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EVOLUTION EQUATIONS WITH CAUSAL OPERATORS

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Abstract. In this paper we present an existence result for causal functional evolution equations. The result is obtained under a condition with respect to the Hausdorff measure of noncompactness. An application with partial differential equations is given to illustrate our main result.

1. Introduction

The study of functional equations with causal operators has recently been developed and some results on existence, stability and control are found in the monographs [6], [12] and [20]. The term causal operators or Volterra abstract operator was introduced by Tonelli [35] (see also Tikhonov [34]). The theory of these operators has the advantage of unifying ordinary differential equations, integrodifferential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, to name but a few. Many papers in the literature address various aspects of the theory of causal operators. In [38] a new and general definition of Volterra operators, including Volterra ones in the sense of Tonelli is given. Control problems involving causal operators were studied in [4], [7], [14] and [32]. A a new class of abstract integral equations has been introduced in [13]. We note that different classes of differential equations with causal operators were studied by several authors, see [1], [3], [8]-[11], [15], [22]-[28] and the references therein. Some properties of the solutions of the differential equations with causal operators were studied in [2], [17], [30], [31], [38].

Let E be a real separable Banach space endowed with the norm $\|\cdot\|$. For $x \in E$ and $r > 0$ let $B_r(x) := \{y \in E; ||y-x|| < r\}$ be the open ball centered at x with radius r, and let $B_r[x]$ be its closure. If $\sigma \geqslant 0$, we denote by $C([-\sigma,b],E)$ the Banach space of continuous bounded functions from $[-\sigma,b]$ into E . If $\sigma > 0$, we denote by C_σ the space $C([-\sigma,0],E)$ endowed with the norm $\|\varphi\|_\sigma = \sup_{-\sigma\leqslant s\leqslant 0}\|\varphi(s)\|$. The space of all (classes of) strongly measurable functions $u(\cdot):[0,b]\to E$ such that

$$\|u(\cdot)\|_p := \left(\int_0^b \|u(t)\|^p\right)^{1/p} < \infty$$

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for $1 \leqslant p < \infty$ and

$$\|u(\cdot)\|_\infty := \operatorname*{ess\,sup}_{t\in[0,b]} \|u(t)\| < \infty,$$

will be denoted by $L^p([0,b],E)$. This is a Banach space with respect to the norm $\|u(\cdot)\|_p$. Let $\sigma \geqslant 0$. The following definition of causal operator was given by Tonelli [35].

DEFINITION 1. An operator $Q : \mathscr{C}([-\sigma,b],E) \to L^p_{loc}([0,b],E)$ is a *causal operator* if, for each $\tau \in [0,b)$ and for all $u(\cdot), v(\cdot) \in \mathscr{C}([-\sigma,b],E)$ with $u(t) = v(t)$ for every $t \in [0,\tau]$, we have $Qu(t) = Qv(t)$ for a.e. $t \in [0,\tau]$.

In this paper we consider the following evolution equation with causal operators in a real separable Banach space E:

$$\begin{cases} u'(t) = Au(t) + (Qu)(t), \text{ for a.e. } t \in [0,b], \\ u(\cdot)|_{[-\sigma,0]} = \varphi(\cdot) \in \mathscr{C}_\sigma, \end{cases} \tag{1.1}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{S(t); t \geqslant 0\}$ and $Q : \mathscr{C}([-\sigma,b],E) \to L^p([0,b],E)$ is a causal operator. Now we provide some examples of evolution equations that can be included in evolution equations with causal operators of the form (1.1). The evolution problems

$$u'(t) = Au(t) + F(t,u(t)), \quad u(0) = u_0,$$

can be considered as a causal evolution equation by identifying $F(t,u(t))$ with $(Qu)(t)$. Another example is the evolution equation with delay argument $\sigma > 0$

$$\begin{cases} u'(t) = Au(t) + F(t,u(t),u(t-\sigma)), \text{ for a.e. } t \in [0,b], \\ u(\cdot)|_{[-\sigma,0]} = \varphi(\cdot) \in \mathscr{C}_\sigma, \end{cases} \tag{1.2}$$

or more generally the functional evolution equation given by

$$\begin{cases} u'(t) = Au(t) + F(t,u(t),u_t), \text{ for a.e. } t \in [0,b], \\ u(\cdot)|_{[-\sigma,0]} = \varphi(\cdot) \in \mathscr{C}_\sigma, \end{cases} \tag{1.3}$$

where $x_t(s) = x(t+s)$, $-\sigma < s < 0$. The next example is the general integro-differential evolution equation

$$\begin{cases} u'(t) = Au(t) + F\left(t,u(t),u_t, \int_{t-\sigma}^t K(t,s,u(s))ds\right), \\ u(\cdot)|_{[-\sigma,0]} = \varphi(\cdot) \in \mathscr{C}_\sigma. \end{cases} \tag{1.4}$$

All the equations (1.2)-(1.4) are examples of causal evolution equations. Also, the evolution equation with "maxima":

$$u'(t) = Au(t) + F\left(t,u(t), \max_{0\leqslant s\leqslant t} u(s)\right), \; u(0) = u_0, \tag{1.5}$$

is another example of a causal evolution equation. Finally, we remark that the Fredholm operator, given by

$$(Qu)(t) = \int_0^a K(t,s,u(s))ds,$$

where $a > 0$ is a fixed real number, is a causal operator if and only if $K(t,s,u) \equiv 0$ for $t < s < a$.

The rest of the paper is organized as follows. In Section 2, we recall some concepts on C_0-semigroups and the measure of noncompactness. In Section 3, we establish an existence result and Section 4 contains an illustrating example.

2. Preliminaries

We denote the space of all bounded linear operators acting on a Banach space E by $\mathscr{L}(E)$. We recall that a family $\{S(t); t \geqslant 0\} \subset \mathscr{L}(E)$ is called a C_0-*semigroup* if the following three properties are satisfied:

(i) $S(0) = I$, the identity operator on E;

(ii) $S(t)S(s) = S(t+s)$ for all $t,s \geqslant 0$;

(iii) $\lim_{t\downarrow 0} S(t)u = u$ for all $u \in E$.

The infinitesimal generator of the C_0-semigroup $\{S(t); t \geqslant 0\}$ is the operator $A : D(A) \subset E \to E$, defined by

$$D(A) = \left\{ u \in E; \lim_{h\downarrow 0} \frac{S(h)u - u}{h} \text{ exists} \right\}$$

and

$$Au = \lim_{h\downarrow 0} \frac{S(h)u - u}{h}, \quad u \in D(A).$$

The generator is always a closed, densely defined operator. Also, we recall that a C_0-semigroup $\{S(t); t \geqslant 0\}$ is said to be equicontinuous if the function $t \mapsto S(t)$ is continuous from $[0,b]$ to $\mathscr{L}(E)$ endowed with the uniform operator norm $\|\cdot\|_{\mathscr{L}(E)}$. In particular, if A is the generator of an uniformly continuous semigroup, a compact semigroup, a differentiable semigroup or an analytic semigroup $\{S(t); t \geqslant 0\}$, then $\{S(t); t \geqslant 0\}$ is an equicontinuous C_0-semigroup (see [37]).

THEOREM 1. *Let $\{(S(t)); t \geqslant 0\}$ be a C_0-semigroup. Then there exist constants $\omega \geqslant 0$ and $N \geqslant 1$ such that*

$$\|S(t)\| \leqslant Ne^{\omega t}, \quad \textit{for all } t \geqslant 0.$$

For further details on the theory of the C_0-semigroups see [21], [29], [37]. We denote by $\beta(A)$ the Hausdorff measure of non-compactness of a nonempty bounded set $A \subset E$, and it is defined by ([16], [19]):

$$\beta(A) = \inf\{\varepsilon > 0; \text{ A admits a finite cover by balls of radius} \leqslant \varepsilon\}.$$

This is equivalent to the measure of non-compactness introduced by Kuratowski (see [16], [19]).

If $\dim(A) = \sup\{||x-y||; x, y \in A\}$ is the diameter of the bounded set A, then we have that $\beta(A) \leqslant \dim(A)$ and $\beta(A) \leqslant 2d$ if $\sup_{x\in A}||x|| \leqslant d$. We recall some properties of β (see [16], [19])). If A, B are bounded subsets of E and $\overline{A}$ denotes the closure of A, then

(i) $\beta(A) = 0$ if and only if $\overline{A}$ is compact;

(ii) $\beta(A) = \beta(\overline{A}) = \beta(\overline{co}(A))$;

(iii) $\beta(\lambda A) = |\lambda|\beta(A)$ for every $\lambda \in \mathbb{R}$;

(iv) $\beta(A) \leqslant \beta(B)$ if $A \subset B$;

(v) $\beta(A+B) = \beta(A) + \beta(B)$.

If for $V \subset C([0,b], E)$ we define

$$\Psi(V) := \sup_{t\in[0,b]} \beta(V(t)),$$

where $V(t) := \{u(t) : u(\cdot) \in V\}$, then Ψ satisfies all the usual properties of the Hausdorff measure of non-compactness except the regularity condition (i). Nevertheless if the family of functions $V \subset C([0,b], E)$ is equicontinuous then $\Psi(V) = \beta_c(V)$, where β_c is the Hausdorff measure of non-compactness in the space $C([0,b], E)$ (see [16], [19]).

We recall the following lemma due to Kisielewicz ([18, Lemma 2.2]).

LEMMA 1. *Let* $\{u_n(\cdot); n \geqslant 1\}$ *be a subset in* $L^1([0,b], E)$ *for which there exists* $m(\cdot) \in L^1([0,b], \mathbb{R}_+)$ *such that* $\|u_n(t)\| \leqslant m(t)$ *for each* $n \geqslant 1$ *and for a.e.* $t \in [0,b]$. *Then the function* $t \mapsto \beta(t) := \beta(\{u_n(t); n \geqslant 1\})$ *is integrable on* $[0,b]$ *and, for each* $t \in [0,b]$, *we have*

$$\beta\left(\left\{\int_0^t u_n(t)dt; n \geqslant 1\right\}\right) \leqslant \int_0^t \beta(t)dt.$$

3. Existence Result

Consider the evolution equation

$$\begin{cases} u'(t) = Au(t) + (Qu)(t), \text{ for a.e. } t \in [0,b], \\ u(\cdot)|_{[-\sigma,0]} = \varphi(\cdot) \in \mathscr{C}_\sigma, \end{cases} \tag{3.1}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{S(t); t \geqslant 0\}$ and $Q : \mathscr{C}([-\sigma,b], E) \to L^p([0,b], E)$ is a causal operator; here $1 \leqslant p \leqslant \infty$. A function $u(\cdot) : [-\sigma, b] \to E$, is a *mild solution* of (3.1) if $u|_{[-\sigma,0]} = \varphi$, $\varphi(0) \in D(A)$ and

$$u(t) = S(t)\varphi(0) + \int_0^t S(t-s)(Qu)(s)ds, \; t \in [0,b].$$

We consider the following assumptions:

(H1) Q is continuous;

(H2) For each $r > 0$ there exist $\psi(\cdot) \in L^p([0,b],\mathbb{R}_+)$ such that,

$$\text{for each } u(\cdot) \in \mathscr{C}([-\sigma,b],E) \text{ with } \sup_{-\sigma \leqslant t \leqslant b} \|u(t)\| \leqslant r,$$

we have

$$\|(Qu)(t)\| \leqslant \psi(t) \text{ for a.e. } t \in [0,b];$$

(H3) There exists a continuous function $L(\cdot) : [0,b] \to \mathbb{R}_+$ such that for each bounded subset $V \subset C([0,b],E)$

$$\beta(S(t)(QV)(s)) \leqslant L(t)\beta(V(s)), \tag{3.2}$$

for all $t \in [0,b]$ and for a.e. $s \in [0,b]$, where $(QV)(s) := \{(Qu)(s) : u(\cdot) \in V\}$.

We remark that (3.2) can be written as

$$\beta_c(S(t)QV) \leqslant L(t)\beta_c(V),$$

for all $t \in [0,b]$, where $QV := \{(Qu)(\cdot) : u(\cdot) \in V\}$. Moreover, if $\{S(t);t \geqslant 0\}$ is compact or there exists $L > 0$ such that for each bounded subset $V \subset C([0,b],E)$

$$\beta((QV)(s)) \leqslant L\beta(V(s)),$$

for a.e. $s \in [0,b]$, then (3.2) is automatically satisfied. For further details see [5, Remark 8.2.1].

THEOREM 2. *Let $Q : \mathscr{C}([-\sigma,b],E) \to L^p([0,b],E)$ be a causal operator such that conditions (H1)-(H3) hold. If A is the generator of an equicontinuous C_0-semigroup $\{S(t);t \geqslant 0\}$ then, for every $\varphi \in \mathscr{C}_\sigma$ with $\varphi(0) \in D(A)$, the evolution equation (3.1) has a mild solution $u(\cdot) : [-\sigma,T] \to E$ on some interval $[-\sigma,T]$ with $T \in (0,b]$.*

Proof. Let $\delta > 0$ be any number. For a given $\varphi \in \mathscr{C}_\sigma$ with $\varphi(0) \in D(A)$, $u^0(\cdot) \in \mathscr{C}([-\sigma,b],E)$ denotes the function defined by

$$u^0(t) = \begin{cases} \varphi(t), \text{ for } t \in [-\sigma,0], \\ S(t)\varphi(0), \text{ for } t \in [0,b]. \end{cases}$$

From Theorem (1) there exists $M \geqslant 1$ such that $\|S(t)\| \leqslant M$ for all $t \in [0,b]$. Then, if we put $r := \|\varphi\|_\sigma + \delta$, it follows that $\sup_{t \in [0,b]} \|u^0(t)\| \leqslant r$ and therefore by (H2) we have $\|(Qu^0)(t)\| \leqslant \psi(t)$ for a.e. $t \in [0,b]$. Since $t \mapsto S(t)\varphi(0)$ continuous on $[0,T]$ (see [29, Corollary 2.3]), $S(0)\varphi(0) = \varphi(0)$, and $t \mapsto \int_0^t \psi(t)dt$ is also continuous on $[0,b]$, then we may find a sufficiently small $T \in (0,b]$ such that

$$\sup_{0 \leqslant t \leqslant T} \|S(t)\varphi(0) - \varphi(0)\| + M\int_0^T \psi(t)dt < \delta.$$

Further, let Ω be the set defined by

$$\Omega = \left\{ u(\cdot) \in \mathscr{C}([-\sigma, T], E); u|_{[-\sigma,0]} = \varphi, \sup_{0 \leqslant t \leqslant T} \|u(t) - u^0(t)\| \leqslant \delta \right\}.$$

If $u(\cdot) \in \Omega$, then it is easy to see that $\sup_{-\sigma \leqslant t \leqslant T} \|u(t)\| \leqslant r$. Consider the operator $\mathscr{P}: \Omega \to \mathscr{C}([-\sigma, T], E)$ given by

$$(\mathscr{P}u)(t) = \begin{cases} \varphi(t), \text{ for } t \in [-\sigma, 0], \\ S(t)\varphi(0) + \int_0^t S(t-s)(Qu)(s)ds, \text{ for } t \in [0, T]. \end{cases}$$

For each $u(\cdot) \in \Omega$ we have

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \|(\mathscr{P}u)(t) - u^0(t)\| \\ &\quad = \sup_{0 \leqslant t \leqslant T} \left\| S(t)\varphi(0) + \int_0^t S(t-s)(Qu)(s)ds - \varphi(0) \right\| \\ &\quad \leqslant \sup_{0 \leqslant t \leqslant T} \|S(t)\varphi(0) - \varphi(0)\| + M \int_0^T \psi(t)dt \\ &\quad < \delta, \end{aligned}$$

and thus $\mathscr{P}(\Omega) \subset \Omega$. Now let $u_m(\cdot) \to u(\cdot)$ in Ω. If $1 \leqslant p < \infty$ and $1/p + 1/q = 1$, then by Hölder's inequality we have

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \|(\mathscr{P}u_m)(t) - (\mathscr{P}u)(t)\| \\ &\quad \leqslant \sup_{0 \leqslant t \leqslant T} \int_0^t \|S(t-s)[(Qu_m)(s) - (Qu)(s)]\| ds \\ &\quad \leqslant \sup_{0 \leqslant t \leqslant T} \int_0^t \|S(t-s)\| \|(Qu_m)(s) - (Qu)(s)\| ds \\ &\quad \leqslant M \int_0^T \|(Qu_m)(s) - (Qu)(s)\| ds \\ &\quad \leqslant MT^{1/q} \left(\int_0^T \|(Qu_m)(s) - (Qu)(s)\|^p ds \right)^{1/p} \end{aligned}$$

and for $p = \infty$ we have

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \|(\mathscr{P}u_m)(t) - (\mathscr{P}u)(t)\| \\ &\quad \leqslant \sup_{0 \leqslant t \leqslant T} \int_0^t \|S(t-s)\| \|(Qu_m)(s) - (Qu)(s)\| ds \\ &\quad \leqslant M \operatorname{ess} \sup_{0 \leqslant t \leqslant T} \|(Qu_m)(t) - (Qu)(t)\|. \end{aligned}$$

Using (H1) it follows that for $1 \leqslant p \leqslant \infty$ we have that

$$\sup_{0 \leqslant t \leqslant T} \|(\mathscr{P}u_m)(t) - (\mathscr{P}u)(t)\| \to 0 \text{ as } m \to \infty.$$

Since for all $m \in \mathbb{N}$, we have $u_m(\cdot)|_{[-\sigma,0]} = \varphi$, it follows that $\mathscr{P} : \Omega \to \Omega$ is a continuous operator. Further, we deduce that $\mathscr{P}(\Omega)$ is uniformly bounded. Next we show that $\mathscr{P}(\Omega)$ is equicontinuous on $[-\sigma, T]$. Since $\mathscr{P}(\Omega)|_{[-\sigma,0]} = \{\varphi(\cdot)\}$ and it follows that $\mathscr{P}(\Omega)$ is equicontinuous on $[-\sigma, 0]$. Let $\varepsilon > 0$ and $s,t \in (0,T]$ with $s < t$. Then we have that

$$\begin{aligned}
&\|(\mathscr{P}u)(t) - (\mathscr{P}u)(t)\| \\
&\quad \leqslant \|S(t)\varphi(0) - S(s)\varphi(0)\| \\
&\qquad + \int_s^t \|S(t-\tau)(Qu)(\tau)\| \, d\tau + \int_0^t \|[S(t-\tau) - S(s-\tau)](Qu)(\tau)\| \, d\tau \\
&\quad \leqslant \|S(t)\varphi(0) - S(s)\varphi(0)\| + M \int_s^t \psi(\tau) d\tau \\
&\qquad + \int_0^t \|S(t-\tau) - S(s-\tau)\| \, \psi(\tau) d\tau.
\end{aligned}$$

The right hand side of previous inequality does not depend on $u(\cdot) \in \Omega$ and tends to zero as $t - s \to 0$, since the equicontinuity of the C_0-semigroup $\{S(t); t > 0\}$ implies that the function $t \mapsto S(t)$ is continuous in the uniform operator norm $\|\cdot\|_{\mathscr{L}(E)}$. Thus $\mathscr{P}(\Omega)$ is uniformly equicontinuous on $[-\sigma, T]$. Next, we construct a sequence $\{u_n(\cdot)\}_{n \geqslant 1}$ of continuous functions $u_n(\cdot) : [-\sigma, T] \to E$ as follows. Given $n \in \mathbb{N}$, for $k = 1, 2, ..., n$, we define $u_n^1(t) = u^0(t)$, $t \in [-\sigma, T/n]$ and

$$u_n^k(t) = \begin{cases} u_n^{k-1}(t), \ \ t \in [-\sigma, (k-1)T/n], \\ S(t)\varphi(0) + \int_0^{t-T/n} S(t - T/n - s)(Qu_n^{k-1})(s) ds, t \in [(k-1)T/n, kT/n], \end{cases}$$

for $k > 1$. It is easy to see that if $k \in \{1, 2, ..., n-1\}$ and $||u_n^k(t)|| \leqslant r$ for $t \in [-\sigma, kT/n]$, then $||u_n^{k+1}(t)|| \leqslant r$ for $t \in [-\sigma, kT/n]$ and, by (H2), $||(Qu_n^k)(t)|| \leqslant \psi(t)$ for a.e. $t \in [-\sigma, kT/n]$. It follows that

$$\begin{aligned}
&||u_n^{k+1}(t) - \varphi(0)|| \\
&\quad = \left\| S(t)\varphi(0) + \int_0^{t-T/n} S(t - T/n - s)(Qu_n^{k-1})(s) ds - \varphi(0) \right\| \\
&\quad \leqslant \sup_{0 \leqslant t \leqslant kT/n} \|S(t)\varphi(0) - \varphi(0)\| + \int_0^{t-T/n} \|S(t - T/n - s)(Qu_n^{k-1})(s)\| ds \\
&\quad \leqslant \sup_{0 \leqslant t \leqslant kT/n} \|S(t)\varphi(0) - \varphi(0)\| + M \int_0^{t-T/n} \psi(s) ds < \delta,
\end{aligned}$$

for all $t \in [-\sigma, (k+1)T/n]$. Since $||u_n^1(t)|| \leqslant r$ for $t \in [-\sigma, T/n]$, then by induction on k we have that $||u_n^k(t)|| \leqslant r$ for all $k = 1, 2, ..., n$, $t \in [-\sigma, kT/n]$. In the following, to simplify the notation, we put $u_n(\cdot) = u_n^n(\cdot)$, $n \in \mathbb{N}$. Since $u_n^n(s) = u_n^{n-1}(s)$ for all $s \in [0,(n-1)T/n]$ and Q is a causal operator, then

$$(Qu_n^n)(s) = (Qu_n^{n-1})(s) \text{ for all } s \in [0,(n-1)T/n].$$

If $t \in [(n-1)T/n, T]$, then

$$t - T/n \in [(n-2)T/n,(n-1)T/n] \text{ and } 0 \leqslant t - T/n - s \leqslant t - T/n$$

and consequently

$$\int_0^{t-T/n} S(t-T/n-s)(Qu_n^n)(s)ds = \int_0^{t-T/n} S(t-T/n-s)(Qu_n^{n-1})(s)ds$$

for $t \in [(n-1)T/n, T]$. It follows that the sequence $\{u_n(\cdot)\}_{n\geqslant 1}$ can be written as

$$u_n(t) = \begin{cases} u^0(t), \ t \in [-\sigma, T/n] \\ S(t)\varphi(0) + \int_0^{t-T/n} S(t-T/n-s)(Qu_n)(s)ds, \ t \in [T/n, T], \end{cases} \tag{3.3}$$

for every $n \in \mathbb{N}$. Moreover, it is easy to see that $u_n(\cdot) \in \Omega$ for all $n \geqslant 1$. Further, if $0 \leqslant t \leqslant T/n$, then we have

$$\begin{aligned} &\|(\mathscr{P}u_n)(t) - u_n(t)\| \\ &\quad = \left\| S(t)\varphi(0) + \int_0^t S(t-s)(Qu_n)(s)ds - \varphi(0) \right\| \\ &\quad \leqslant \sup_{0\leqslant t\leqslant T/n} \|S(t)\varphi(0) - \varphi(0)\| + \int_0^{T/n} \|S(t-s)(Qu_n)(s)\|ds \\ &\quad \leqslant \sup_{0\leqslant t\leqslant T/n} \|S(t)\varphi(0) - \varphi(0)\| + M\int_0^{T/n} \psi(s)ds. \end{aligned}$$

If $T/n \leqslant t \leqslant T$, then we have

$$\begin{aligned} &\|(\mathscr{P}u_n)(t) - u_n(t)\| \\ &\quad \leqslant \int_{t-T/n}^t \|S(t-T/n-s)(Qu_n)(s)\| ds \\ &\qquad + \int_0^{t-T/n} \|[S(t-s) - S(t-T/n-s)]\,(Qu_n)(s)\| ds \\ &\quad \leqslant M\int_{t-T/n}^t \psi(s)ds + \int_0^{t-T/n} \|S(t-s) - S(t-T/n-s)\|\,\psi(s)ds. \end{aligned}$$

Since the function $t \mapsto S(t)$ is continuous in the uniform operator norm $\|\cdot\|_{\mathscr{L}(E)}$, then the last two inequalities imply

$$\sup_{0\leqslant t\leqslant T} \|(\mathscr{P}u_n)(t) - u_n(t)\| \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Let $V=\{u_n(\cdot);n\geqslant 1\}$. Consider the identity mapping I on Ω. By (3.4) we deduce that $(I-\mathscr{P})(V)$ is an equicontinuous subset of Ω. As $V\subset(I-\mathscr{P})(V)+\mathscr{P}(V)$ and the set $\mathscr{P}(V)$ is equicontinuous, then it follows that V is also equicontinuous on $[-\sigma,T]$. Define $V(t)=\{u_n(t);n\geqslant 1\}$ for $t\in[0,T]$. Then, by (3.3) and the properties (1), (3) and (4) of the measure of non-compactness we have

$$\beta(V(t))\leqslant\beta\left(\int_0^t S(t-s)(QV)(s)ds\right)+\beta\left(\int_{t-T/n}^t S(t-s)(QV)(s)ds\right).$$

Note that, given $\varepsilon>0$, we can find $n(\varepsilon)>0$ such that $\int_{t-T/n}^t\psi(s)ds<\varepsilon/2M$ for $t\in[0,T]$ and $n\geqslant n(\varepsilon)$. Since

$$\|S(t-s)(Qu_n)(s)\|\leqslant M\|(Qu_n)(s)\|\leqslant M\psi(s)$$

for a.e. $s\in[0,T]$ and $n\geqslant 1$, then we have that

$$\begin{aligned}\beta&\left(\int_{t-T/n}^t S(t-s)(QV)(s)ds\right)\\&=\beta\left(\left\{\int_{t-T/n}^t S(t-s)(Qu_n)(s)ds;n\geqslant n(\varepsilon)\right\}\right)\\&\leqslant 2\sup_{n\geqslant n(\varepsilon)}\int_{t-T/n}^t\psi(s)ds<\varepsilon.\end{aligned}$$

Using the last inequality, Lemma (1) and (H3), we obtain that

$$\begin{aligned}\beta(V(t))&\leqslant\beta\left(\int_0^t S(t-s)(QV)(s)ds\right)\leqslant\int_0^t\beta\left(S(t-s)(QV)(s)\right)ds\\&\leqslant\int_0^t L(t-s)\beta\left(V(s)\right)ds\leqslant\int_0^t\sup_{\theta\in[0,T]}L(\theta)\beta\left(V(s)\right)ds\\&\leqslant L\int_0^t\beta\left(V(s)\right)ds,\end{aligned}$$

where $L:=\sup_{\theta\in[0,T]}L(\theta)$. Since $\beta(V(0))=0$, then by Gronwall's lemma we must have that $\beta(V(t))=0$ for every $t\in[0,T]$. Moreover, since $\beta_c(V)=\sup_{0\leqslant t\leqslant T}\beta(V(t))$ and $V|_{[-\sigma,0]}=\{\varphi\}$ we deduce that $\beta_c(V)=0$. Therefore, V is a relatively compact subset of $\mathscr{C}([-\sigma,T],E)$. Then, by the Arzela-Ascoli theorem (see [19], Theorem 1.1.5), and extracting a subsequence if necessary, we may assume that the sequence $\{u_n(\cdot)\}_{n\geqslant 1}$ converges uniformly on $[0,T]$ to a continuous function $u(\cdot)\in\Omega$. Therefore, since

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(\mathscr{P}u)(t)-u(t)\|&\leqslant\sup_{0\leqslant t\leqslant T}\|(\mathscr{P}u)(t)-(\mathscr{P}u_n)(t)\|\\&\quad+\sup_{0\leqslant t\leqslant T}\|(\mathscr{P}u_n)(t)-u_n(t)\|+\sup_{0\leqslant t\leqslant T}\|u_n(t)-u(t)\|,\end{aligned}$$

then by the continuity of $\mathscr{P}$ and (3.4), we get $\sup_{0\leqslant t\leqslant T}\|(\mathscr{P}u)(t)-u(t)\|=0$. It follows that

$$u(t)=(\mathscr{P}u)(t) \text{ for all } t\in[0,T].$$

Thus

$$u(t)=\begin{cases}\varphi(t), \text{ for } t\in[-\sigma,0],\\ S(t)\varphi(0)+\int_0^t S(t-s)(Qu)(s)ds, \text{ for } t\in[0,T],\end{cases}$$

is a solution of the causal evolution equation (3.1). □

4. An Application

Consider the reaction-diffusion equation with delay

$$\begin{cases}\frac{\partial w}{\partial t}(t,x)=\frac{\partial^2 w}{\partial x^2}(t,x)+f(t,w_t(\cdot,x)),\ 0\leqslant x\leqslant\pi,\ 0\leqslant t\leqslant b\\ w(t,0)=w(t,\pi)=0,\ 0\leqslant t\leqslant b,\\ w(s,x)=\varphi(s)(x),\ 0\leqslant x\leqslant\pi,\ -\sigma\leqslant s\leqslant 0,\end{cases}\tag{4.1}$$

where $\varphi(\cdot)\in\mathscr{C}_\sigma:=\mathscr{C}([-\sigma,0],E)$, $f:[0,b]\times\mathscr{C}_\sigma\to\mathbb{R}$ is a given continuous function, $w_t(s,x)=w(t+s,x)$ for $s\in[-\sigma,0]$ and $t\in[0,b]$, and $E=L^2([0,\pi])$.

Let $A:E\to E$ be defined by $Ay=y''$ with the domain

$$D(A)=\{y\in E: y,\ y' \text{ are absolutely continuous, } y''\in E \text{ and } y(0)=y(\pi)=0\}.$$

Then the operator A is the infinitesimal generator of a compact C_0-semigroup $\{S(t),t\geqslant 0\}$ (see [36, Example 5.2]). Moreover, the operator A can be written as

$$Ay=-\sum_{n=1}^{\infty}n^2\langle y,y_n\rangle y_n,\ \ y\in D(A),$$

where $\{y_n(x)=(\sqrt{\tfrac{2}{\pi}})\sin nx;\ n=1,2,...\}$ is the orthogonal set of the eigenvectors of A and the C_0-semigroup $\{S(t),t\geqslant 0\}$ is given by

$$S(t)y=\sum_{n=1}^{\infty}\exp(-n^2t)\langle y,y_n\rangle y_n,\ \ y\in E.$$

Let us define $u(\cdot):[0,b]\to E$, $F:[0,b]\times\mathscr{C}_\sigma\to E$, and $Q:\mathscr{C}([-\sigma,b],E)\to\mathscr{C}([0,b],E)$ respectively by

$$u(t)(x):=w(x,t),\ F(t,\varphi(\cdot))(x)=f(t,\varphi(\cdot)(x)) \quad\text{and}\quad (Qu)(t)=F(t,u_t(\cdot)),$$

where $u_t(\cdot)\in\mathscr{C}_\sigma$ is given by $u_t(\cdot)(x)=w_t(\cdot,x)$ for $t\in[0,b]$. Then the reaction-diffusion equation (4.1) can be written in the abstract form

$$\begin{cases}u'(t)=Au(t)+(Qu)(t) \text{ for a.e. } t\in[0,b]\\ u(\cdot)|_{[-\sigma,0]}=\varphi(\cdot)\in\mathscr{C}_\sigma.\end{cases}$$

It is easy to check that Q is a causal operator and satisfies conditions (H1) and (H2). Since $\{S(t), t \geqslant 0\}$ is a compact C_0-semigroup, then condition (H3) is automatically satisfied (see [5, Remark 8.2.1]). Therefore, there exists a mild solution of (4.1).

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