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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF STRONGLY MONOTONE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. Two types of nonlinear differential systems

$$\text{(A)}\quad x' + p(t)y^{\alpha} = 0, \quad y' + q(t)x^{\beta} = 0; \qquad \text{(B)}\quad x' - p(t)y^{\alpha} = 0, \quad y' - q(t)x^{\beta} = 0$$

are considered under the assumption that α and β are positive constants such that $\alpha\beta < 1$ and $p(t)$ and $q(t)$ are continuous regularly varying functions on a neighborhood of infinity. An attempt is made to obtain precise information on the existence and asymptotic behavior of strongly monotone regularly varying solutions $(x(t), y(t))$ of (A) and (B) whose x-components or y-components are slowly varying. It is shown that the results thus obtained are applied to the generalized Thomas-Fermi equations of the form $(p(t)|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x$ to provide new useful knowledge of their strongly monotone solutions. The present paper is designed to supplement the pioneering results on the asymptotic analysis of (A) and (B) by means of regular variation developed in the paper [4].

1. Introduction

In a recent paper by Jaroš and Kusano [4] the authors considered two-dimensional cyclic systems of first order nonlinear differential equations of the forms

$$x' + p(t)y^{\alpha} = 0, \qquad y' + q(t)x^{\beta} = 0, \tag{A}$$

$$x' - p(t)y^{\alpha} = 0, \qquad y' - q(t)x^{\beta} = 0, \tag{B}$$

under the assumption that α and β are positive constants such that $\alpha\beta < 1$ and $p(t)$ and $q(t)$ are continuous regularly varying functions on $[a,\infty)$, and established sharp results on the existence and asymptotic behavior of some particular classes of *strongly monotone solutions* which are regularly varying for systems (A) and (B). Moreover they showed that their results can be effectively applied to acquire new information on *strongly monotone regularly varying solutions* of second order Thomas-Fermi type nonlinear differential equations of the type

$$(p(t)|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x, \tag{C}$$

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where α and β are positive constants such that $\alpha > \beta$ and $p(t)$ and $q(t)$ are continuous regularly varying functions on $[a,\infty)$. The prototype of (C) is the equation

$$x'' = t^{-\frac{1}{2}}x^{\frac{3}{2}}$$

which arises in nuclear physics as a dimensionless form of the radially symmetric Poisson equation describing the potential of electrons considered as a degenerated gas around the nucleus of an atom of large atomic number. The study of this equation under the singular boundary conditions $x(0) = 1$ and $x(\infty) = 0$ by Thomas [7] and Fermi [2] (see also [3]) has motivated intensive investigations of asymptotic behavior of solutions of nonlinear differential equations including (C) from various viewpoints.

By a positive solution of (A) or (B) we mean a vector function $(x(t),y(t))$ both components of which are positive and satisfy the system (A) or (B) in a neighborhood of infinity. A positive solution $(x(t),y(t))$ of (A) (resp. of (B)) is said to be *strongly decreasing* (resp. *strongly increasing*) if $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$ (resp. $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \infty$). In this paper we are concerned exclusively with positive solutions of (A) and (B) both components of which are regularly varying functions (in the sense of Karamata). Such a solution $(x(t),y(t))$ is called *regularly varying of index* (ρ,σ) if $x(t)$ and $y(t)$ are regularly varying of indices $\rho(\in\mathbb{R})$ and $\sigma(\in\mathbb{R})$, respectively, and is denoted by $(x,y) \in \mathrm{RV}(\rho,\sigma)$. (For the definition and some basic properties of regularly varying functions the reader is referred to Section 2 of the paper [4].)

The paper [4] is devoted to the analysis of strongly decreasing regularly varying solutions $(x,y) \in \mathrm{RV}(\rho,\sigma)$ with non-zero ρ and σ (more precisely, $\rho < 0$ and $\sigma < 0$ for system (A), and $\rho > 0$ and $\sigma > 0$ for system (B)). As is easily observed, however, one cannot exclude the possibility that (A) and (B) have strongly monotone regularly varying solutions $(x,y) \in \mathrm{RV}(\rho,\sigma)$ with either or both of ρ and σ zero. The aim of this paper is to supplement the results developed in the paper [2] by providing necessary and sufficient conditions for the existence of strongly monotone regularly varying solutions $(x,y) \in \mathrm{RV}(\rho,\sigma)$ of (A) and (B) with either $\rho = 0$ or $\sigma = 0$, in which case either $x(t)$ or $y(t)$ is slowly varying, and then by applying the results thus obtained to the generalized Thomas-Fermi equation (C) to show that it may possess strongly monotone solutions with the asymptotic behavior distinctly different from the ones constructed in [4].

2. Strongly decreasing solutions with slowly varying component

With regard to systems (A) and (B) we assume throughout the paper that α and β are positive constants such that $\alpha\beta < 1$ and that $p \in \mathrm{RV}(\lambda)$ and $q \in \mathrm{RV}(\mu)$ and they are expressed as

$$p(t) = t^{\lambda} l(t), \qquad q(t) = t^{\mu} m(t), \qquad l,\ m \in \mathrm{RV}(0). \tag{2.1}$$

And we always seek strongly monotone solutions $(x(t),y(t))$ of (A) and (B) which are regularly varying of index (ρ,σ) and are represented in the form

$$x(t) = t^{\rho}\xi(t), \qquad y(t) = t^{\sigma}\eta(t), \qquad \xi,\ \eta \in \mathrm{RV}(0). \tag{2.2}$$

This section concerns strongly decreasing regularly varying solutions of indices (ρ,σ) with $(\rho = 0,\ \sigma < 0)$ and $(\rho < 0,\ \sigma = 0)$, and shows that the existence of these types of solutions can be fully characterized and moreover that their asymptotic behavior as $t \to \infty$ can be determined accurately.

THEOREM 2.1. *System* (A) *possesses strongly decreasing regularly varying solutions* $(x,y) \in \mathrm{RV}(\rho,\sigma)$ *with* $\rho = 0$ *and* $\sigma < 0$ *if and only if*

$$\mu+1<0, \qquad \lambda+1+\alpha(\mu+1)=0, \tag{2.3}$$

and

$$\int_a^\infty p(t)(tq(t))^\alpha dt < \infty, \tag{2.4}$$

in which case $\sigma = \mu+1$ *and any such solution* $(x(t),y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$\begin{aligned} x(t) &\sim \left[(1-\alpha\beta)\int_t^\infty p(s)\left(\frac{sq(s)}{-\sigma}\right)^\alpha ds\right]^{\frac{1}{1-\alpha\beta}}, \\ y(t) &\sim \frac{tq(t)}{-\sigma}\left[(1-\alpha\beta)\int_t^\infty p(s)\left(\frac{sq(s)}{-\sigma}\right)^\alpha ds\right]^{\frac{\beta}{1-\alpha\beta}}, \qquad t\to\infty. \end{aligned} \tag{2.5}$$

THEOREM 2.2. *System* (A) *possesses strongly decreasing regularly varying solutions* $(x,y) \in \mathrm{RV}(\rho,\sigma)$ *with* $\rho < 0$ *and* $\sigma = 0$ *if and only if*

$$\lambda+1<0, \qquad \beta(\lambda+1)+\mu+1=0, \tag{2.6}$$

and

$$\int_a^\infty (tp(t))^\beta q(t) dt < \infty, \tag{2.7}$$

in which case $\rho = \lambda+1$ *and any such solution* $(x(t),y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$\begin{aligned} x(t) &\sim \frac{tp(t)}{-\rho}\left[(1-\alpha\beta)\int_t^\infty \left(\frac{sp(s)}{-\rho}\right)^\beta q(s)ds\right]^{\frac{\alpha}{1-\alpha\beta}}, \\ y(t) &\sim \left[(1-\alpha\beta)\int_t^\infty \left(\frac{sp(s)}{-\rho}\right)^\beta q(s)ds\right]^{\frac{1}{1-\alpha\beta}}, \qquad t\to\infty. \end{aligned} \tag{2.8}$$

Since Theorem 2.1 implies Theorem 2.2 and vice versa, we need only to prove Theorem 2.1.

Proof of Theorem 2.1. Suppose that (A) has a strongly decreasing solution $(x,y) \in \mathrm{RV}(\rho,\sigma)$ with $\rho = 0$ and $\sigma < 0$. Note that $(x(t),y(t))$ satisfies the integral equations

$$x(t) = \int_t^\infty p(s)y(s)^\alpha ds, \qquad y(t) = \int_t^\infty q(s)x(s)^\beta ds, \quad t \gg 1. \tag{2.9}$$

Using (2.1) and (2.2), we rewrite (2.9) as

$$x(t) = \int_t^\infty s^{\lambda+\alpha\sigma} l(s)\eta(s)^\alpha ds, \qquad y(t) = \int_t^\infty s^{\mu+\beta\rho} m(s)\xi(s)^\beta ds, \tag{2.10}$$

and applying Karamata's integration theorem (cf. [4, Proposition 2.5]) to (2.10), we see that $\rho = 0$ and $\sigma < 0$ if and only if $\lambda + \alpha\sigma = -1$ and $\mu + \beta\rho = \mu < -1$, which implies (2.3) and $\sigma = \mu + 1$, so that (2.10) is converted into

$$x(t) = \int_t^\infty s^{-1} l(s)\eta(s)^\alpha ds, \tag{2.11}$$

and

$$y(t) \sim \frac{t^{\mu+1} m(t)}{-(\mu+1)} \xi(t)^\beta \implies \eta(t) \sim \frac{m(t)}{-\sigma} \xi(t)^\beta \tag{2.12}$$

as $t \to \infty$. Combining (2.11) with (2.12) gives

$$x(t) = \xi(t) \sim \int_t^\infty s^{-1} l(s) \left(\frac{m(s)}{-\sigma}\right)^\alpha \xi(s)^{\alpha\beta} ds, \quad t \to \infty. \tag{2.13}$$

Let $u(t)$ denote the integral in (2.13). Then, (2.13) is transformed into the following asymptotic relation

$$-u(t)^{-\alpha\beta} u'(t) \sim t^{-1} l(t) \left(\frac{m(t)}{-\sigma}\right)^\alpha = p(t) \left(\frac{tq(t)}{-\sigma}\right)^\alpha, \quad t \to \infty,$$

and so integrating the above over $[t, \infty)$, we obtain

$$x(t) = \xi(t) \sim u(t) \sim \left[(1-\alpha\beta) \int_t^\infty p(s) \left(\frac{sq(s)}{-\sigma}\right)^\alpha ds\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty,$$

which, combined with (2.12), establishes the asymptotic formulas (2.5) for $(x(t), y(t))$. Naturally the convergence condition (2.4) has been confirmed.

Conversely, assume that (2.3) and (2.4) hold. Put $\sigma = \mu + 1$ and define the vector function $(X, Y) \in \mathrm{RV}(0, \sigma)$ by

$$X(t) = \left[(1-\alpha\beta) \int_t^\infty p(s) \left(\frac{sq(s)}{-\sigma}\right)^\alpha ds\right]^{\frac{1}{1-\alpha\beta}},$$

$$Y(t) = \frac{tq(t)}{-\sigma} \left[(1-\alpha\beta) \int_t^\infty p(s) \left(\frac{sq(s)}{-\sigma}\right)^\alpha ds\right]^{\frac{\beta}{1-\alpha\beta}} = \frac{tq(t)}{-\sigma} X(t)^\beta \tag{2.14}$$

for $t \geqq a$. It can be checked that $(X(t), Y(t))$ satisfies the system of asymptotic relations

$$\int_t^\infty p(s) Y(s)^\alpha ds \sim X(t), \qquad \int_t^\infty q(s) X(s)^\beta ds \sim Y(t), \qquad t \to \infty. \tag{2.15}$$

In fact, we see that

$$\int_t^\infty p(s)Y(s)^\alpha ds = \int_t^\infty p(s)\left(\frac{sq(s)}{-\sigma}\right)^\alpha X(s)^{\alpha\beta} ds = \int_t^\infty (-X'(s))ds = X(t)$$

and

$$\int_t^\infty q(s)X(s)^\beta ds = \int_t^\infty s^\mu m(s)X(s)^\beta ds \ \sim \ \frac{t^{\mu+1}m(t)}{-(\mu+1)}X(t)^\beta = \frac{tq(t)}{-\sigma}X(t)^\beta = Y(t)$$

as $t \to \infty$.

Because of (2.15) there exists $T > a$ such that

$$\frac{1}{2}X(t) \leqq \int_t^\infty p(s)Y(s)^\alpha ds \leqq 2X(t), \quad \frac{1}{2}Y(t) \leqq \int_t^\infty q(s)X(s)^\beta ds \leqq 2Y(t) \tag{2.16}$$

for $t \geqq T$. Clearly, $X(t)$ is a decreasing function. We may assume that $Y(t)$ is also decreasing on $[T,\infty)$ because a regularly varying function of negative index is asymptotically equivalent to a decreasing regularly varying function of the same index. (cf. Theorem 1.5.3 of [1]). Denote by $\mathscr{V}$ the set of vector functions $(x(t),y(t))$ satisfying

$$hX(t) \leqq x(t) \leqq HX(t), \qquad kY(t) \leqq y(t) \leqq KY(t), \qquad \text{for } t \geqq T, \tag{2.17}$$

where h,H,k,K are positive constants satisfying

$$H \geqq 2K^\alpha, \quad K \geqq 2H^\beta, \quad h \leqq \frac{1}{2}k^\alpha, \quad k \leqq \frac{1}{2}h^\beta. \tag{2.18}$$

It suffices, for example, to choose

$$H = 2^{\frac{\alpha+1}{1-\alpha\beta}}, \quad K = 2^{\frac{\beta+1}{1-\alpha\beta}}, \quad h = 2^{-\frac{\alpha+1}{1-\alpha\beta}}, \quad k = 2^{-\frac{\beta+1}{1-\alpha\beta}}.$$

It is clear that $\mathscr{V}$ is a closed convex subset of the locally convex space $C[T,\infty)\times C[T,\infty)$. We define the map $\Phi : \mathscr{V} \to C[T,\infty)\times C[T,\infty)$ by

$$\Phi(x,y)(t) = (Fy(t), Gx(t)), \quad t \geqq T, \tag{2.19}$$

where F and G denote the integral operators

$$Fy(t) = \int_t^\infty p(s)y(s)^\alpha ds, \qquad Gx(t) = \int_t^\infty q(s)x(s)^\beta ds, \qquad t \geqq T. \tag{2.20}$$

Using (2.16) – (2.20) it is easy to verify that $(x,y) \in \mathscr{V}$ implies

$$hX(t) \leqq Fy(t) \leqq HX(t), \qquad kY(t) \leqq Gx(t) \leqq KY(t), \qquad \text{for } t \geqq T,$$

so that Φ maps $\mathscr{V}$ into itself. Furthermore, as in the proof of [4, Theorem 3.1] one can prove that Φ is a continuous map and that $\Phi(\mathscr{V})$ is a relatively compact subset of $C[T,\infty)\times C[T,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists $(x,y) \in \mathscr{V}$ such that $(x,y) = \Phi(x,y) = (Fy, Gx)$, which means that $(x(t),y(t))$

satisfies the system of integral equations (2.9), and hence gives a strongly decreasing solution of system (A) on $[T,\infty)$. At this point it is only known that this solution is nearly regularly varying (cf. [4, Definition 2.2]), and so we have to show that the solution is really regularly varying of index $(0,\sigma)$. But this can be done with the help of the generalized L'Hospital's rule by following exactly the same procedure as described in the proof of Theorem 3.2 of [4]. The details may be omitted. This completes the proof. □

EXAMPLE 2.3. Consider the system (A) in which $p(t)$ and $q(t)$ are given by

$$p(t) = \frac{t^{\alpha-1}}{2\sqrt{\log t}} \exp\left(-(\alpha+1)\sqrt{\log t}\right), \qquad q(t) = t^{-2}\exp\left(\sqrt{\log t}\right).$$

This is a special case of (A) with $\lambda = \alpha - 1$ and $\mu = -2$. Since $\mu + 1 = -1 < 0$, $\lambda + 1 + \alpha(\mu+1) = 0$, and

$$\int_t^\infty p(s)(sq(s))^\alpha ds = \exp\left(-\sqrt{\log t}\right),$$

we conclude from Theorem 2.1 that this system possesses strongly decreasing regularly varying solutions $(x(t), y(t))$ of index $(0,-1)$ all of which enjoy one and the same asymptotic behavior

$$x(t) \sim (1-\alpha\beta)^{\frac{1}{1-\alpha\beta}} \exp\left(-\frac{\sqrt{\log t}}{1-\alpha\beta}\right),$$

$$y(t) \sim (1-\alpha\beta)^{\frac{\beta}{1-\alpha\beta}} t^{-1} \exp\left(-\frac{1-(\alpha+1)\beta}{1-\alpha\beta}\sqrt{\log t}\right)$$

as $t \to \infty$.

3. Strongly increasing solutions with slowly varying component

In this section our attention is focused on strongly increasing solutions of system (B) whose x-components or y-components are slowly varying. As the following theorems assert the existence of such solutions can be completely characterized and the formulas governing the asymptotic growth of all such solutions can be determined precisely and explicitly.

THEOREM 3.1. *System* (B) *possesses strongly increasing regularly varying solutions* $(x,y) \in \mathrm{RV}(\rho,\sigma)$ *with* $\rho = 0$ *and* $\sigma > 0$ *if and only if*

$$\mu + 1 > 0, \qquad \lambda + 1 + \alpha(\mu+1) = 0, \tag{3.1}$$

and

$$\int_a^\infty p(t)(tq(t))^\alpha dt = \infty, \tag{3.2}$$

in which case $\sigma = \mu + 1$ *and any such solution* $(x(t), y(t))$ *of* (B) *has one and the same asymptotic behavior*

$$x(t) \sim \left[(1-\alpha\beta)\int_a^t p(s)\left(\frac{sq(s)}{\sigma}\right)^{\alpha} ds\right]^{\frac{1}{1-\alpha\beta}},$$
$$y(t) \sim \frac{tq(t)}{\sigma}\left[(1-\alpha\beta)\int_a^t p(s)\left(\frac{sq(s)}{\sigma}\right)^{\alpha} ds\right]^{\frac{\beta}{1-\alpha\beta}}, \qquad t \to \infty. \tag{3.3}$$

THEOREM 3.2. *System* (B) *possesses strongly increasing regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho > 0$ *and* $\sigma = 0$ *if and only if*

$$\lambda + 1 > 0, \qquad \beta(\lambda+1) + \mu + 1 = 0, \tag{3.4}$$

and

$$\int_a^{\infty} (tp(t))^{\beta} q(t)dt = \infty, \tag{3.5}$$

in which case $\rho = \lambda + 1$ *and any such solution* $(x(t), y(t))$ *of* (B) *has one and the same asymptotic behavior*

$$x(t) \sim \frac{tp(t)}{\rho}\left[(1-\alpha\beta)\int_a^t \left(\frac{sp(s)}{\rho}\right)^{\beta} q(s)ds\right]^{\frac{\alpha}{1-\alpha\beta}},$$
$$y(t) \sim \left[(1-\alpha\beta)\int_a^t \left(\frac{sp(s)}{\rho}\right)^{\beta} q(s)ds\right]^{\frac{1}{1-\alpha\beta}}, \qquad t \to \infty. \tag{3.6}$$

Proof of Theorem 3.2. We only give a proof of Theorem 3.2. Suppose that (B) has a strongly increasing regularly varying solution $(x(t), y(t))$ of index $(\rho, 0)$ with $\rho > 0$. From the integral equations

$$x(t) = x_0 + \int_T^t p(s)y(s)^{\alpha} ds, \qquad y(t) = y_0 + \int_T^t q(s)x(s)^{\beta} ds, \qquad t \geqq T, \tag{3.7}$$

$T > a$, $x_0 > 0$ and $y_0 > 0$ being some constants, we obtain the system of asymptotic relations

$$x(t) \sim \int_T^t p(s)y(s)^{\alpha} ds, \qquad y(t) \sim \int_T^t q(s)x(s)^{\beta} ds, \qquad t \to \infty. \tag{3.8}$$

Using (2.1), (2.2) we rewrite (3.8) as

$$x(t) \sim \int_T^t s^{\lambda} l(s)\eta(s)^{\alpha} ds, \qquad y(t) \sim \int_T^t s^{\mu+\beta\rho} m(s)\xi(s)^{\beta} ds, \quad t \to \infty, \tag{3.9}$$

and applying Karamata's integration theorem we see that $\rho > 0$ if and only if $\lambda + 1 > 0$ and $\mu + \beta\rho + 1 = 0$, in which case (3.9) reduces to

$$x(t) \sim \frac{t^{\lambda+1} l(t)\eta(t)^{\alpha}}{\lambda+1}, \qquad y(t) \sim \int_T^t s^{-1} m(s)\xi(s)^{\beta} ds, \quad t \to \infty. \tag{3.10}$$

This means that $\rho = \lambda + 1$ and so $\beta(\lambda+1) + \mu + 1 = 0$, and moreover that

$$y(t) = \eta(t) \ \sim \ \int_T^t s^{-1}\left(\frac{l(s)}{\rho}\right)^{\beta} m(s)\eta(s)^{\alpha\beta}ds, \quad t \to \infty. \tag{3.11}$$

Letting $v(t)$ denote the right-hand side of (3.11), transform (3.11) into the asymptotic relation

$$v(t)^{-\alpha\beta}v'(t) \ \sim \ t^{-1}\left(\frac{l(t)}{\rho}\right)^{\beta} m(t) = \left(\frac{tp(t)}{\rho}\right)^{\beta} q(t), \quad t \to \infty, \tag{3.12}$$

and integrate the above relation on $[T,t]$. We then obtain

$$y(t) = \eta(t) \ \sim \ v(t) \ \sim \ \left[(1-\alpha\beta)\int_T^t \left(\frac{sp(s)}{\rho}\right)^{\beta} q(s)ds\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty,$$

which, combined with the first relation of (3.10), gives

$$x(t) \ \sim \ \frac{tp(t)}{\rho}\left[(1-\alpha\beta)\int_T^t \left(\frac{sp(s)}{\rho}\right)^{\beta} q(s)ds\right]^{\frac{\alpha}{1-\alpha\beta}}, \quad t \to \infty.$$

Since $v(t) \to \infty$ as $t \to \infty$, (3.12) implies that $\int_T^\infty (tp(t))^{\beta} q(t)dt = \infty$, which is equivalent to (3.5).

Conversely, assume that (3.4) and (3.5) hold, put $\rho = \lambda + 1$ and define the functions $X(t)$ and $Y(t)$ by

$$\begin{gathered} Y(t) = \left[(1-\alpha\beta)\int_a^t \left(\frac{sp(s)}{\rho}\right)^{\beta} q(s)ds\right]^{\frac{1}{1-\alpha\beta}} \in \mathrm{RV}(0), \\ X(t) = \frac{tp(t)}{\rho} Y(t)^{\alpha} \in \mathrm{RV}(\rho). \end{gathered} \tag{3.13}$$

As is easily seen, $(X(t), Y(t))$ satisfies the system of asymptotic relations

$$\int_b^t p(s)Y(s)^{\alpha}ds \ \sim \ X(t), \qquad \int_b^t q(s)X(s)^{\beta}ds \ \sim \ Y(t), \qquad t \to \infty \tag{3.14}$$

for any $b \geqq a$. From (3.14) it follows that there exists $T_0 > a$ such that

$$\int_{T_0}^t p(s)Y(s)^{\alpha}ds \leqq 2X(t), \qquad \int_{T_0}^t q(s)X(s)^{\beta}ds \leqq 2Y(t), \qquad t \geqq T_0. \tag{3.15}$$

Note that $Y(t)$ is an increasing function. Without loss of generality $X(t)$ may be assumed to be increasing on $[T_0, \infty)$ because a regularly varying function of positive index is asymptotically equivalent to an increasing regularly varying function of the same index (cf. [1, Theorem 1.5.3]).

Furthermore since (3.14) holds for $b = T_0$, there exists $T_1 > T_0$ such that

$$\int_{T_0}^{t} p(s)Y(s)^{\alpha} ds \geqq \frac{1}{2}X(t), \qquad \int_{T_0}^{t} q(s)X(s)^{\beta} ds \geqq \frac{1}{2}Y(t), \qquad t \geqq T_1. \tag{3.16}$$

Let h, H, k, K denote the constants defined by

$$H = (4\Gamma)^{\frac{1+\alpha}{1-\alpha\beta}}, \qquad K = (4\Gamma)^{\frac{1+\beta}{1-\alpha\beta}}, \qquad h = \left(\frac{\gamma}{2}\right)^{\frac{1+\alpha}{1-\alpha\beta}}, \qquad k = \left(\frac{\gamma}{2}\right)^{\frac{1+\beta}{1-\alpha\beta}}, \tag{3.17}$$

where the constants γ and Γ, $0 < \gamma < 1 < \Gamma$, are chosen so that

$$2hX(T_1) \leqq HX(T_0), \qquad 2kY(T_1) \leqq KY(T_0). \tag{3.18}$$

Consider the set $\mathscr{W}$ consisting of continuous vector functions $(x(t), y(t))$ on $[T_0, \infty)$ satisfying

$$hX(t) \leqq x(t) \leqq HX(t), \qquad kY(t) \leqq y(t) \leqq KY(t), \qquad t \geqq T_0, \tag{3.19}$$

and define the map $\Psi : \mathscr{W} \to C[T_0, \infty) \times C[T_0, \infty)$ by

$$\Psi(x, y)(t) = (\mathscr{F}y(t), \mathscr{G}x(t)), \qquad t \geqq T_0, \tag{3.20}$$

where

$$\mathscr{F}y(t) = x_0 + \int_{T_0}^{t} p(s)y(s)^{\alpha} ds, \qquad \mathscr{G}x(t) = y_0 + \int_{T_0}^{t} q(s)x(s)^{\beta} ds, \qquad t \geqq T_0, \tag{3.21}$$

and x_0 and y_0 are positive constants such that

$$hX(T_1) \leqq x_0 \leqq \frac{1}{2}HX(T_0), \qquad kY(T_1) \leqq y_0 \leqq \frac{1}{2}KY(T_0). \tag{3.22}$$

Using (3.15)–(3.21), we see that if $(x, y) \in \mathscr{W}$, then

$$\begin{aligned} \mathscr{F}y(t) &\leqq x_0 + \Gamma \int_{T_0}^{t} p(s)(KY(s))^{\alpha} ds \leqq \frac{1}{2}HX(T_0) + 2\Gamma K^{\alpha} X(t) \\ &= \frac{1}{2}HX(T_0) + \frac{1}{2}HX(t) \leqq \frac{1}{2}HX(t) + \frac{1}{2}HX(t) = HX(t), \quad t \geqq T_0, \\ \mathscr{F}y(t) &\geqq x_0 \geqq hX(T_1) \geqq hX(t), \quad T_0 \leqq t \leqq T_1, \end{aligned}$$

and

$$\mathscr{F}y(t) \geqq \gamma \int_{T_0}^{t} p(s)(kY(s))^{\alpha} ds \geqq \frac{\gamma}{2} k^{\alpha} X(t) = hX(t), \quad t \geqq T_1,$$

implying that $hX(t) \leqq \mathscr{F}y(t) \leqq HX(t)$ for $t \geqq T_0$. Similarly, it can be shown that $kY(t) \leqq \mathscr{G}x(t) \leqq KY(t)$ for $t \geqq T_0$. It follows therefore that Ψ maps $\mathscr{W}$ into itself. From this point on, proceeding as in the proof of Theorem 4.1 in [4] one can prove that Ψ is a continuous map and sends $\mathscr{W}$ into a relatively compact subset of

$C[T_0,\infty)\times C[T_0,\infty)$. Consequently, the Schauder-Tychonoff fixed point theorem ensures the existence of strongly increasing solutions of (B) which are nearly regularly varying of index $(\rho,0)$. It remains to verify the regularity of the nearly regularly varying solutions of (B) thus constructed. But to do so it suffices to repeat the proof of Theorem 4.2 of [4] based on an effective application of the generalized L'Hospital's rule. The details may be omitted. This completes the proof of Theorem 3.2. $\square$

EXAMPLE 3.3. Consider system (B) in which $p(t)$ and $q(t)$ are given by

$$p(t)=t^{\alpha-1}\left(\frac{\log t}{\log\log t}\right)^{\alpha}, \qquad q(t)=(t\log t)^{-1-\alpha\beta}.$$

Here $\lambda=\alpha-1$ and $\mu=-1-\alpha\beta$ satisfy $\lambda+1=\alpha>0$ and $\beta(\lambda+1)+\mu+1=0$, and furthermore $p(t)$ and $q(t)$ satisfy

$$\int_a^t (sp(s))^{\beta}q(s)ds \ \sim\ \frac{(\log\log t)^{1-\alpha\beta}}{1-\alpha\beta}\to\infty, \qquad t\to\infty,$$

where $a=\exp(e)$. Therefore, by Theorem 3.2 this system possesses strongly increasing regularly varying solutions $(x(t),y(t))$ of index $(\rho,0)$ with $\rho=\alpha$, and all such solutions enjoy one and the same asymptotic behavior

$$x(t)\ \sim\ (\alpha\beta)^{-1}t^{\alpha}(\log t)^{\alpha}, \qquad y(t)\ \sim\ \log\log t, \qquad t\to\infty.$$

4. Application to the Thomas – Fermi type differential equations

We consider the Thomas – Fermi type differential equation

$$(p(t)|x'|^{\alpha-1}x')'=q(t)|x|^{\beta-1}x, \tag{C}$$

under the assumptions:

(a) α and β are positive constants such that $\alpha>\beta$;

(b) $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$ which are regularly varying of indices λ and μ, respectively, and are represented in the form (2.1).

We are interested in positive solutions of (C) which exist in a neighborhood of infinity. The following two cases will be discussed separately:

$$\text{Case (I):}\quad \int_a^{\infty} p(t)^{-\frac{1}{\alpha}}dt=\infty; \qquad \text{Case (II):}\quad \int_a^{\infty} p(t)^{-\frac{1}{\alpha}}dt<\infty.$$

A positive solution $x(t)$ of (C) is called *strongly increasing* if

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}p(t)x'(t)^{\alpha}=\infty,$$

which is equivalent to

$$\lim_{t\to\infty}\frac{x(t)}{P(t)}=\infty \quad \text{in Case (I)}, \qquad \lim_{t\to\infty}x(t)=\infty \quad \text{in Case (II)},$$

and *strongly decreasing* if $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}p(t)(-x'(t))^{\alpha}=0$, which is equivalent to

$$\lim_{t\to\infty}x(t)=0 \quad \text{in Case (I)}, \qquad \lim_{t\to\infty}\frac{x(t)}{\pi(t)}=0 \quad \text{in Case (II)},$$

where

$$P(t)=\int_a^t p(s)^{-\frac{1}{\alpha}}ds, \qquad \pi(t)=\int_t^{\infty}p(s)^{-\frac{1}{\alpha}}ds.$$

We assume that the regularity index λ of $p(t)$ is different from α, excluding the border case $\lambda=\alpha$ which seems to be difficult to analyze from our consideration. It is easy to see that if $\lambda<\alpha$ (resp. $\lambda>\alpha$), then Case (I) (resp. Case (II)) occurs and the function $P(t)$ (resp. $\pi(t)$) has the asymptotic behavior

$$P(t)\ \sim\ \frac{\alpha}{\alpha-\lambda}t^{\frac{\alpha-\lambda}{\alpha}}l(t)^{-\frac{1}{\alpha}} \quad \left(\text{resp.}\quad \pi(t)\ \sim\ \frac{\alpha}{\lambda-\alpha}t^{\frac{\alpha-\lambda}{\alpha}}l(t)^{-\frac{1}{\alpha}}\right) \quad \text{as } t\to\infty.$$

Let $x(t)$ be a strongly increasing (resp. strongly decreasing) solution of (C). Then, it is clear that $x'(t)>0$ (resp. $x'(t)<0$) for all large t and satisfies $\lim_{t\to\infty}p(t)x'(t)^{\alpha}=\infty$ (resp. $\lim_{t\to\infty}p(t)(-x'(t))^{\alpha}=0$), so that by putting $y(t)=p(t)x'(t)^{\alpha}$ (resp. $y(t)=p(t)(-x'(t))^{\alpha}$) we see that the vector function $(x(t),y(t))$ is a strongly increasing solution of the system of differential equations

$$x'-p(t)^{-\frac{1}{\alpha}}y^{\frac{1}{\alpha}}=0, \qquad y'-q(t)x^{\beta}=0, \tag{4.1}$$

$$\left(\text{resp.}\quad x'+p(t)^{-\frac{1}{\alpha}}y^{\frac{1}{\alpha}}=0, \qquad y'+q(t)x^{\beta}=0.\right) \tag{4.2}$$

Conversely, if $(x(t),y(t))$ is a strongly increasing (resp. strongly decreasing) solution of system (4.1) (resp. (4.2)), then its first component $x(t)$ gives a strongly increasing (strongly decreasing) solution of equation (C). On the basis of this observation, applying Theorems 3.1 and 3.2 to system (4.1) we obtain the following Theorems 4.1 and 4.2, respectively, supplementing Theorem 5.2 of [2].

THEOREM 4.1. *Let $\lambda>\alpha$. Equation* (C) *possesses strongly increasing slowly varying solution if and only if*

$$\mu=\lambda-\alpha-1 \qquad \text{and} \qquad \int_a^{\infty}\left(\frac{1}{p(t)}\int_a^t q(s)ds\right)^{\frac{1}{\alpha}}dt=\infty,$$

in which case the asymptotic behavior of any such solution $x(t)$ obeys the unique growth law

$$x(t)\ \sim\ \left[\frac{\alpha-\beta}{\alpha}\int_a^t\left(\frac{1}{p(s)}\int_a^s q(r)dr\right)^{\frac{1}{\alpha}}ds\right]^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty.$$

THEOREM 4.2. *Let* $\lambda < \alpha$. *Equation* (C) *possesses strongly increasing regularly varying solutions of index* $\frac{\alpha-\lambda}{\alpha}$ *if and only if*

$$\mu = \frac{\beta}{\alpha}\lambda - \beta - 1 \qquad \text{and} \qquad \int_a^\infty P(t)^\beta q(t)dt = \infty,$$

in which case the asymptotic behavior of any such solution $x(t)$ *obeys the unique growth law*

$$x(t) \sim P(t)\left[\frac{\alpha-\beta}{\alpha}\int_a^t P(s)^\beta q(s)ds\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

On the other hand, from Theorems 2.1 and 2.2 applied to (4.2) there follow Theorems 4.3 and 4.4, respectively, which supplement Theorem 5.1 of [4].

THEOREM 4.3. *Let* $\lambda < \alpha$. *Equation* (C) *possesses strongly decreasing slowly varying solutions if and only if*

$$\mu = \lambda - \alpha - 1 \qquad \text{and} \qquad \int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty q(s)ds\right)^{\frac{1}{\alpha}} ds < \infty,$$

in which case the asymptotic behavior of any such solution $x(t)$ *obeys the unqique decay law*

$$x(t) \sim \left[\frac{\alpha-\beta}{\alpha}\int_t^\infty \left(\frac{1}{p(s)}\int_s^\infty q(r)dr\right)^{\frac{1}{\alpha}} ds\right]^{\frac{\alpha}{\alpha-\beta}}, \quad t \to \infty.$$

THEOREM 4.4. *Let* $\lambda > \alpha$. *Equation* (C) *possesses strongly decreasing regularly varying solution of index* $\frac{\alpha-\lambda}{\alpha}$ *if and only if*

$$\mu = \frac{\beta}{\alpha}\lambda - \beta - 1 \qquad \text{and} \qquad \int_a^\infty \pi(t)^\beta q(t)dt < \infty,$$

in which case the asymptotic behavior of any such solution $x(t)$ *obeys the unique decay law*

$$x(t) \sim \pi(t)\left[\frac{\alpha-\beta}{\alpha}\int_t^\infty \pi(s)^\beta q(s)ds\right]^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

5. Concluding remarks

(1) We have excluded from our consideration strongly monotone solutions $(x(t), y(t))$ of index $(0,0)$ of (A) and (B), that is, those solutions both components of which are slowly varying. Such solutions exist only if $p(t)$ and $q(t)$ are regularly varying functions of index -1, and as observed in the paper [5] this fact seems to cause the difficulty in characterizing the existence of slowly varying solutions for systems (A) and (B).

(2) The study of Thomas-Fermi equations of the form (C) in the framework of regular variation was first attempted in the paper [6] for the special case where $p(t) \equiv 1$. It

would be of interest to observe from the papers [4, 5] that almost complete information on possible regularly varying solutions of the general case of (C) is provided through the analysis of simple two-dimensional cyclic systems (A) and (B) of first order differential equations.

(3) Systems (A) and (B) are studied under the assumption that $\alpha\beta < 1$, so that the consideration of equation (C) is restricted to the case where $\alpha > \beta$. It is natural to ask whether systems (A) and (B) with $\alpha\beta > 1$, and hence equation (C) with $\alpha < \beta$, can possibly be investigated by means of regularly varying functions.

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