frac{2}{3} P_{n-1}(t) u(t) \leqslant k_1 + \sup_{t_0 \leqslant s \leqslant t} [P_{n-1}(s) u(s)] \frac{1}{3}$$

and

$$\frac{|y_1(t)|}{P_1(t) \cdots P_{n-2}(t)} \leqslant P_{n-1}(t) u(t) \leqslant 3k_1 \quad \text{for all } t \geqslant t^*. \tag{2.14}$$

This implies part (1) of the theorem.

To show part (2) of the theorem, note that by (H4), (2.3) and $h_n(s) \geqslant s$, we have

$$\begin{aligned}
&\int_{t_0}^{t} p_{n-1}(s) |y_n(h_n(s))|\, ds \\
&\leqslant \int_{t_0}^{t} p_{n-1}(s) \int_{s}^{\infty} |f_n(\xi, y_1(h_1(\xi)))|\, d\xi\, ds \\
&\leqslant \int_{t_0}^{t} P_{n-1}(s) \omega\big(s, y_1(h_1(s))\big)\, ds + P_{n-1}(t) \int_{t_0}^{t} \omega\big(s, y_1(h_1(s))\big)\, ds.
\end{aligned}$$

Since $\lim_{t\to\infty} P_{n-1}(t) = \infty$ there exists t_1 such that $1/P_{n-1}(h_1(t)) \leqslant 1$ for all $t \geqslant t_1$. From (2.14) and (2.7), with $a = 1/P_{n-1}(h_1)$, we have

$$\omega\big(s, y_1(h_1(s))\big) \leqslant \frac{1}{P_{n-1}(h_1(s))} \omega\big(s, 3k_1 P_1(h_1) \cdots P_{n-1}(h_1)\big). \tag{2.15}$$

Then using (2.3) and that P_{n-1} is non-decreasing,

$$\begin{aligned}
\int_{t_0}^{t} p_{n-1}(s) |y_n(h_n(s))|\, ds \leqslant & \int_{t_0}^{t} \frac{P_{n-1}(s)}{P_{n-1}(h_1)} \omega(s, 3k_1 P_1(h_1) \cdots P_{n-1}(h_1))\, ds \\
& + \int_{t}^{\infty} \frac{P_{n-1}(s)}{P_{n-1}(h_1(s))} \omega(s, 3k_1 P_1(h_1) \cdots P_{n-1}(h_1))\, ds.
\end{aligned}$$

By assumption (2.5), the right-hand side is bounded. This allows us to define $\alpha_{n-1} = y_{n-1}(t_0) + \int_{t_0}^{\infty} p_{n-1}(s) f_{n-1}\big(y_n(h_n(s))\big)\, ds$. Then part (2) of the theorem follows.

To proof part (3), note that by (2.15),

$$\begin{aligned}
P_{n-1}(t) |y_n(t)| &\leqslant P_{n-1}(t) \int_{t}^{\infty} \omega(s, |y_1(h_1)|) \\
&\leqslant \int_{t}^{\infty} \frac{P_{n-1}(s)}{P_{n-1}(h_1)} \omega(s, 3k_1 P_1(h_1) \cdots P_{n-1}(h_1))\, ds.
\end{aligned}$$

By (2.5) the integral converges; thus $\lim_{t\to\infty} P_{n-1}(t) |y_n(t)| = 0$, which completes the proof.

Kitamura [8], initially, assumed (2.4) (with $n=2$), and later in a second approach assumed that:

$$\begin{aligned} &\omega(t,x)/x \quad \text{is non-increasing,} \\ &\int_0^\infty \max\{1,P_1(h_1(s))\}\omega(s,c)\,ds<\infty \end{aligned} \tag{2.16}$$

for all constant c. However, we do not gain any generality because (2.16) implies (2.4) and (2.5) (with $n=2$).

REMARK 2.1. If $\lim_{t\to\infty} y_{n-1}(t)=\alpha_{n-1}\neq 0$, then by the same process as in Lemma 2.3, we have

$$\lim_{t\to\infty}\frac{y_{n-2}(t)}{P_{n-2}(t)}=f_{n-2}(\alpha_{n-1})\neq 0,\dots,\ \lim_{t\to\infty}\frac{y_1(t)}{P_{1,n-2}(t)}=f_1\big(\cdots\big(f_{n-2}(\alpha_{n-1})\big)\big)\neq 0.$$

On the other hand if $\lim_{t\to\infty} y_{n-1}(t)=\alpha_{n-1}=0$, we have the following results.

THEOREM 2.2. *Let $(y_1,\dots,y_n)$ be a solution of* (1.2) *with $h_i(t)=t$ for $i=2,\dots,n-1$ and $h_n(t)\geqslant t$. Assume* (H1)–(H4), (2.2), (2.4), (2.5), *and that*

$$\int_{t_0}^\infty \max\Big\{1,\frac{P_{n-2}(s)P_{n-1}(s)}{P_{n-2}(h_1(s))P_{n-1}(h_1(s))}\Big\}\omega\big(s,MP_1(h_1(s))\cdots P_{n-1}(h_1(s))\big)\,ds<\infty \tag{2.17}$$

for each positive constant M. If $\lim_{t\to\infty} y_n(t)=0$ and $\lim_{t\to\infty} y_{n-1}(t)=0$ then:

(1) $\limsup_{t\to\infty}\frac{|y_1(t)|}{P_1(t),\dots,P_{n-3}(t)}<\infty$,

(2) y_{n-2} *has finite limit at infinity, with*

$$\lim_{t\to\infty} y_{n-2}(t)=\alpha_{n-2},\quad y_{n-2}(t)=\alpha_{n-2}-\int_t^\infty p_{n-2}(s)f_{n-1}\big(y_{n-1}(s)\big)\,ds,$$

(3) y_{n-1} *converges to zero at least at the rate given by* $\lim_{t\to\infty} P_{n-2}(t)y_{n-1}(t)=0$.

Proof. Using that the functions $P_i(s)=\int_{t_0}^s p_i(s)\,ds$ are non-decreasing, we have

$$\int_{t_0}^s P_{n-2}(\xi)p_{n-1}(\xi)\,d\xi\leqslant P_{n-2}(s)\int_{t_0}^s p_{n-1}(\xi)\,d\xi=P_{n-2}(s)P_{n-1}(s).$$

Using this inequality and integrating by parts (three times), we have

$$\begin{aligned} &\int_{t_0}^t p_{n-2}(s)\int_s^\infty p_{n-1}(\xi)\int_\xi^\infty |f_n|\,dx\,d\xi\,ds \\ &=\int_{t_0}^t P_{n-2}(s)p_{n-1}(s)\int_s^\infty |f_n|\,dx\,ds+P_{n-2}(t)\int_t^\infty p_{n-1}(s)\int_s^\infty |f_n|\,dx\,ds \\ &\leqslant 2\int_{t_0}^t P_{n-2}(s)P_{n-1}(s)|f_n|\,ds+2P_{n-2}(t)P_{n-1}(t)\int_t^\infty |f_n|\,ds. \end{aligned}$$

As in the proof of Theorem 2.1, we find a bound for y_1 :

$$\begin{aligned}
&\frac{|y_1(t)|}{P_1(t)\cdots P_{n-1}(t)}\\
&\leqslant \frac{|y_1(t_0)|}{P_1(t)\cdots P_{n-1}(t)}+\frac{\beta_1|y_2(t_0)|}{P_2(t)\cdots P_{n-1}(t)}+\frac{\beta_1\beta_2|y_3(t_0)|}{P_3(t)\cdots P_{n-1}(t)}+\cdots\\
&\quad+\frac{(\beta_1\cdots\beta_{n-3})|y_{n-2}(t_0)|}{P_{n-2}(t)P_{n-1}(t)}+2\frac{(\beta_1\cdots\beta_{n-1})}{P_{n-2}(t)P_{n-1}(t)}\int_{t_0}^{t}P_{n-2}(s)P_{n-1}(s)|f_n|\,ds\\
&\quad+2(\beta_1\cdots\beta_{n-1})\int_t^\infty |f_n|\,ds.
\end{aligned}$$

By (2.2), there exist M and $t^*>t_0$, such that $|y_1(t)|\leqslant MP_1(t)\cdots P_{n-1}(t)$ for all $t\geqslant t^*$. Then by (2.17), we select $t_2\geqslant t^*$ such that

$$\begin{aligned}
&\frac{(\beta_1\cdots\beta_{n-1})}{M}\int_{t_1}^{\infty}\max\Big\{1,\frac{P_{n-2}(s)P_{n-2}(s)}{P_{n-2}(h_1(s))P_{n-1}(h_1(s))}\Big\}\\
&\qquad\times\omega\big(s,MP_1(h_1(s))\cdots P_{n-1}(h_1(s))\big)\,ds<\frac{1}{6}
\end{aligned}$$

proceed as in Theorem 2.1 to obtain

$$\frac{|y_1(t)|}{P_1(t)\cdots P_{n-3}(t)}\leqslant P_{n-2}(t)P_{n-1}(t)u(t)\leqslant 3k_2\quad\text{for all } t\geqslant t^*. \tag{2.18}$$

This implies part (1) of the theorem. The proof of parts (2) and (3) are similar to the proof in Theorem 2.1, hence omitted.

The above process can be repeated for the two cases: $\alpha_{n-2}\neq 0$ obtaining results similar to Remark 2.1, and $\alpha_{n-2}=0$ obtaining results similar to Theorem 2.2.

3. Result for $a(t)\geqslant 0$

In this section, we apply results from the previous section for the general equation (1.1), with the following assumptions:

(H5) There exists a constant a_0 such that $0\leqslant a(t)\leqslant a_0$, and $\lim_{t\to\infty}g(t)=\infty$.

Also we assume that the component y_1 is non-oscillatory, but no assumption is made on the other components of the solution. Note that $f_n(t,y_1(h_1(t)))$ may be oscillatory, even when y_1 is non-oscillatory. Therefore, under our assumptions the component y_1 can be non-oscillatory, while the other components are oscillatory.

For short notation, let

$$z_1(t)=y_1(t)+a(t)y_1(g(t)).$$

THEOREM 3.1. *Assume* (H1)–(H3), (H5). *Let* $(y_1,\dots,y_n)$ *be a solution* (1.1) *with* y_1 *non-oscillatory. If*

$$\limsup_{t\to\infty}\frac{|z_1(t)|}{P_{1,\dots,n-1}(t)}=\infty,$$

then

$$\limsup_{t\to\infty}\frac{|y_2(t)|}{P_{2,\dots,n-1}(t)}=\infty,\ \dots,\ \limsup_{t\to\infty}\frac{|y_{n-1}(t)|}{P_{n-1}(t)}=\infty,\quad \limsup_{t\to\infty}|y_n(t)|=\infty$$

and $\limsup_{t\to\infty} z_1(t)=\infty$.

Proof. By contradiction assume that there are constants M and $t^*>t_0$ such that $|y_2(t)|/P_{2,\dots,n-1}(t)\leqslant M$ for all $t\geqslant t*$. By (1.1) and (H3), for $t\geqslant t^*$,

$$|z_1(t)|\leqslant|y_1(t_0)|+\beta_2M\int_{z_0}^{t}p_1(s)P_{2,\dots,n-1}(s)\,ds=|y_1(t_0)|+\beta_2MP_{1,\dots,n-1}(t),$$

which contradicts the assumption on $z_1/P_{1,\dots,n-1}$.

That $|y_2(t)|/P_{2,\dots,n-1}(t)$ implies $|y_3(t)|/P_{3,\dots,n-1}(t)$ is done as is in Lemma 2.1. Then using the same proof as in Lemma 2.1, we obtain the desired results.

THEOREM 3.2. *Let* $(y_1,\dots,y_n)$ *be a solution of* (1.1) *with* y_1 *non-oscillatory. Assume that* (H1)–(H5) *and* (2.1) *hold. If*

$$\limsup_{t\to\infty}\frac{|z_1(t)|}{P_{1,\dots,n-1}(t)}<\infty, \tag{3.1}$$

then y_n *has finite limit at infinity, with*

$$\lim_{t\to\infty}y_n(t)=\alpha_n,\quad y_n(t)=\alpha_n-\int_t^\infty f_n\big(s,y_1(h_1(s))\big)\,ds.$$

Proof. Note that

$$\frac{|z_1(t)|}{P_{1,\dots,n-1}(t)}\geqslant\frac{|y_1(t)|}{P_{1,\dots,n-1}(t)}.$$

Taking the limit superior on both sides, we have $\limsup_{t\to\infty}\frac{|y_1(t)|}{P_{1,\dots,n-1}(t)}<\infty$. Then using the same technique as in the proof of Lemma 2.2, we obtain the desired results.

THEOREM 3.3. *Let* $(y_1,\dots,y_n)$ *be a solution of* (1.2) *with* y_1 *non-oscillatory. Assume* (H1)–(H5), (2.1) *and* (3.1). *If* $\lim_{t\to\infty}y_n(t)=\alpha_n\neq0$, *then*

$$\lim_{t\to\infty}\frac{y_{n-1}(t)}{P_{n-1}(t)}=f_{n-1}(\alpha_n)\neq0,\quad \lim_{t\to\infty}\frac{|y_{n-2}(t)|}{P_{n-2,n-1}(t)}=f_{n-2}\big(f_{n-1}(\alpha_n)\big)\neq0,$$
$$\dots,\ \lim_{t\to\infty}\frac{z_1(t)}{P_{1,\dots,n-1}(t)}=f_1\big(\dots f_{n-2}\big(f_{n-1}(\alpha_n)\big)\big)\neq0.$$

The proof of the above theorem follows the same steps in Lemma 2.3. Note that $\lim_{t\to\infty}\frac{y_1(t)}{P_{1,\dots,n-1}(t)}$ may or may not exist; If the limit exists, then it is not equal to zero.

THEOREM 3.4. *Let $h_i(t) = t$ for $i = 2, \dots, n-1$ and $h_n(t) \geqslant t$, and let $(y_1, \dots, y_n)$ be a solution of* (1.2) *with y_1 non-oscillatory. Assume* (H1)–(H4), (2.2), (2.4), *and* (2.5). *If $\lim_{t\to\infty} y_n(t) = 0$, then:*

(1) $\limsup_{t\to\infty} \frac{|z_1(t)|}{P_1(t),\dots,P_{n-2}(t)} < \infty$,

(2) y_{n-1} *has finite limit at infinity, with*

$$\lim_{t\to\infty} y_{n-1}(t) = \alpha_{n-1}, \quad y_{n-1}(t) = \alpha_{n-1} - \int_t^\infty p_{n-1}(s) f_n\big(y_n(h_n(s))\big)\, ds,$$

(3) y_n *converges to zero at least at a rate given by* $\lim_{t\to\infty} P_{n-1}(t) y_n(t) = 0$.

Proof. Note that because y_1 does not change signs and $0 \leqslant a(t) \leqslant a_0$, we have $|z_1(t)| = |y_1(t)| + a(t)|y_1(g(t))|$; also the bounds for $|z_1(t)|$ yield bounds for $|y_1(t)|$, and viceversa. Then (2.8) still holds with a different constant k. This leads to bound for y_1, which can be used for proving parts (2) and (3) as in Theorem 2.1.

The assumptions that y_1 does not change signs and that $0 \leqslant a(t) \leqslant a_0$ allow us to obtain bounds for $|y_1(t)|$ from the bounds of $|z_1(t)|$. From this fact, we can prove results similar to those in Remark 2.1 and Theorem 2.2 for the general equation (1.1).

EXAMPLE 3.1. As a simple example of functions satisfying the hypotheses (H1)–(H5), we have the following functions: $a(t) = 1 + \cos(t)$, $f_i(t) = t/2$ (with $\beta_i = 1$), $f_n(t,x) = tx\sin(x)$ (with $\omega(t,x) = tx$), $g(t) = \sqrt{t}$, $h_i(t) = t + \cos(t)$, and $p_i(t) = (t + \cos(t))/t$.

To conclude this article, we state that the cases $a(t) \leqslant 0$ and $a(t)$ having sign changes are possible extensions of our results.

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