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THE RICCATI EQUATION METHOD WITH VARIABLE EXPANSION COEFFICIENTS. III. SOLVING THE NEWELL–WHITEHEAD EQUATION

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Abstract. The Riccati equation method with variable expansion coefficients, introduced in previous papers, is used to find traveling wave solutions to the Newell-Whitehead (NW) equation $u_t = u_{xx} + au - bu^3$. The ξ-dependent coefficients A and B of the Riccati equation $Y' = A + BY^2$ are either proportional each other or their product is equals to an exponential function. They are determined as solutions of ODEs they satisfy and their solutions are expressed either in terms of Bessel's functions or in terms of functions already found in Paper I. The same situation occurs for the expansion coefficients as well. The function Y which is a solution of Riccati's equation, is expressed in terms of Bessel functions or it is a constant quantity.

1. Introduction

Nonlinear partial differential equations arise in a number of areas of Mathematics and Physics in an attempt to model physical processes, like Chemical Kinetics (Gray and Scott [36]), Fluid Mechanics (Whitham [94]), or biological processes like Population Dynamics (Murray [64]). In the recent past there are a number of new methods which have been invented in solving these equations. Among the new methods are the *inverse scattering method* (AKNS [11], Ablowitz and Clarkson [12], Ablowitz and Segur [13], Novikov, Manakov, Pitaevskii and Zakharov [69]), *Hirota's bilinear method* (Hirota [42] and [43]), the *algebro-geometric approach* (Belokolos et al [20]), the *tanh-coth method* (Malfliet [58] and [59], Malfliet and Hereman [60] and [61], El-Wakil and Abdou [27], Fan [31], Griffiths and Sciesser [37], Fan and Hon [32], Parkes and Duffy [73], Parkes, Zhou, Duffy and Huang [75], Wazwaz [90]), the *sn-cn method* (Baldwin et al [18]), the F*-expansion method* (Abdou [4] and [7], Wang and Li [87]), the *Jacobi elliptic function method* (Abbott, Parkes and Duffy [1], Abdou and Elhanbaly [10], Chen and Zhang [23], Chen and Wang [24], Fan and Zhang [33], Inc and Ergüt [45], Liu, Fu, Liu and Zhao [54], [55], Lu and Shi [56], Parkes, Duffy and Abbott [74]), the *Riccati equation method* (Zhang and Zhang [101], Abdou [3], Antoniou [15] and [16]), the *Weierstrass elliptic function method* (Kudryashov [50], [52]), the *exp-function method* (He and Wu [40], Abdou [8], Aslan [17], Bekir and Boz [19], Ebaid

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[25], El-Wakil, Abdou and Hendi [28], He and Abdou [39], Naher, Abdullah and Akbar [65] and [66]), the *Bäcklund transformation method* (Rogers and Shadwick [77]), the (G'/G)- *expansion method* (Borhanibar and Moghanlu [22], Feng, Li and Wan [34], Jabbari, Kheiri and Bekir [46], Naher, Abdullah and Akbar [67], Ozis and Aslan [72], Wang, Li and Zhang [88], Zayed [97], Zayed and Gepreel [99], Antoniou [15] and [16]), the *homogeneous balance method* (Fan [30], Wang, Zhou and Li [86], El-Wakil, Abulwafa, Elhanbaly and Abdou [26]), the *direct algebraic method* (Soliman and Abdou [82]), the *basic equation method* (Kudryashov [51]) and its variants, like the *simplest equation method* (Abdou [6], Jawad, Petkovich and Biswas [47], Vitanov [85], Yefimova [96], Zayed [98]), the *first integral method* (Feng [35], Raslan [76]), the integral bifurcation method (Rui, Xie, Long and He [79]), the *reduced differential transform method* (Keskin and Oturanc [48]), the *Cole-Hopf* transformation method (Salas and Gomez [80]), the *Adomian decomposition method* (Adomian [14], Abdou [2], Wazwaz [89] and [91]), the *Painlevé truncated method* (Weiss, Tabor and Carnevale [92] and [93]), the *homotopy perturbation method* (Taghizadeh, Akbari and Ghelichzadeh [84], Yahya et al [95], Liao [53], El-Wakil and Abdou [29]), the *Lie symmetry method* (Lie point symmetries, potential symmetries, nonclassical symmetries, the direct method) (Bluman and Kumei [21], Hydon [44], Olver [70], Ovsiannikov [71], Stephani [83]), the *variational iteration method* (He [38], Abdou [5], Abdou and Soliman [9], Wazwaz [91]). A more detailed, although not complete set of references of the above methods, appears in Antoniou [15].

The implementation of most of these methods was made possible only using Symbolic Languages like *Mathematica, Macsyma, Maple*, etc.

In this paper we implement the Riccati equation method with variable expansion coefficients introduced previously (Antoniou [15]) and we find traveling wave solutions of the Newell-Whitehead equation.

The paper is organized as follows: In Section 2 we introduce the basic ingredients of the method used. In Section 3 we consider Newell-Whitehead equation and Riccati's equation method of solution, where the expansion coefficients depend on the variable ξ. We find that the ξ-dependent coefficients A and B of the Riccati equation $Y' = A + BY^2$ are either proportional each other or their product is an exponential function and satisfy their own ODEs. Their solutions are expressed either in terms of Bessel's functions or in terms of functions already found in Paper I (Antoniou [15]). The same situation occurs for the expansion coefficients as well. The solution Y which satisfies Riccati's equation is expressed either in terms of the proportionality factor of A and B or in terms of Bessel's functions.

The traveling wave solutions of the Newell-Whitehead equation are expressed by Theorems I, II, III and IV cited at the end of Sections 3.I, 3.II, 3.III and 3.IV respectively.

2. The Method

We consider an evolution equation of the general form

$$u_t = G(u, u_x, u_{xx}, \ldots) \quad \text{or} \quad u_{tt} = G(u, u_x, u_{xx}, \ldots), \tag{2.1}$$

where u is a smooth function. We introduce a new variable ξ given by

$$\xi = k(x - \omega t), \tag{2.2}$$

where k and ω are constants. Changing variables and introducing a new function $U(\xi)$ by $u(x,t) = U(k(x-\omega t)) \equiv U(\xi)$ since

$$u_t = (-\omega k)U'(\xi), u_x = kU'(\xi),\ u_{xx} = k^2 U''(\xi), \dots \tag{2.3}$$

equation (2.1) becomes an ordinary differential equation

$$(-\omega k)\frac{dU}{d\xi} = G\Big(u, k\frac{dU}{d\xi}, k^2\frac{d^2U}{d\xi^2}, \dots\Big) \tag{2.4}$$

or

$$\omega^2 k^2 \frac{dU}{d\xi} = G\Big(u, k\frac{dU}{d\xi}, k^2\frac{d^2U}{d\xi^2}, \dots\Big). \tag{2.5}$$

Equations (2.4) or (2.5) will be solved considering expansions of the form

$$U(\xi) = \sum_{k=0}^{n} a_k Y^k \tag{2.6}$$

or

$$U(\xi) = \sum_{k=0}^{n} a_k Y^k + \sum_{k=0}^{n} \frac{b_k}{Y^k}, \tag{2.7}$$

where all the *expansion coefficients* depend on the variable ξ,

$$a_k \equiv a_k(\xi),\ b_k \equiv b_k(\xi),\ \text{for every } k = 0, 1, 2, \dots n$$

contrary to the previously considered cases, where the expansion coefficients were considered as constants. The function $Y(\xi)$ satisfies Riccati's equation

$$Y'(\xi) = A + BY^2, \tag{2.8}$$

where again the coefficients A and B depend on the variable ξ.

In solving equations (2.4) or (2.5), we consider the expansions (2.6) or (2.7) and then we balance the nonlinear term with the highest derivative of the function $U(\xi)$ which determines n (the number of the expansion terms). Equating similar powers of the function $Y(\xi)$ we can determine the various coefficients and thus find the solution of the equation considered.

In our two previous papers we have applied successfully this method into two notable equations, the Burgers equation and the KdV equation. We have also considered the accompanied (G'/G)- method with variable expansion coefficients which can only be applied once the Riccati equation method has applied before. A remarkable feature of the method is that we obtain quite new solutions and apart of that, we get a proliferation of solutions, once we continue to apply the method in all the intermediate equations, leading to the final solution of the original equation.

3. The NW equation and its solutions

The NW equation was introduced by Newell and Whitehead [68] (see also Segel [81]) and belongs to the general class of reaction-diffusion equations. This equation describes in particular the Rayleigh-Benard convection. The NW equation has also been considered and solved through a variety of methods by Kheiri et al [49], Lu et al [57], Malik et al [62], Malomed [63], and by Zhang [100]. The reader can also consult the very interesting review by Rojas, Elias and Clerc [78] about the many application aspects of the NW equation. In this paper we consider the NW equation in the form $(a>0, b>0)$,

$$u_t = u_{xx} + au - bu^3 \tag{3.1}$$

and try to find traveling wave solutions of this equation. We introduce a new variable ξ given by

$$\xi = x - \omega t, \tag{3.2}$$

where ω is a non-zero constant, $\omega \neq 0$. Changing variables and introducing a new function $U(\xi)$ by $u(x,t) = U(k(x-\omega t)) \equiv U(\xi)$ since

$$u_t = (-\omega k)U'(\xi), u_x = kU'(\xi),\ u_{xx} = k^2 U''(\xi), \ldots \tag{3.3}$$

equation (3.1) becomes an ordinary differential equation

$$U''(\xi) + \omega U'(\xi) + aU(\xi) - bU^3(\xi) = 0. \tag{3.4}$$

We consider the *extended* Riccati equation method in solving equation (3.4), since the Riccati equation method leads to trivial results.

We thus consider the expansion

$$U(\xi) = \sum_{k=0}^{n} a_k Y^k + \sum_{k=0}^{n} \frac{b_k}{Y^k}$$

and balance the second order derivative term with the third order of (3.4). We then find that $n = 1$. The proof might go as follows: The first derivative $U'(\xi)$ contains the highest order term $Y^{n-1}Y'$ and upon the substitution $Y' \to A + BY^2$ the highest order term becomes Y^{n+1}. The second derivative term $U''(\xi)$ contains the highest order term $Y^n Y'$ and upon the substitution $Y' \to A + BY^2$ the highest order term becomes Y^{n+2}. The nonlinear term U^3 contains the highest order term U^{3n}. Therefore balancing the second derivative term $U''(\xi)$ with the nonlinear term $U^3(\xi)$ leads to the equation $n+2 = 3n$ from which we obtain $n = 1$. We thus have

$$U(\xi) = a_0 + a_1 Y + \frac{b_1}{Y}, \tag{3.5}$$

where again all the coefficients a_0, a_1 and b_1 depend on ξ, and Y satisfies Riccati's equation $Y' = A + BY^2$. From equation (3.5) we obtain (taking into account $Y' = A + BY^2$)

$$U'(\xi) = (a_0' - b_1 B + a_1 A) + a_1 BY^2 + \frac{b_1'}{Y} - \frac{b_1 A}{Y^2}, \tag{3.6}$$

$$U''(\xi) = (a_0'' + 2a_1'A + a_1A' - b_1B' - 2b_1'B) + (a_1'' + 2a_1AB)Y + (a_1B' + 2a_1'B)Y^2 + (2a_1B^2)Y^3 + \frac{b_1'' + 2b_1AB}{Y} - \frac{2b_1'A + b_1A'}{Y^2} + \frac{2b_1A^2}{Y^3}. \tag{3.7}$$

Therefore equation (3.4), under the substitution (3.5), (3.6) and (3.7), becomes

$$\begin{aligned}(a_0'' &+ 2a_1'A + a_1A' - b_1B' - 2b_1'B) \\ &+ (a_1'' + 2a_1AB)Y + (a_1B' + 2a_1'B)Y^2 + (2a_1B^2)Y^3 \\ &+ \frac{b_1'' + 2b_1AB}{Y} - \frac{2b_1'A + b_1A'}{Y^2} + \frac{2b_1A^2}{Y^3} \\ &+ \omega\Big((a_0' - b_1B + a_1A) + a_1'Y + a_1BY^2 + \frac{b_1'}{Y} - \frac{b_1A}{Y^2}\Big) \\ &+ a\Big(a_0 + a_1Y + \frac{b_1}{Y}\Big) - b\Big(a_0 + a_1Y + \frac{b_1}{Y}\Big)^3 = 0. \end{aligned} \tag{3.8}$$

Upon expanding and equating the coefficients of Y to zero, we obtain a system of differential equations from which we can determine the various expansion coefficients and the coefficients A and B of Riccati's equation. We obtain

coefficient of Y^3:

$$-ba_1^3 + 2a_1B^2 = 0, \tag{3.9}$$

coeffficient of Y^2:

$$2a_1'B - 3ba_0a_1^2 + a_1B' + \omega a_1B = 0, \tag{3.10}$$

coefficient of Y:

$$a_1'' + aa_1 + \omega a_1' - 3b(b_1a_1^2 + a_1a_0^2) + 2a_1AB = 0, \tag{3.11}$$

coefficient of Y^0:

$$a_0'' + aa_0 + 2a_1'A - b_1B' + a_1A' - 2b_1'B + \omega(a_0' + a_1A - b_1B) - b(a_0^3 + 6a_0a_1b_1) = 0, \tag{3.12}$$

coefficient of Y^{-1}:

$$b_1'' + ab_1 + 2b_1AB + \omega b_1' - 3b(a_0^2b_1 + a_1b_1^2) = 0, \tag{3.13}$$

coefficient of Y^{-2}:

$$-3ba_0b_1^2 - \omega Ab_1 - b_1A' - 2b_1'A = 0, \tag{3.14}$$

coefficient of Y^{-3}:

$$2b_1A^2 - bb_1^3 = 0. \tag{3.15}$$

We now have to solve the system of equations (3.9)-(3.15) supplemented by the Riccati equation $Y' = A + BY^2$.

From equations (3.9) and (3.15), ignoring the trivial solutions, we obtain

$$a_1 = \pm\sqrt{\frac{2}{b}}B \quad \text{and} \quad b_1 = \pm\sqrt{\frac{2}{b}}A \tag{3.16}$$

respectively. We thus consider the following four cases.

3.I. Case I. For $a_1 = \sqrt{\frac{2}{b}}B$ and $b_1 = \sqrt{\frac{2}{b}}A$, we obtain
from (3.14),

$$6a_0 = -\sqrt{\frac{2}{b}}\left(\omega + 3\frac{A'}{A}\right), \tag{3.17}$$

from (3.10)

$$6a_0 = \sqrt{\frac{2}{b}}\left(\omega + 3\frac{B'}{B}\right), \tag{3.18}$$

from (3.11)

$$\frac{B''}{B} + \omega\frac{B'}{B} - 4AB + a - 3ba_0^2 = 0, \tag{3.19}$$

from (3.13)

$$\frac{A''}{A} + \omega\frac{A'}{A} - 4AB + a - 3ba_0^2 = 0, \tag{3.20}$$

from (3.12)

$$a_0'' + \omega a_0' + aa_0 - 12a_0AB - ba_0^3 + \sqrt{\frac{2}{b}}(AB' - A'B) = 0. \tag{3.21}$$

Equating the two different expressions of a_0, equations (3.17) and (3.18), we obtain the equation $3\left(\frac{A'}{A} + \frac{B'}{B}\right) = -2\omega$, which by integration gives

$$AB = K\exp\left(-\frac{2\omega}{3}\xi\right), \tag{3.22}$$

where K is a positive constant, $K > 0$. Using (3.18) and (3.22), we obtain from (3.19) the equation

$$\frac{B''}{B} - \frac{3}{2}\left(\frac{B'}{B}\right)^2 - 4K\exp\left(-\frac{2\omega}{3}\xi\right) + a - \frac{\omega^2}{6} = 0. \tag{3.23}$$

The previous equation, under the substitution

$$F = \frac{B'}{B} \tag{3.24}$$

transforms into the equation

$$F' = \frac{1}{2}F^2 + 4K\exp\left(-\frac{2\omega}{3}\xi\right) - a + \frac{\omega^2}{6} \tag{3.25}$$

which is a Riccati differential equation. Under the standard transformation

$$F = -2\frac{w'}{w} \tag{3.26}$$

equation (3.25) takes on the form of a second order linear differential equation

$$w'' + \left(2K\exp\left(-\frac{2\omega}{3}\xi\right) - \frac{a}{2} + \frac{\omega^2}{12}\right)w = 0. \tag{3.27}$$

The substitution $z = e^{\mu\xi}$ transforms (3.27) into

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + \frac{1}{\mu^2}\left(2Kz^{-\frac{2\omega}{3\mu}} - \frac{a}{2} + \frac{\omega^2}{12}\right)w = 0. \tag{3.28}$$

The choice $\mu = -\frac{\omega}{3}$ converts $z^{-\frac{2\omega}{3\mu}}$ into z^2 and thus equation (3.28) takes on the form

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + \left(\frac{18K}{\omega^2}z^2 + \frac{3}{4} - \frac{9a}{2\omega^2}\right)w = 0. \tag{3.29}$$

The substitution

$$\rho^2 = \frac{18K}{\omega^2} \quad \text{and} \quad -\nu^2 = \frac{3}{4} - \frac{9a}{2\omega^2} \tag{3.30}$$

converts (3.29) into

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (\rho^2 z^2 - \nu^2)w = 0 \tag{3.31}$$

which is Bessel's differential equation with general solution

$$w = C_1 J_\nu(\rho z) + C_2 Y_\nu(\rho z). \tag{3.32}$$

Going back to the original variables, we have

$$w(\xi) = C_1 J_\nu\left(\frac{3\sqrt{2K}}{\omega}e^{-\frac{\omega}{3}\xi}\right) + C_2 Y_\nu\left(\frac{3\sqrt{2K}}{\omega}e^{-\frac{\omega}{3}\xi}\right), \tag{3.33}$$

where

$$\nu = \frac{\sqrt{18a - 3\omega^2}}{2\omega} \quad (6a > \omega^2). \tag{3.34}$$

Bessel's functions are known to be defined by

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}z^2\right)^k}{\Gamma(\nu+k+1)k!} \quad \text{and} \quad Y_\nu = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$

respectively. The constants C_1 and C_2 in (3.33) are chosen such that $w(\xi) \neq 0,\ \forall\xi$. It is not trivial to ensure that since the Bessel functions are oscillating at infinity. However if the constant $\omega > 0$, then we have not such a problem in (3.33) $w(\xi) \neq 0, \forall\xi$ because

$\frac{3\sqrt{2}}{\omega}e^{-\frac{\omega}{3}\xi} \to 0$ as $\xi \to \infty$. Combining (3.24) and (3.26), we derive the equation $\frac{B'}{B} + 2\frac{w'}{w} = 0$, from which by integration we obtain

$$B(\xi) = \frac{C}{w^2(\xi)} \tag{3.35}$$

with w given by (3.33). Using equations (3.22) and (3.35), we determine the coefficient A:

$$A(\xi) = \frac{K}{C}w^2(\xi)\exp(-\frac{2\omega}{3}\xi). \tag{3.36}$$

We thus get

$$a_1 = C\sqrt{\frac{2}{b}}\frac{1}{w^2(\xi)} \quad \text{and} \quad b_1 = C\sqrt{\frac{2}{b}}w^2(\xi)\exp\Big(-\frac{2\omega}{3}\xi\Big) \tag{3.37}$$

and from (3.18), since $\frac{B'}{B} + 2\frac{w'}{w} = 0$,

$$a_0(\xi) = \frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega - 6\frac{w'(\xi)}{w(\xi)}\Big). \tag{3.38}$$

Equation (3.21), using (3.18) and (3.22), can be expressed in terms of $F = \frac{B'}{B}$ as

$$F'' - \frac{1}{2}F^3 + \omega\Big(F' - \frac{1}{2}F^2\Big) + \Big(a - \frac{\omega^2}{6}\Big)F - 8KFe^{-\frac{2\omega}{3}\xi} - \frac{8K}{3}\omega e^{-\frac{2\omega}{3}\xi} + \frac{\omega}{54}(18a - \omega^2) = 0. \tag{3.39}$$

The above equation should be compatible to (3.25) (i.e. the solution (3.33) should satisfy (3.39) given that F is connected to $w(\xi)$ through (3.26)). According to the results of Appendix A, we have the constraints

$$\nu^2 = \frac{1}{4}, \quad \omega^2 = \frac{9a}{2}$$

and

$$J_\nu\Big(\frac{3\sqrt{2K}}{\omega}\Big) \times Y_{\nu+1}\Big(\frac{3\sqrt{2K}}{\omega}\Big) - J_{\nu+1}\Big(\frac{3\sqrt{2K}}{\omega}\Big) \times Y_\nu\Big(\frac{3\sqrt{2K}}{\omega}\Big) = 0.$$

We turn now to the determination of the function Y which satisfies Riccati's equation. Under the substitution

$$Y = -\frac{1}{B}\frac{u'}{u}, \tag{3.40}$$

Riccati's equation $Y' = A + BY^2$ becomes

$$u'' - \Big(\frac{B'}{B}\Big)u' + ABu = 0. \tag{3.41}$$

The substitution

$$u = \sqrt{B}y \tag{3.42}$$

converts (3.41) into

$$y'' + \frac{1}{2}\Big[\frac{B''}{B} - \frac{3}{2}\Big(\frac{B'}{B}\Big)^2\Big]y + K\exp\Big(-\frac{2\omega}{3}\xi\Big)y = 0 \tag{3.43}$$

taking also into account (3.22). Since $B = \frac{C}{w^2(\xi)}$, we have $\frac{B''}{B} - \frac{3}{2}\Big(\frac{B'}{B}\Big)^2 = -2\frac{w''}{w}$ and using (3.32), we obtain, using the various formulas for the derivatives and recurrence relations of Bessel's functions, that

$$-2\frac{w''}{w} = \frac{2}{9}\Big(\rho^2\exp\Big(-\frac{2\omega}{3}\xi\Big) - \nu^2\Big)\omega^2.$$

Therefore equation (3.43) takes on the form

$$y'' + \Big[\lambda^2\exp\Big(-\frac{2\omega}{3}\xi\Big) - \Big(\frac{\omega\nu}{3}\Big)^2\Big]y = 0, \tag{3.44}$$

where

$$\lambda^2 = \frac{1}{9}\rho^2\omega^2 + K = 3K. \tag{3.45}$$

Equation (3.44) can be solved by converting it to Bessel's equation, along the lines of reasoning in transforming (3.27) into (3.31). We thus get the following solution of equation (3.44):

$$y = \tilde{C}_1 J_\nu\Big(\frac{3\sqrt{3K}}{\omega}e^{-\frac{\omega}{3}\xi}\Big) + \tilde{C}_2 Y_\nu\Big(\frac{3\sqrt{3K}}{\omega}e^{-\frac{\omega}{3}\xi}\Big). \tag{3.46}$$

The same index ν appears in both expressions of the Bessel's function (3.46) and (3.33), and is given by (3.34). We then obtain from (3.40), using (3.42) and (3.35)

$$Y = -\frac{1}{C}w^2(\xi)\Big(\frac{y'(\xi)}{y(\xi)} - \frac{w'(\xi)}{w(\xi)}\Big). \tag{3.47}$$

Therefore we arrive at the following

Solution I. The solution of equation (3.4)

$$U''(\xi) + \omega U'(\xi) + aU(\xi) - bU^3(\xi) = 0$$

is given by (3.5), $U(\xi) = a_0 + a_1 Y + \frac{b_1}{Y}$, where

$$a_0(\xi) = \frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega - 6\frac{w'(\xi)}{w(xi)}\Big), \quad a_1 = C\sqrt{\frac{2}{b}}\frac{1}{w^2(\xi)},$$

$$b_1 = \frac{K}{C}\sqrt{\frac{2}{b}}w^2(\xi)\exp\Big(-\frac{2\omega}{3}\xi\Big) \quad \text{and} \quad Y = -\frac{1}{C}w^2(\xi)\Big(\frac{y'(\xi)}{y(\xi)} - \frac{w'(\xi)}{w(\xi)}\Big),$$

with $w(\xi)$ and $y(\xi)$ being expressed in terms of Bessel's functions by (3.33) and (3.46) respectively. Therefore

$$U(\xi)=\frac{1}{6}\sqrt{\frac{2}{b}}\Big[\omega-6\frac{y'(\xi)}{y(\xi)}-6K\frac{\exp\left(-\frac{2\omega}{3}\xi\right)}{\left(\frac{y'(\xi)}{y(\xi)}-\frac{w'(\xi)}{w(\xi)}\right)}\Big].$$

In the above solution we have to take into account the condition $\frac{w'(\xi)}{w(\xi)}=\frac{\omega}{6}$, according to (4.8) of Appendix A. We thus have the following solution expressed by Theorem I:

THEOREM I. The traveling wave solutions of the Newell-Whitehead equation $u_t = u_{xx}+au-bu^3$, derived under the substitution $\xi = x-\omega t$, i.e. the solutions of the equation $U''(\xi)+\omega U'(\xi)+aU(\xi)-bU^3(\xi)=0$, are given by

$$U(\xi)=\frac{1}{6}\sqrt{\frac{2}{b}}\Big[\omega-6\frac{y'(\xi)}{y(\xi)}-6K\Big(\frac{y'(\xi)}{y(\xi)}-\frac{\omega}{6}\Big)^{-1}\times\exp\Big(-\frac{2\omega}{3}\xi\Big)\Big]. \tag{3.48}$$

The above solution is subject to the constraints

$$\nu^2=\frac{1}{4},\quad \omega^2=\frac{9a}{2} \tag{3.49}$$

and

$$J_\nu\Big(\frac{3\sqrt{2K}}{\omega}\Big)\times Y_{\nu+1}\Big(\frac{3\sqrt{2K}}{\omega}\Big)-J_{\nu+1}\Big(\frac{3\sqrt{2K}}{\omega}\Big)\times Y_\nu\Big(\frac{3\sqrt{2K}}{\omega}\Big)=0. \tag{3.50}$$

3.II. Case II. For $a_1=\sqrt{\frac{2}{b}}B$ and $b_1=-\sqrt{\frac{2}{b}}A$, we obtain
from (3.14),

$$6a_0=\sqrt{\frac{2}{b}}\Big(\omega+3\frac{A'}{A}\Big), \tag{3.51}$$

from (3.10)

$$6a_0=\sqrt{\frac{2}{b}}\Big(\omega+3\frac{B'}{B}\Big), \tag{3.52}$$

from (3.11)

$$\frac{B''}{B}+\omega\frac{B'}{B}+8AB-3ba_0^2+a=0, \tag{3.53}$$

from (3.13)

$$\frac{A''}{A}+\omega\frac{A'}{A}+8AB-3ba_0^2+a=0, \tag{3.54}$$

from (3.12)

$$a_0''+\omega a_0'+aa_0+12a_0AB-ba_0^3+3\sqrt{\frac{2}{b}}(AB)'+2\omega\sqrt{\frac{2}{b}}AB=0. \tag{3.55}$$

Equating the two different expressions for a_0,equations (3.51) and (3.52), we find that $\frac{A'}{A} = \frac{B'}{B}$ and by integration

$$B = -s^2 A, \tag{3.56}$$

where s is real. Equation (3.54) then, using (3.51) and (3.56), becomes

$$AA'' - \frac{3}{2}(A')^2 + 2m^2 A^2 - 8s^2 A^4 = 0, \tag{3.57}$$

where

$$2m^2 = a - \frac{\omega^2}{6}, \quad m \in \mathbb{R}. \tag{3.58}$$

Equation (3.57) has been solved by a variety of methods in Paper I (Antoniou [15], Appendices A, B and C). In Appendix C of this paper we list all the solution methods and the corresponding solutions we have already found in Paper I.

Equation (3.55) expressed in terms of $G = \frac{A'}{A}$ gives, taking into account (3.51)

$$G'' - \frac{1}{2}G^3 + \omega\Big(G' - \frac{1}{2}G^2\Big) + \Big(a - \frac{\omega^2}{6}\Big)G \\ - 24s^2 AA' - 8\omega s^2 A^2 + \frac{\omega}{54}(18a - \omega^2) = 0. \tag{3.59}$$

All the solutions which satisfy (3.57) should be substituted in the above equation. The resulting expressions will provide compatibility conditions between the various parameters and constants. According to the results of Appendix B, we find that (3.57) (expressed in terms of G, see (5.2)) is compatible to (3.59) if

$$\omega^2 = \frac{9a}{2}. \tag{3.60}$$

We now turn to the determination of the function Y which satisfies Riccati's equation. Under the substitution

$$Y = -\frac{1}{B}\frac{v'}{v} \tag{3.61}$$

Riccati's equation $Y' = A + BY^2$ becomes

$$v'' - \Big(\frac{B'}{B}\Big)v' + ABv = 0 \tag{3.62}$$

and because of (3.56),

$$v'' - \Big(\frac{A'}{A}\Big)v' - s^2 A^2 v = 0. \tag{3.63}$$

From the above equation we obtain $\frac{v'}{v} = \pm sA$ and then (see eqs (3.53)-(3.60) of paper I, Antoniou [15])

$$Y(\xi) = \pm\frac{1}{s}. \tag{3.64}$$

Collecting the results of this Section, we arrive at

Solution II. The solution of equation (3.4)

$$U''(\xi)+\omega U'(\xi)+aU(\xi)-bU^3(\xi)=0$$

is given by (3.5), $U(\xi)=a_0+a_1Y+\frac{b_1}{Y}$, where

$$a_0=\frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega+3\frac{A'}{A}\Big),\quad a_1=-s^2\sqrt{\frac{2}{b}}A,\quad b_1=-\sqrt{\frac{2}{b}}A\quad\text{and}\quad Y=\pm\frac{1}{s}$$

and A is any solution of the equation (3.57). We thus have the following solution expressed by Theorem II:

THEOREM II. The traveling wave solutions of the Newell-Whitehead equation $u_t=u_{xx}+au-bu^3$, derived under the substitution $\xi=x-\omega t$, i.e. the solutions of the equation $U''(\xi)+\omega U'(\xi)+aU(\xi)-bU^3(\xi)=0$, are given by

$$U(\xi)=\frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega+3\frac{A'(\xi)}{A(\xi)}\Big)-2s\sqrt{\frac{2}{b}}A(\xi),\quad Y=\frac{1}{s},\tag{3.65}$$

$$U(\xi)=\frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega+3\frac{A'(\xi)}{A(\xi)}\Big)+2s\sqrt{\frac{2}{b}}A(\xi),\quad Y=-\frac{1}{s},\tag{3.66}$$

with $\omega^2=\frac{9a}{2}$ while the various functions $A(\xi)$ which are solutions of (3.57) are given in Appendix C.

3.III. Case III. For $a_1=-\sqrt{\frac{2}{b}}B$ and $b_1=\sqrt{\frac{2}{b}}A$, we obtain from (3.14),

$$6a_0=-\sqrt{\frac{2}{b}}\Big(\omega+3\frac{A'}{A}\Big),\tag{3.67}$$

from (3.10)

$$6a_0=-\sqrt{\frac{2}{b}}\Big(\omega+3\frac{B'}{B}\Big),\tag{3.68}$$

from (3.11)

$$\frac{B''}{B}+\omega\frac{B'}{B}+8AB-3ba_0^2+a=0,\tag{3.69}$$

from (3.13)

$$\frac{A''}{A}+\omega\frac{A'}{A}+8AB-3ba_0^2+a=0,\tag{3.70}$$

from (3.12)

$$a_0''+\omega a_0'+aa_0+12a_0AB-ba_0^3-3\sqrt{\frac{2}{b}}(AB)'-2\omega\sqrt{\frac{2}{b}}AB=0.\tag{3.71}$$

Equating the two different expressions for a_0,equations (3.67) and (3.68), we conclude that $B=-s^2A$, which is (3.56). We thus obtain again $Y=\pm\frac{1}{s}$. Equation (3.70) gives, because of (3.67), equation (3.57). Equation (3.71) also gives, taking into account (3.67) and (3.56)

$$G'' - \frac{1}{2}G^3 + \omega\Big(G' - \frac{1}{2}G^2\Big) + \Big(a - \frac{\omega^2}{6}\Big)G \\ - 24s^2AA' - 8\omega s^2A^2 - \frac{\omega}{54}(18a - \omega^2) = 0. \tag{3.72}$$

All the solutions which satisfy (3.70) (i.e.(3.57)) should be substituted in the above equation. The resulting expressions will provide compatibility conditions between the various parameters and constants. According to the results of Appendix B, we find $\omega^2 = \frac{9a}{2}$. We thus obtain the following

Solution III.. The solution of equation (3.4)

$$U''(\xi) + \omega U'(\xi) + aU(\xi) - bU^3(\xi) = 0$$

is given by (3.5), $U(\xi) = a_0 + a_1Y + \frac{b_1}{Y}$, where

$$a_0 = -\frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega + 3\frac{A'}{A}\Big), \quad a_1 = s^2\sqrt{\frac{2}{b}}A, \quad b_1 = \sqrt{\frac{2}{b}}A \quad \text{and} \quad Y = \pm\frac{1}{s}$$

and A is any solution of the equation (3.57). We thus have the following solution expressed by Theorem III:

THEOREM III. The traveling wave solutions of the Newell-Whitehead equation $u_t = u_{xx} + au - bu^3$, derived under the substitution $\xi = x - \omega t$, i.e. the solutions of the equation $U''(\xi) + \omega U'(\xi) + aU(\xi) - bU^3(\xi) = 0$, are given by

$$U(\xi) = -\frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega + 3\frac{A'(\xi)}{A(\xi)}\Big) + 2s\sqrt{\frac{2}{b}}A(\xi), \quad Y = \frac{1}{s}, \tag{3.73}$$

$$U(\xi) = -\frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega + 3\frac{A'(\xi)}{A(\xi)}\Big) - 2s\sqrt{\frac{2}{b}}A(\xi), \quad Y = -\frac{1}{s}, \tag{3.74}$$

with $\omega^2 = \frac{9a}{2}$ while the various functions $A(\xi)$ which are solutions of (3.57) are given in Appendix C.

3.IV. Case IV. For $a_1 = -\sqrt{\frac{2}{b}}B$ and $b_1 = -\sqrt{\frac{2}{b}}A$, we obtain from (3.14),

$$6a_0 = \sqrt{\frac{2}{b}}\Big(\omega + 3\frac{A'}{A}\Big), \tag{3.75}$$

from (3.10)

$$6a_0 = -\sqrt{\frac{2}{b}}\Big(\omega + 3\frac{B'}{B}\Big), \tag{3.76}$$

from (3.11)

$$\frac{B''}{B} + \omega\frac{B'}{B} - 4AB + a - 3ba_0^2 = 0, \tag{3.77}$$

from (3.13)

$$\frac{A''}{A}+\omega\frac{A'}{A}-4AB+a-3ba_0^2=0, \tag{3.78}$$

from (3.12)

$$a_0''+\omega a_0'+aa_0-12a_0AB-ba_0^3-\sqrt{\frac{2}{b}}(AB'-A'B)=0. \tag{3.79}$$

Equating the two different expressions of a_0, equations (3.75) and (3.76), we obtain the equation $3\left(\frac{A'}{A}+\frac{B'}{B}\right)=-2\omega$, which by integration gives (3.22). We then obtain from (3.77), using (3.76) and (3.22), the equation (3.23), which can be converted to equation (3.31) with solution given by (3.33). We also obtain again (3.47) as the solution of Riccati's equation. Equation (3.79) can be expressed in terms of $F=\frac{B'}{B}$ as

$$F''-\frac{1}{2}F^3+\omega\Big(F'-\frac{1}{2}F^2\Big)+\Big(a-\frac{\omega^2}{6}\Big)F-8KFe^{-\frac{2\omega}{3}\xi}$$
$$-\frac{8K}{3}\omega e^{-\frac{2\omega}{3}\xi}-\frac{\omega}{54}(18a-\omega^2)=0.$$

The solution (3.33) is to be substituted in the above equation to get a compatibility condition between the parameters and the constants. We obtain as in **Case I**, the same set of compatibility conditions. Therefore we arrive at the following

Solution IV. The solution of equation (3.4),

$$U''(\xi)+\omega U'(\xi)+aU(\xi)-bU^3(\xi)=0$$

is given by (3.5), $U(\xi)=a_0+a_1Y+\frac{b_1}{Y}$, where

$$a_0(\xi)=-\frac{1}{6}\sqrt{\frac{2}{b}}\Big(\omega-6\frac{w'(\xi)}{w(xi)}\Big),\quad a_1=-C\sqrt{\frac{2}{b}}\frac{1}{w^2(\xi)},$$

$$b_1=-\frac{K}{C}\sqrt{\frac{2}{b}}w^2(\xi)\exp\Big(-\frac{2\omega}{3}\xi\Big)\quad\text{and}\quad Y=-\frac{1}{C}w^2(\xi)\Big(\frac{y'(\xi)}{y(\xi)}-\frac{w'(\xi)}{w(\xi)}\Big)$$

with $w(\xi)$ and $y(\xi)$ being expressed in terms of Bessel's functions by (3.33) and (3.46) respectively. Therefore

$$U(\xi)=\frac{1}{6}\sqrt{\frac{2}{b}}\left[-\omega+6\frac{y'(\xi)}{y(\xi)}+6K\Big(\frac{y'(\xi)}{y(\xi)}-\frac{w'(\xi)}{w(\xi)}\Big)^{-1}\times\exp\Big(-\frac{2\omega}{3}\xi\Big)\right].$$

In the above solution we have to take into account the condition $\frac{w'(\xi)}{w(\xi)}=\frac{\omega}{6}$, according to (4.8) of Appendix A. We thus have the following solution expressed by Theorem IV:

THEOREM IV. The traveling wave solutions of the Newell-Whitehead equation $u_t=u_{xx}+au-bu^3$, derived under the substitution $\xi=x-\omega t$, i.e. the solutions of the equation $U''(\xi)+\omega U'(\xi)+aU(\xi)-bU^3(\xi)=0$, are given by

$$U(\xi)=\frac{1}{6}\sqrt{\frac{2}{b}}\Big[-\omega+6\frac{y'(\xi)}{y(\xi)}+6K\Big(\frac{y'(\xi)}{y(\xi)}-\frac{\omega}{6}\Big)^{-1}\times\exp\Big(-\frac{2\omega}{3}\xi\Big)\Big]. \tag{3.80}$$

The above solution is subject to the constraints

$$\nu^2 = \frac{1}{4}, \quad \omega^2 = \frac{9a}{2} \tag{3.81}$$

and

$$J_\nu\Big(\frac{3\sqrt{2K}}{\omega}\Big) \times Y_{\nu+1}\Big(\frac{3\sqrt{2K}}{\omega}\Big) - J_{\nu+1}\Big(\frac{3\sqrt{2K}}{\omega}\Big) \times Y_\nu\Big(\frac{3\sqrt{2K}}{\omega}\Big) = 0. \tag{3.82}$$

We list below the derivative of the function $y(\xi)$ which appears in Solutions I and IV. Using the general formulas for the derivatives of Bessel's functions, we obtain

$$\begin{aligned} y'(\xi) = \sqrt{3K}e^{-\frac{\omega}{3}\xi}\Big[\tilde{C}_1 J_\nu\Big(\frac{3\sqrt{3K}}{\omega}e^{-\frac{\omega}{3}\xi}\Big) + \tilde{C}_2 Y_\nu\Big(\frac{3\sqrt{3K}}{\omega}e^{-\frac{\omega}{3}\xi}\Big)\Big] \\ -\frac{1}{3}\omega\nu\Big[\tilde{C}_1 J_\nu\Big(\frac{3\sqrt{3K}}{\omega}e^{-\frac{\omega}{3}\xi}\Big) + \tilde{C}_2 Y_\nu\Big(\frac{3\sqrt{3K}}{\omega}e^{-\frac{\omega}{3}\xi}\Big)\Big]. \end{aligned} \tag{3.83}$$

4. Appendix A

In this Appendix we consider the issue of expressing the compatibility condition (3.39) in terms of the constants and the various parameters. We first consider (3.39),

$$\begin{aligned} F'' - \frac{1}{2}F^3 + \omega\Big(F' - \frac{1}{2}F^2\Big) + \Big(a - \frac{1}{6}\omega^2\Big)F - 8Ke^{-\frac{2\omega}{3}\xi}F \\ -\frac{8}{3}K\omega e^{-\frac{2\omega}{3}\xi} + \frac{\omega}{54}(18a - \omega^2) = 0. \end{aligned} \tag{4.1}$$

We consider the above equation with (3.25):

$$F' - \frac{1}{2}F^2 = 4Ke^{-\frac{2\omega}{3}\xi} - \Big(a - \frac{\omega^2}{6}\Big). \tag{4.2}$$

Substituting $F' - \frac{1}{2}F^2$ in (4.1) by the expression given in (4.2), we obtain

$$\begin{aligned} F'' - \frac{1}{2}F^3 + \Big(a - \frac{1}{6}\omega^2\Big)F + \frac{4}{3}K\omega e^{-\frac{2\omega}{3}\xi} - 8Ke^{-\frac{2\omega}{3}\xi}F \\ + \frac{2}{3}\omega\Big(\frac{2}{9}\omega^2 - a\Big) = 0. \end{aligned} \tag{4.3}$$

We multiply by F equation (4.2),

$$-\frac{1}{2}F^3 = -FF' + 4Ke^{-\frac{2\omega}{3}\xi}F + \Big(\frac{\omega^2}{6} - a\Big)F \tag{4.4}$$

and we substitute $-\frac{1}{2}F^3$ given by (4.4) into (4.3) and we obtain the equation

$$F'' - FF' + \frac{4}{3}K\omega e^{-\frac{2\omega}{3}\xi} - 4Ke^{-\frac{2\omega}{3}\xi}F + \frac{2}{3}\omega\Big(\frac{2}{9}\omega^2 - a\Big) = 0. \tag{4.5}$$

Upon differentiation of (4.2) with respect to ξ, we find

$$F'' - FF' = -\frac{8}{3}K\omega e^{-\frac{2\omega}{3}\xi}. \tag{4.6}$$

Equation (4.5), because of (4.6), takes on the form

$$-4K\left(\frac{\omega}{3} + F\right)e^{-\frac{2\omega}{3}\xi} + \frac{2}{3}\omega\left(\frac{2}{9}\omega^2 - a\right) = 0. \tag{4.7}$$

The above equation should hold for every ξ. We thus have $\frac{\omega}{3} + F = 0$ and $\frac{2}{9}\omega^2 - a = 0$, i.e.

$$\omega w(\xi) - 6w'(\xi) = 0 \tag{4.8}$$

and

$$\omega^2 = \frac{9a}{2} \tag{4.9}$$

respectively, where $w(\xi)$ is given by (3.33). Combining (4.9) with the second of (3.30), we obtain

$$\nu^2 = \frac{1}{4}. \tag{4.10}$$

Upon expanding (4.8) in a series around $\xi = 0$ (this is best facilitated using any of the known Computer Algebra Systems) we find that (4.8) is true to every order, provided that

$$C_1 J_\nu\left(\frac{3\sqrt{2K}}{\omega}\right) + C_2 Y_\nu\left(\frac{3\sqrt{2K}}{\omega}\right) = 0 \tag{4.11}$$

and

$$C_1 J_{\nu+1}\left(\frac{3\sqrt{2K}}{\omega}\right) + C_2 Y_{\nu+1}\left(\frac{3\sqrt{2K}}{\omega}\right) = 0. \tag{4.12}$$

We thus have

$$\frac{C_1}{C_2} = -\frac{Y_\nu\left(\frac{3\sqrt{2K}}{\omega}\right)}{J_\nu\left(\frac{3\sqrt{2K}}{\omega}\right)} = -\frac{Y_{\nu+1}\left(\frac{3\sqrt{2K}}{\omega}\right)}{J_{\nu+1}\left(\frac{3\sqrt{2K}}{\omega}\right)}. \tag{4.13}$$

From the above relations we find that

$$J_\nu\left(\frac{3\sqrt{2K}}{\omega}\right) \times Y_{\nu+1}\left(\frac{3\sqrt{2K}}{\omega}\right) - J_{\nu+1}\left(\frac{3\sqrt{2K}}{\omega}\right) \times Y_\nu\left(\frac{3\sqrt{2K}}{\omega}\right) = 0. \tag{4.14}$$

The above equation provides essentially a relation which determines the constant K in terms of a (since ω is expressed through a by (4.9)).

Considering **Case IV**, we find the same compatibility conditions as in **Case I.**

5. Appendix B

In this Appendix we consider the issue of expressing the compatibility condition (3.59) in terms of the constants and the various parameters. We first consider (3.59),

$$G'' - \frac{1}{2}G^3 + \omega\Big(G' - \frac{1}{2}G^2\Big) + \Big(a - \frac{\omega^2}{6}\Big)G - 24s^2AA' - 8\omega s^2A^2 + \frac{\omega}{54}(18a - \omega^2) = 0. \tag{5.1}$$

We similarly express (3.57) in terms of G:

$$G' - \frac{1}{2}G^2 = 8s^2A^2 - \Big(a - \frac{\omega^2}{6}\Big). \tag{5.2}$$

Upon substituting $G' - \frac{1}{2}G^2$ given by (5.2) into (5.1), we obtain

$$G'' - \frac{1}{2}G^3 + \Big(a - \frac{\omega^2}{6}\Big)G - 24s^2AA' + \frac{2\omega}{3}\Big(\frac{2\omega^2}{9} - a\Big) = 0. \tag{5.3}$$

From (5.2), multiplying by G, we find

$$-\frac{1}{2}G^3 + \Big(a - \frac{\omega^2}{6}\Big)G = -GG' + 8s^2A^2G. \tag{5.4}$$

Equation (5.3) then gives, because of (5.4),

$$G'' - GG' + 8s^2A^2G - 24s^2AA' + \frac{2\omega}{3}\Big(\frac{2\omega^2}{9} - a\Big) = 0. \tag{5.5}$$

Differentiating (5.2) with respect to ξ, we derive

$$G'' - GG' = 16s^2AA'. \tag{5.6}$$

Therefore, because of (5.6), equation (5.5) becomes

$$16s^2AA' + 8s^2A^2G - 24s^2AA' + \frac{2\omega}{3}\Big(\frac{2\omega^2}{9} - a\Big) = 0$$

which, because of $G = \frac{A'}{A}$, gives us the relation $\frac{2\omega}{3}\Big(\frac{2\omega^2}{9} - a\Big) = 0$ from which we obtain

$$\omega^2 = \frac{9a}{2}. \tag{5.7}$$

Considering **Case III**, we find that (3.70) and (3.71) are compatible if equation (5.7) holds true as well.

6. Appendix C

In this Appendix we list the solution methods and the corresponding solutions of the equation (3.57),

$$A(\xi)A''(\xi)-\frac{3}{2}(A'(\xi))^2+2m^2A^2(\xi)-8s^2A^4(\xi)=0. \tag{6.1}$$

A.I. First Method. We consider an expansion of the form

$$A(\xi)=a_0+a_1\varphi(\xi), \tag{6.2}$$

where $\varphi(\xi)$ satisfies Jacobi's differential equation

$$\frac{d}{d\xi}\varphi(\xi)=\sqrt{\lambda+\mu\varphi^2+\rho^2\varphi^4}, \tag{6.3}$$

where λ, μ and ρ are constant real parameters. Upon substituting (6.2) into (6.1), taking into account (6.3) and equating to zero the coefficients of the different powers of φ, we can determine the coefficients a_0, a_1 and the function $\varphi(\xi)$. We find that the coefficients a_0 and a_1 are given by the relations

$$a_1^2=\frac{\rho^2}{16s^2},\quad a_0^2=\frac{m^2}{16s^2} \tag{6.4}$$

and the function $\varphi(\xi)$ by

$$\varphi(\xi)=-\frac{m}{\rho}\cdot\frac{C\tanh(m\xi)+1}{C+\tanh(m\xi)}, \tag{6.5}$$

where C is an arbitrary constant. We thus have, using (6.2), the four values (6.4) and equation (6.5), the following expressions for $A(\xi)$:

$$\begin{aligned}
A(\xi)&=\frac{\rho}{4s}+\frac{m}{4s}\Big(-\frac{m}{\rho}\cdot\frac{C\tanh(m\xi)+1}{C+\tanh(m\xi)}\Big),\\
A(\xi)&=\frac{\rho}{4s}-\frac{m}{4s}\Big(-\frac{m}{\rho}\cdot\frac{C\tanh(m\xi)+1}{C+\tanh(m\xi)}\Big),\\
A(\xi)&=-\frac{\rho}{4s}+\frac{m}{4s}\Big(-\frac{m}{\rho}\cdot\frac{C\tanh(m\xi)+1}{C+\tanh(m\xi)}\Big),\\
A(\xi)&=-\frac{\rho}{4s}-\frac{m}{4s}\Big(-\frac{m}{\rho}\cdot\frac{C\tanh(m\xi)+1}{C+\tanh(m\xi)}\Big).
\end{aligned} \tag{6.6}$$

A.II. Second Method.. We consider the $(G'/G)-$ expansion of the form

$$A(\xi)=a_0+a_a\Big(\frac{G'}{G}\Big) \tag{6.7}$$

with a_0 and a_1 constants and G a function depending on $\xi : G = G(\xi)$. Upon substituting (6.7) into (6.1) and equating to zero the coefficients of the different powers of G, and solving the resulting system, we get $a_1^2 = \frac{1}{16s^2}$, $a_0^2 = \frac{m^2}{4s^2}$ and

$$G = C_1 + C_2\xi + C_3\exp\left(-4m^2\frac{a_1}{a_0}\xi\right). \tag{6.8}$$

Therefore

$$\frac{G'}{G} = \frac{D_1 - 4m^2 a_1 \exp\left(-4m^2\frac{a_1}{a_0}\xi\right)}{D_2 + D_1\xi + a_0\exp\left(-4m^2\frac{a_1}{a_0}\xi\right)}$$

with D_1 and D_2 constants. We thus have

$$A(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) = a_0 + a_1\left[\frac{D_1 - 4m^2 a_1 \exp\left(-4m^2\frac{a_1}{a_0}\xi\right)}{D_2 + D_1\xi + a_0\exp\left(-4m^2\frac{a_1}{a_0}\xi\right)}\right].$$

The above expression gives us the following four expressions (corresponding to the four combinations of signs for the coefficients a_0 and a_1):

$$\begin{aligned}
A(\xi) &= \frac{m}{2s} + \frac{1}{4s}\left[\frac{D_1 - \frac{m^2}{s}\exp(-2m\xi)}{D_2 + D_1\xi + \frac{m}{2s}\exp(-2m\xi)}\right],\\
A(\xi) &= \frac{m}{2s} - \frac{1}{4s}\left[\frac{D_1 + \frac{m^2}{s}\exp(2m\xi)}{D_2 + D_1\xi + \frac{m}{2s}\exp(2m\xi)}\right],\\
A(\xi) &= -\frac{m}{2s} + \frac{1}{4s}\left[\frac{D_1 - \frac{m^2}{s}\exp(2m\xi)}{D_2 + D_1\xi - \frac{m}{2s}\exp(2m\xi)}\right],\\
A(\xi) &= -\frac{m}{2s} - \frac{1}{4s}\left[\frac{D_1 + \frac{m^2}{s}\exp(-2m\xi)}{D_2 + D_1\xi - \frac{m}{2s}\exp(-2m\xi)}\right].
\end{aligned} \tag{6.9}$$

A.III. Third Method. Using the transformation

$$A(\xi) = \frac{1}{w^2(\xi)} \tag{6.10}$$

equation (6.1) transforms into the Ermakov equation

$$w''(\xi) - m^2 w(\xi) = -4s^2 w^{-3}(\xi). \tag{6.11}$$

The solution of Ermakov's equation is given by

$$w(\xi)^2 = \frac{(2mC_2 \pm C_1 e^{2m\xi})^2 - 16m^2 s^2}{4m^2 C_1} \times e^{-2m\xi}.$$

We thus obtain, using the above expression and (6.10), that $A(\xi)$ is given by

$$A(\xi) = \frac{4m^2 C_1 \cdot e^{2m\xi}}{(2mC_2 \pm C_1 e^{2m\xi})^2 - 16m^2 s^2}. \tag{6.12}$$

The above expression can also be written in terms of $\tanh(m\xi)$ as

$$A(\xi)=\frac{4m^2C_1[1-\tanh^2(m\xi)]}{[(2mC_2\pm C_1)-(2mC_2\mp C_1)\tanh(m\xi)]^2-16m^2s^2[1-\tanh(m\xi)]^2}.$$

A.IV. Fourth Method. In this case we use the projective Riccati equation method. We consider the expansion

$$A(\xi)=a_0+a_1f(\xi)+b_1g(\xi), \tag{6.13}$$

where the functions $f(\xi)$ and $g(\xi)$ satisfy the system

$$f'(\xi)=pf(\xi)g(\xi), \tag{6.14}$$

$$g'(\xi)=q+pg^2(\xi)-rf(\xi), \tag{6.15}$$

$$g^2(\xi)=-\frac{1}{p}\left[q-2rf(\xi)+\frac{r^2+\delta}{q}f^2(\xi)\right]. \tag{6.16}$$

The system of equations (6.14) and (6.15) admit five families of solutions depending on the values of the parameters δ, p, q and r.

(I) If $\delta=\lambda^2-\mu^2$ and $pq<0$ then

$$f(\xi)=\frac{q}{r+\lambda\cdot\sinh\sqrt{-pq\xi}+\mu\cdot\cosh\sqrt{-pq\xi}}, \tag{6.17}$$

$$g(\xi)=\frac{\sqrt{-pq}}{p}\cdot\frac{\lambda\cdot\cosh\sqrt{-pq\xi}+\mu\cdot\sinh\sqrt{-pq\xi}}{r+\lambda\cdot\sinh\sqrt{-pq\xi}+\mu\cdot\cosh\sqrt{-pq\xi}} \tag{6.18}$$

with

$$g^2(\xi)=-\frac{1}{p}\left[q-2rf(\xi)+\frac{r^2+\lambda^2-\mu^2}{q}f^2(\xi)\right]. \tag{6.19}$$

(II) If If $\delta=-\lambda^2-\mu^2$ and $pq>0$ then

$$f(\xi)=\frac{q}{r+\lambda\cdot\sin\sqrt{pq\xi}+\mu\cdot\cos\sqrt{pq\xi}}, \tag{6.20}$$

$$g(\xi)=\frac{\sqrt{pq}}{p}\cdot\frac{\lambda\cdot\cos\sqrt{pq\xi}+\mu\cdot\sin\sqrt{pq\xi}}{r+\lambda\cdot\sin\sqrt{pq\xi}+\mu\cdot\cos\sqrt{pq\xi}} \tag{6.21}$$

with

$$g^2(\xi)=-\frac{1}{p}\left[q-2rf(\xi)+\frac{r^2-\lambda^2-\mu^2}{q}f^2(\xi)\right]. \tag{6.22}$$

(III) If $q=0$, then

$$f(\xi)=\frac{1}{\frac{pr}{2}\xi^2+\sigma\xi+\zeta}, \tag{6.23}$$

$$g(\xi)=-\frac{1}{p}\cdot\frac{pr\xi+\sigma}{\frac{pr}{2}\xi^2+\sigma\xi+\zeta} \tag{6.24}$$

with

$$g^2(\xi) = \frac{2r}{p} f(\xi) + \left[\frac{\sigma^2}{p^2} - \frac{2r\zeta}{p}\right] f^2(\xi), \tag{6.25}$$

where σ and ζ are free parameters.

(IV) If $p = \pm 1$ and $\delta = -r^2$ then

$$f(\xi) = \frac{q}{6r} + \frac{2}{pr}\psi(\xi), \tag{6.26}$$

$$g(\xi) = \frac{12\psi'(\xi)}{q + 12\psi'(\xi)}, \tag{6.27}$$

where $\psi(\xi)$ satisfies the Weierstrass equation

$$(\psi'(\xi))^2 = 4\psi^3(\xi) - \frac{q^2}{12}\psi(\xi) - \frac{pq^3}{216} \tag{6.28}$$

with solution $\psi(\xi) = \wp(\xi)$.

The relation between f and g is given by

$$g^2(\xi) = \frac{2r}{p} f(\xi) - \frac{p}{q}. \tag{6.29}$$

(V) If $p = \pm 1$ and $\delta = -\frac{r^2}{25}$ then

$$f(\xi) = \frac{5q}{6r} + \frac{5pq^2}{72\psi(\xi)}, \tag{6.30}$$

$$g(\xi) = -\frac{q\psi'(\xi)}{[pq + 12\psi(\xi)]\psi(\xi)} \tag{6.31}$$

with $\psi(\xi) = \wp(\xi)$ and

$$g^2(\xi) = -\frac{1}{p}\left(q - 2rf(\xi) + \frac{24r^2}{25q} f^2(\xi)\right). \tag{6.32}$$

Upon substituting (6.13) into (6.1) and equating to zero the coefficients of f^i $(i = 0,1,2,3,4)$, g and $f^i g$ $(i = 1,2,3)$, we find eight solutions, where to every solution correspond two or three sub-families of solutions.

Family I. This family corresponds to the Solution I

$$a_0 = \pm\frac{m}{4s^2}, \quad a_1 = \frac{\rho b_1}{2m}, \quad b_1 = b_1, \quad p = 8s^2 b_1, \quad q = -\frac{m^2}{2s^2 b_1},$$

where ρ is any root of the equation $\rho^2 = \delta + r^2$. From (6.13) and Solution I, we obtain

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{\rho b_1}{2m} f(\xi) + b_1 g(\xi) \tag{6.33}$$

with $p = 8s^2 b_1$, $q = -\frac{m^2}{2s^2 b_1}$ with ρ satisfying the equation $\rho^2 = \delta + r^2$.

Sub-family Ia. Since $pq = -4m^2 < 0$, we consider the choice $\delta = \lambda^2 - \mu^2$. The functions $f(\xi)$ and $g(\xi)$ in (6.33) are given by (6.17) and (6.18) respectively:

$$f(\xi) = -\frac{m^2}{2s^2 b_1} \cdot \frac{1}{U(\xi)} \quad \text{and} \quad g(\xi) = -\frac{m^2}{2s^2 b_1} \cdot \frac{2\sqrt{\xi}U'(\xi)}{U(\xi)},$$

where

$$U(\xi) = r + \lambda \cdot \sinh(2|m|\sqrt{\xi}) + \mu \cdot \cosh(2|m|\sqrt{\xi}). \tag{6.34}$$

We thus obtain

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{1}{4s^2} \cdot \frac{\rho m - \sqrt{\xi} \cdot U'(\xi)}{U(\xi)}, \tag{6.35}$$

where ρ is any root of the equation $\rho^2 = \lambda^2 - \mu^2 + r^2$.

Sub-family Ib. If $p = \pm 1$ and $\delta = -r^2$, we have $\rho = 0$ and then $a_1 = 0$. In this case we have $A(\xi) = \pm\frac{m}{4s^2} + b_1 g(\xi)$, where $g(\xi)$ is given by (6.27).

(Ib.a) For $p = 1$, we have $b_1 = \frac{1}{8s^2}$ and $q = -4m^2$. Therefore

$$g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) - 4m^2}$$

and thus

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{3}{2s^2} \cdot \frac{\wp'(\xi)}{12\wp'(\xi) - 4m^2}, \tag{6.36}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27} \tag{6.37}$$

(Ib.b) For $p = -1$, we have $b_1 = -\frac{1}{8s^2}$ and $q = 4m^2$. Therefore

$$g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) + 4m^2}$$

and thus

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{3}{2s^2} \cdot \frac{\wp'(\xi)}{12\wp'(\xi) + 4m^2}. \tag{6.38}$$

Sub-family Ic. If $p = \pm 1$ and $\delta = -\frac{r^2}{25}$, ρ satisfies the equation $\rho^2 = \frac{24r^2}{25}$ while $f(\xi)$ and $g(\xi)$ are given by (6.30) and (6.31) respectively.

(Ic.a) For $p = 1$, we have $a_1 = \frac{\rho}{2m} \cdot \frac{1}{8s^2}$, $b_1 = \frac{1}{8s^2}$ and $q = -4m^2$. Therefore

$$f(\xi) = -\frac{10m^2}{3r} + \frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi) = \frac{4m^2\wp'(\xi)}{[12\wp(\xi) - 4m^2]\wp(\xi)}.$$

We thus have the following expression for $A(\xi)$:

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{5\rho m}{24s^2}\left(\frac{m^2}{3\wp(\xi)} - \frac{1}{r}\right) + \frac{m^2}{2s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[12\wp(\xi)-4m^2]}, \tag{6.39}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27}. \tag{6.40}$$

(Ic.b) For $p=-1$, we have $a_1 = -\frac{\rho}{2m}\cdot\frac{1}{8s^2}$, $b_1 = -\frac{1}{8s^2}$ and $q = 4m^2$. Therefore

$$f(\xi) = \frac{10m^2}{3r} - \frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi) = -\frac{4m^2\wp'(\xi)}{[12\wp(\xi)-4m^2]\wp(\xi)}.$$

We thus have the following expression for $A(\xi)$:

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{5\rho m}{24s^2}\left(\frac{m^2}{3\wp(\xi)} - \frac{1}{r}\right) + \frac{m^2}{2s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[12\wp(\xi)-4m^2]}. \tag{6.41}$$

Note. If $\delta = -\lambda^2-\mu^2$ and $pq>0$ leads to $m^2<0$. The case $q=0$ cannot be considered either, since it leads to $m=0$ and then the coefficient of $f(\xi)$ in (6.33) becomes infinite.

Family II. This family corresponds to the Solution II,

$$a_0 = \pm\frac{m}{4s^2}, \quad a_1 = \frac{\rho b_1}{2m}, \quad b_1 = b_1, \quad p = -8s^2 b_1, \quad q = -\frac{m^2}{2s^2 b_1},$$

where ρ is any root of the equation $\rho^2 = \delta + r^2$. From (6.13) and Solution II, we obtain

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{\rho b_1}{2m} f(\xi) + b_1 g(\xi) \tag{6.42}$$

with $p = -8s^2 b_1$, $\quad q = -\frac{m^2}{2s^2 b_1}$ and ρ satisfies the equation $\rho^2 = \delta + r^2$.

Sub-family IIa. Since $pq = 4m^2 > 0$, we first consider the case $\delta = -\lambda^2-\mu^2$. The functions $f(\xi)$ and $g(\xi)$ are given by (6.20) and (6.21) respectively:

$$f(\xi) = -\frac{m^2}{2s^2 b_1}\cdot\frac{1}{V(\xi)} \quad \text{and} \quad g(\xi) = -\frac{2|m|}{4s^2 b_1}\cdot\frac{Z(\xi)}{V(\xi)},$$

where

$$Z(\xi) = \lambda\cdot\cos(2|m|\sqrt{\xi}) + \mu\cdot\sin(2|m|\sqrt{\xi}), \tag{6.43}$$

$$V(\xi) = r + \lambda\cdot\sin(2|m|\sqrt{\xi}) + \mu\cdot\cos(2|m|\sqrt{\xi}). \tag{6.44}$$

We thus obtain

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{1}{4s^2}\cdot\frac{\rho m + |m|Z(\xi)}{V(\xi)}, \tag{6.45}$$

where ρ is any root of the equation $\rho^2 = r^2 - \lambda^2 - \mu^2$.

Sub-family IIb. If $p = \pm 1$ and $\delta = -r^2$ then $\rho = 0$ and thus $a_1 = 0$. In this case we have $A(\xi) = \pm\frac{m}{4s^2} + b_1 g(\xi)$, where $g(\xi)$ is given by (6.27).

(IIb.a) For $p = 1$, we have $b_1 = -\frac{1}{8s^2}$ and $q = 4m^2$. Therefore

$$g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) + 4m^2}$$

and thus

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{3}{2s^2} \cdot \frac{\wp'(\xi)}{12\wp'(\xi) + 4m^2}, \tag{6.46}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) - \frac{8m^6}{27}. \tag{6.47}$$

(IIb.b) For $p = -1$, we have $b_1 = \frac{1}{8s^2}$ and $q = -4m^2$. Therefore

$$g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) - 4m^2}$$

and thus

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{3}{2s^2} \cdot \frac{\wp'(\xi)}{12\wp'(\xi) - 4m^2}. \tag{6.48}$$

Sub-family IIc. If $p = \pm 1$ and $\delta = -\frac{r^2}{25}$ then ρ satisfies the equation $\rho^2 = \frac{24r^2}{25}$. The functions $f(\xi)$ and $g(\xi)$ are given by (6.30) and (6.31) respectively.

(IIc.a) For $p = 1$, we have we have $b_1 = -\frac{1}{8s^2}$, $a_1 = -\frac{\rho}{2m} \cdot \frac{1}{8s^2}$ and $q = 4m^2$. We then have the following expressions for $f(\xi)$ and $g(\xi)$,

$$f(\xi) = \frac{10m^2}{3r} + \frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi) = -\frac{4m^2\wp'(\xi)}{[4m^2 + 12\wp(\xi)]\wp(\xi)}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{5\rho m}{24s^2}\left[\frac{1}{r} + \frac{m^2}{3\wp(\xi)}\right] + \frac{m^2}{2s^2} \cdot \frac{\wp'(\xi)}{\wp(\xi)[4m^2 + 12\wp(\xi)]}, \tag{6.49}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) - \frac{8m^6}{27}. \tag{6.50}$$

(IIc.b) For $p = -1$, we have we have $b_1 = \frac{1}{8s^2}$, $a_1 = \frac{\rho}{2m} \cdot \frac{1}{8s^2}$ and $q = -4m^2$. We then have the following expressions for $f(\xi)$ and $g(\xi)$

$$f(\xi) = -\frac{10m^2}{3r} - \frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi) = \frac{4m^2\wp'(\xi)}{[4m^2 + 12\wp(\xi)]\wp(\xi)}.$$

Therefore

$$A(\xi)=\pm\frac{m}{4s^2}-\frac{5\rho m}{24s^2}\left[\frac{1}{r}+\frac{m^2}{3\wp(\xi)}\right]+\frac{m^2}{2s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[4m^2+12\wp(\xi)]}. \tag{6.51}$$

Note. If $\delta=\lambda^2-\mu^2$ and $pq<0$ leads to $m^2<0$. The case $q=0$ cannot be considered either, since it leads to $m=0$ and then the coefficient of $f(\xi)$ in (6.42) becomes infinite.

Family III. This family corresponds to the Solution III

$$a_0=\pm\frac{m}{4s^2},\quad a_1=a_1,\quad b_1=b_1,\quad \delta=\frac{4m^2a_1^2-r^2b_1^2}{b_1^2},\quad p=8s^2b_1,\quad q=-\frac{m^2}{2s^2b_1}.$$

From (6.13) and Solution III, we obtain

$$A(\xi)=\pm\frac{m}{4s^2}+a_1f(\xi)+b_1g(\xi), \tag{6.52}$$

where $\delta=\frac{4m^2a_1^2-r^2b_1^2}{b_1^2}$, $p=8s^2b_1$, $q=-\frac{m^2}{2s^2b_1}$.

Sub-family IIIa. Since $pq=-4m^2<0$ for $\delta=\lambda^2-\mu^2$, i.e.

$$\lambda^2-\mu^2=\frac{4m^2a_1^2-r^2b_1^2}{b_1^2},$$

the functions $f(\xi)$ and $g(\xi)$ are given by (6.17) and (6.18) respectively:

$$f(\xi)=-\frac{m^2}{2b_1s^2}\cdot\frac{1}{U(\xi)},\quad g(\xi)=\frac{1}{4s^2}\cdot\frac{\sqrt{\xi}U'(\xi)}{U(\xi)},$$

where

$$U(\xi)=r+\lambda\cdot\sinh(2|m|\sqrt{\xi})+\mu\cdot\cosh(2|m|\sqrt{\xi}). \tag{6.53}$$

Therefore

$$A(\xi)=\pm\frac{m}{4s^2}-\frac{2m^2a_1}{4b_1s^2}\cdot\frac{1}{U(\xi)}+\frac{1}{4s^2}\cdot\frac{\sqrt{\xi}U'(\xi)}{U(\xi)}, \tag{6.54}$$

where the following relation holds $4m^2a_1^2=(\lambda^2-\mu^2+r^2)b_1^2$.

Sub-family IIIb. If $p=\pm1$ and $\delta=-r^2$, i.e. $-r^2=\frac{4m^2a_1^2-r^2b_1^2}{b_1^2}$, then $a_1=0$ and $g(\xi)$ is given by (6.27).

(IIIb.a) If $p=1$, then $b_1=\frac{1}{8s^2}$, $q=-4m^2$ and $g(\xi)=\frac{12\wp'(\xi)}{12\wp'(\xi)-4m^2}$. Therefore

$$A(\xi)=\pm\frac{m}{4s^2}+\frac{3}{8s^2}\cdot\frac{\wp'(\xi)}{12\wp'(\xi)-4m^2}, \tag{6.55}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27} \tag{6.56}$$

(IIIb.b) If $p=-1$, then $b_1 = -\frac{1}{8s^2}$, $q=4m^2$ and $g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi)+4m^2}$. Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{3}{8s^2}\cdot\frac{\wp'(\xi)}{3\wp'(\xi)+m^2} \tag{6.57}$$

Sub-family IIIc. If $p=\pm1$ and $\delta = -\frac{r^2}{25}$, i.e. $-\frac{r^2}{25} = \frac{4m^2a_1^2-r^2b_1^2}{b_1^2}$ then $f(\xi)$ and $g(\xi)$ are given by (6.30) and (6.31) respectively.

(IIIc.a) For $p=1$, we have $b_1 = \frac{1}{8s^2}$ and $q=-4m^2$ and thus

$$f(\xi) = -\frac{10m^2}{3r} + \frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi) = \frac{4m^2\wp'(\xi)}{[12\wp(\xi)-4m^2]\wp(\xi)}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{10a_1m^2}{3}\left(-\frac{1}{r} + \frac{m^2}{3\wp(\xi)}\right) + \frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi)-m^2]}, \tag{6.58}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27} \tag{6.59}$$

with $100m^2a_1^2 = 24r^2b_1^2$.

(IIIc.b) For $p=-1$, we have $b_1 = -\frac{1}{8s^2}$ and $q=4m^2$ and thus

$$f(\xi) = \frac{10m^2}{3r} - \frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi) = -\frac{4m^2\wp'(\xi)}{[12\wp(\xi)+4m^2]\wp(\xi)}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{10a_1m^2}{3}\left(\frac{1}{r} - \frac{m^2}{3\wp(\xi)}\right) + \frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi)+m^2]}. \tag{6.60}$$

Note. If $\delta = -\lambda^2-\mu^2$ and $pq>0$ leads to $m^2<0$. The choice $q=0$ leads to $m=0$.

Family IV. This family corresponds to the Solution IV,

$$a_0 = \pm\frac{m}{4s^2}, \quad a_1 = a_1, \quad b_1 = b_1, \quad \delta = \frac{4m^2a_1^2-r^2b_1^2}{b_1^2}, \quad p=-8s^2b_1, \quad q = \frac{m^2}{2s^2b_1}.$$

From (6.13) and Solution IV, we obtain

$$A(\xi) = \pm\frac{m}{4s^2} + a_1 f(\xi) + b_1 g(\xi), \tag{6.61}$$

where $\delta = \frac{4m^2a_1^2 - r^2b_1^2}{b_1^2}$, $p = -8s^2b_1$, $q = \frac{m^2}{2s^2b_1}$.

Sub-family IVa. Since $pq = -4m^2 < 0$ for $\delta = \lambda^2 - \mu^2$, i.e.

$$\lambda^2 - \mu^2 = \frac{4m^2a_1^2 - r^2b_1^2}{b_1^2},$$

then $f(\xi)$ and $g(\xi)$ are given by (6.17) and (6.18) respectively:

$$f(\xi) = \frac{m^2}{2s^2b_1}\cdot\frac{1}{U(\xi)}, \quad g(\xi) = -\frac{1}{4s^2}\cdot\frac{\sqrt{\xi}U'(\xi)}{U(\xi)},$$

where

$$U(\xi) = r + \lambda\cdot\sinh(2|m|\sqrt{\xi}) + \mu\cdot\cosh(2|m|\sqrt{\xi}). \tag{6.62}$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{m^2a_1}{2s^2b_1}\cdot\frac{1}{U(\xi)} - \frac{1}{4s^2}\cdot\frac{\sqrt{\xi}U'(\xi)}{U(\xi)} \tag{6.63}$$

with $4m^2a_1^2 = (\lambda^2 - \mu^2 + r^2)b_1^2$.

Sub-family IVb. If $p = \pm 1$ and $\delta = -r^2$, i.e. $-r^2 = \frac{4m^2a_1^2 - r^2b_1^2}{b_1^2}$, we have $a_1 = 0$.

(IVb.a) For $p = 1$, we have $b_1 = -\frac{1}{8s^2}$, $q = -4m^2$ and thus from (6.27) we get $g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) - 4m^2}$ and then

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{3}{8s^2}\cdot\frac{\wp'(\xi)}{3\wp'(\xi) - m^2}, \tag{6.64}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27}. \tag{6.65}$$

(IVb.b) For $p = -1$, we have $b_1 = \frac{1}{8s^2}$, $q = 4m^2$ and thus form (6.27) we get $g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) + 4m^2}$ and then

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{3}{8s^2}\cdot\frac{\wp'(\xi)}{3\wp'(\xi) + m^2}. \tag{6.66}$$

Sub-family IVc. If $p = \pm 1$ and $\delta = -\frac{r^2}{25}$, i.e. $-\frac{r^2}{25} = \frac{4m^2a_1^2 - r^2b_1^2}{b_1^2}$ then $f(\xi)$ and $g(\xi)$ are given by (6.30) and (6.31) respectively.

(IVc.a) For $p=1$, we have $b_1=-\frac{1}{8s^2}$, $q=-4m^2$ and then

$$f(\xi)=-\frac{10m^2}{3r}+\frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi)=\frac{4m^2\wp'(\xi)}{[12\wp(\xi)-4m^2]\wp(\xi)}.$$

Therefore

$$A(\xi)=\pm\frac{m}{4s^2}+\frac{10a_1m^2}{3}\left(-\frac{1}{r}+\frac{m^2}{3\wp(\xi)}\right)-\frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi)-m^2]}, \tag{6.67}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2=4\wp^3(\xi)-\frac{4m^4}{3}\wp(\xi)+\frac{8m^6}{27} \tag{6.68}$$

with $a_1^2=\frac{3r^2}{800m^2s^4}$. This last equation comes from equating the two different expressions of δ and using the value of b_1.

(IVc.b) For $p=-1$, we have $b_1=\frac{1}{8s^2}$, $q=4m^2$ and then

$$f(\xi)=\frac{10m^2}{3r}-\frac{10m^4}{9\wp(\xi)} \quad \text{and} \quad g(\xi)=-\frac{4m^2\wp'(\xi)}{[12\wp(\xi)+4m^2]\wp(\xi)}.$$

Therefore

$$A(\xi)=\pm\frac{m}{4s^2}+\frac{10a_1m^2}{3}\left(\frac{1}{r}-\frac{m^2}{3\wp(\xi)}\right)-\frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi)+m^2]}. \tag{6.69}$$

Note. The case $\delta=-\lambda^2-\mu^2$ and $pq>0$ leads to $m^2<0$. The choice $q=0$ leads to $m=0$.

Family V. This family corresponds to the Solution V

$$a_0=\pm\frac{m}{4s^2}, a_1=0, b_1=b_1, p=4s^2b_1, q=-\frac{m^2}{4s^2b_1}.$$

From (6.13)and Solution V, we obtain

$$A(\xi)=\pm\frac{m}{4s^2}+b_1g(\xi), \tag{6.70}$$

where $p=4s^2b_1$, $q=-\frac{m^2}{4s^2b_1}$.

Sub-family Va. Since $pq=-4m^2<0$, the choice $\delta=\lambda^2-\mu^2$, the function $g(\xi)$ is given by (6.18):

$$g(\xi)=\frac{m}{4s^2b_1}\cdot\frac{2\sqrt{\xi}\cdot X'\xi)}{X(\xi)},$$

where

$$X(\xi)=r+\lambda\cdot\sinh(|m|\sqrt{\xi})+\mu\cdot\cosh(|m|\sqrt{\xi}). \tag{6.71}$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{m}{2s^2} \cdot \frac{\sqrt{\xi}X'(\xi)}{X(\xi)}. \tag{6.72}$$

Sub-family Vb. If $p = \pm 1$ and $\delta = -r^2$, then $g(\xi)$ is given by (6.27).

(Vb.a) For $p = 1$, we have $b_1 = \frac{1}{4s^2}$, $q = -m^2$ and then

$$g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) - m^2}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{3}{s^2} \cdot \frac{\wp'(\xi)}{12\wp'(\xi) - m^2}, \tag{6.73}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{m^4}{12}\wp(\xi) + \frac{m^6}{216}. \tag{6.74}$$

(Vb.b) For $p = -1$, we have $b_1 = -\frac{1}{4s^2}$, $q = m^2$ and then

$$g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) + m^2}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{3}{s^2} \cdot \frac{\wp'(\xi)}{12\wp'(\xi) - m^2}. \tag{6.75}$$

Sub-family Vc. If $p = \pm 1$ and $\delta = -\frac{r^2}{25}$ then $g(\xi)$ is given by (6.31).

(Vc.a) For $p = 1$, we get $b_1 = \frac{1}{4s^2}$, $q = -m^2$ and then

$$g(\xi) = \frac{m^2\wp'(\xi)}{\wp(\xi)[12\wp'(\xi) - m^2]}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{m^2}{4s^2} \cdot \frac{\wp'(\xi)}{\wp(\xi)[12\wp(\xi) - m^2]}. \tag{6.76}$$

(Vc.b) For $p = -1$, we get $b_1 = -\frac{1}{4s^2}$, $q = m^2$ and then

$$g(\xi) = \frac{m^2\wp'(\xi)}{\wp(\xi)[12\wp(\xi) + m^2]}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{m^2}{4s^2} \cdot \frac{\wp'(\xi)}{\wp(\xi)[12\wp(\xi) + m^2]}. \tag{6.77}$$

Note. The case $\delta = -\lambda^2 - \mu^2$ and $pq > 0$ leads to $m^2 < 0$. The case $q = 0$ leads to $m = 0$.

Family VI. This family corresponds to Solution VI

$$a_0 = \pm\frac{m}{4s^2}, a_1 = 0, b_1 = b_1, p = 8s^2 b_1, q = -\frac{m^2}{2s^2 b_1}$$

and r is any root of $r^2+\delta=0$. From (6.13)and Solution VI, we obtain

$$A(\xi) = \pm\frac{m}{4s^2} + b_1 g(\xi), \tag{6.78}$$

where $p = 8s^2 b_1$, $q = -\frac{m^2}{2s^2 b_1}$ and r is any root of $r^2+\delta=0$.

Sub-family VIa. Since $pq = -4m^2 < 0$, the choice $\delta = \lambda^2 - \mu^2$, i.e. $r^2 + \lambda^2 - \mu^2 = 0$ and $g(\xi)$ is given by (6.18): $g(\xi) = \frac{1}{4s^2 b_1} \cdot \frac{\sqrt{\xi} \cdot U'(\xi)}{U(\xi)}$, where

$$U(\xi) = r + \lambda \cdot \sinh(2|m|\sqrt{\xi}) + \mu \cdot \cosh(2|m|\sqrt{\xi}). \tag{6.79}$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{m}{4s^2} \cdot \frac{\sqrt{\xi}U'(\xi)}{U(\xi)}, \tag{6.80}$$

where r satisfies the equation $r^2 + \lambda^2 - \mu^2 = 0$.

Sub-family VIb. If $p = \pm 1$ and $\delta = -r^2$ (i.e. r is any real) then $g(\xi)$ is given by (6.27).

(VIb.a) For $p=1$, we get $b_1 = \frac{1}{8s^2}$, $q = -4m^2$ and thus $g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) - 4m^2}$.

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{3}{8s^2} \cdot \frac{\wp'(\xi)}{3\wp'(\xi) - m^2}, \tag{6.81}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27}. \tag{6.82}$$

(VIb.b) For $p=-1$, we get $b_1 = -\frac{1}{8s^2}$, $q = 4m^2$ and thus $g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi)+4m^2}$.

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{3}{8s^2} \cdot \frac{\wp'(\xi)}{3\wp'(\xi) + m^2}. \tag{6.83}$$

Sub-family VIc. If $p = \pm 1$ and $\delta = -\frac{r^2}{25}$, i.e. $r^2 - \frac{r^2}{25} = 0$ then $g(\xi)$ is given by (6.31).

(VIc.a) For $p=1$, we get $b_1 = \frac{1}{8s^2}$, $q = -4m^2$ and then

$$g(\xi) = \frac{4m^2\wp'(\xi)}{\wp(\xi)[12\wp'(\xi) - 4m^2]}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi) - m^2]}. \tag{6.84}$$

(VIc.b) For $p = -1$, we get $b_1 = -\frac{1}{8s^2}$, $q = 4m^2$ and then

$$g(\xi) = -\frac{4m^2\wp'(\xi)}{\wp(\xi)[12\wp(\xi) + 4m^2]}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi) + m^2]}. \tag{6.85}$$

Note. The case $\delta = -\lambda^2 - \mu^2$ and $pq > 0$ leads to $m^2 < 0$. The case $q = 0$ leads to $m = 0$.

Family VII. This family corresponds to the Solution VII

$$a_0 = \pm\frac{m}{4s^2}, a_1 = 0, b_1 = b_1, p = -8s^2b_1, q = \frac{m^2}{2s^2b_1}$$

and r is any root of $r^2 + \delta = 0$. From (6.13)and Solution VII, we obtain

$$A(\xi) = \pm\frac{m}{4s^2} + b_1 g(\xi), \tag{6.86}$$

where $p = -8s^2b_1$, $q = \frac{m^2}{4s^2b_1}$ and r is any root of $r^2 + \delta = 0$.

Sub-family VIIa. Since $pq = -4m^2 < 0$, the choice $\delta = \lambda^2 - \mu^2$, i.e. $r^2 + \lambda^2 - \mu^2 = 0$, then $g(\xi)$ is given by (6.18):

$$g(\xi) = -\frac{1}{4s^2b_1}\cdot\frac{\sqrt{\xi}\cdot U'(\xi)}{U(\xi)},$$

where

$$U(\xi) = r + \lambda\cdot\sinh(2|m|\sqrt{\xi}) + \mu\cdot\cosh(2|m|\sqrt{\xi}).$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{1}{4s^2}\cdot\frac{\sqrt{\xi}U'(\xi)}{U(\xi)}, \tag{6.87}$$

where r satisfies the equation $r^2 + \lambda^2 - \mu^2 = 0$.

Sub-family VIIb. If $p = \pm 1$ and $\delta = -r^2$ (i.e. r is any real) then $g(\xi)$ is given by (6.27).

(VIIb.a) For $p = 1$, we have $b_1 = -\frac{1}{8s^2}$, $q = -4m^2$ and thus $g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi) - 4m^2}$. Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{3}{8s^2}\cdot\frac{\wp'(\xi)}{3\wp'(\xi) - m^2}, \tag{6.88}$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27}. \tag{6.89}$$

(VIIb.b) For $p=-1$, we have $b_1 = \frac{1}{8s^2}$, $q = 4m^2$ and thus $g(\xi) = \frac{12\wp'(\xi)}{12\wp'(\xi)+4m^2}$. Therefore

$$A(\xi) = \pm\frac{m}{4s^2} + \frac{3}{8s^2}\cdot\frac{\wp'(\xi)}{3\wp'(\xi)+m^2}. \tag{6.90}$$

Sub-family VIIc. If $p=\pm 1$ and $\delta = -\frac{r^2}{25}$, i.e. $r^2 - \frac{r^2}{25} = 0$ then $g(\xi)$ is given by (6.31).

(VIIc.a) For $p=1$, we get $b_1 = -\frac{1}{8s^2}$, $q=-4m^2$ and then

$$g(\xi) = \frac{4m^2\wp'(\xi)}{\wp(\xi)[12\wp(\xi)-4m^2]}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi)-m^2]}. \tag{6.91}$$

(VIIc.b) For $p=-1$, we get $b_1 = \frac{1}{8s^2}$, $q=4m^2$ and then

$$g(\xi) = -\frac{4m^2\wp'(\xi)}{\wp(\xi)[12\wp(\xi)+4m^2]}.$$

Therefore

$$A(\xi) = \pm\frac{m}{4s^2} - \frac{m^2}{8s^2}\cdot\frac{\wp'(\xi)}{\wp(\xi)[3\wp(\xi)+m^2]}. \tag{6.92}$$

Note. The case $\delta = -\lambda^2 - \mu^2$ and $pq > 0$ leads to $m^2 < 0$. The case $q=0$ leads to $m=0$.

Family VIII. This family corresponds to the Solution VIII

$$a_0 = 0, a_1 = \frac{\rho p}{8s^2 m}, b_1 = 0, p = p, q = -\frac{4m^2}{p}$$

and ρ is any root of $\rho^2 = \delta + r^2$. From (6.13) and Solution VIII, we obtain

$$A(\xi) = \frac{\rho p}{8s^2 m} f(\xi), \tag{6.93}$$

where $p=p$, $q = -\frac{4m^2}{p}$ and ρ is any root of the equation $\rho^2 = \delta + r^2$.

Sub-family VIIIa. Since $pq = -4m^2 < 0$, the choice $\delta = \lambda^2 - \mu^2$, i.e. $\rho^2 = \lambda^2 - \mu^2 + r^2$, then $f(\xi)$ is given by (6.17): $f(\xi) = -\frac{4m^2}{p}\cdot\frac{1}{U(\xi)}$, where

$$U(\xi) = r + \lambda\cdot\sinh(2|m|\sqrt{\xi}) + \mu\cdot\cosh(2|m|\sqrt{\xi}). \tag{6.94}$$

Therefore

$$A(\xi) = -\frac{\rho m}{2s^2} \cdot \frac{1}{U(\xi)}, \tag{6.95}$$

where ρ satisfies the equation $\rho^2 = \lambda^2 - \mu^2 + r^2$.

Sub-family VIIIb. If $p = \pm 1$ and $\delta = -\frac{r^2}{25}$, leads to $\rho^2 = \frac{24}{25} r^2$ and then $f(\xi)$ is given by (6.30).

(VIIIb.a) For $p = 1$, we have $q = -4m^2$ and thus

$$f(\xi) = -\frac{10m^2}{3r} + \frac{10m^4}{9\wp(\xi)},$$

where $\wp(\xi)$ is the Weierstrass function, satisfying the equation

$$(\wp'(\xi))^2 = 4\wp^3(\xi) - \frac{4m^4}{3}\wp(\xi) + \frac{8m^6}{27}. \tag{6.96}$$

Therefore

$$A(\xi) = \frac{5\rho m}{12s^2}\Big(-\frac{1}{r} + \frac{m^2}{3\wp(\xi)}\Big) \tag{6.97}$$

and ρ is any root of the equation $\rho^2 = \frac{24r^2}{25}$.

(VIIIb.b) For $p = -1$, we have $q = 4m^2$ and thus

$$f(\xi) = \frac{10m^2}{3r} - \frac{10m^4}{9\wp(\xi)}.$$

Therefore

$$A(\xi) = -\frac{5\rho m}{12s^2}\Big(\frac{1}{r} - \frac{m^2}{3\wp(\xi)}\Big). \tag{6.98}$$

Note. The case $\delta = -\lambda^2 - \mu^2$ and $pq > 0$ leads to $m^2 < 0$. The case $q = 0$ leads to $m = 0$. The case $\delta = -r^2$ cannot be considered since in this case $\rho = 0$ and then $a_1 = 0$ (i.e. $A(\xi) = 0$).

A.V. Fifth Method. We consider now an expression for $A(\xi)$ of the form

$$A(\xi) = \frac{a_1 e^{m\xi} + a_0 + b_1 e^{-m\xi}}{c_1 e^{m\xi} + c_0 + d_1 e^{-m\xi}} \tag{6.99}$$

and substitute back into (6.1). We then derive the following set of solutions:

$$A(\xi) = \frac{e^{m\xi}}{C_1 e^{m\xi} + C_2 e^{-m\xi}}, \tag{6.100}$$

$$A(\xi) = \frac{me^{-m\xi}}{mC_1 e^{m\xi} - 2se^{-m\xi}}, \tag{6.101}$$

$$A(\xi) = \frac{me^{-m\xi}}{mC_1e^{m\xi} + 2se^{-m\xi}}, \tag{6.102}$$

$$A(\xi) = \frac{m^2C_1e^{m\xi} + 2msC_2e^{-m\xi} + 2ms}{2msC_1e^{m\xi} + 4s^2C_2e^{-m\xi} + 4s^2}. \tag{6.103}$$

A.VI. Sixth Method. We consider an expansion of the form

$$A(\xi) = a_0 + a_1\varphi(\xi), \tag{6.104}$$

where $\varphi(\xi)$ satisfies Jacobi's differential equation

$$\frac{d}{d\xi}\varphi(\xi) = \sqrt{n_0 + n_1\varphi + n_2\varphi^2 + n_3\varphi^3 + n_4\varphi^4}. \tag{6.105}$$

Upon substituting (6.104) into (6.1), taking into account (6.105) and equating to zero the coefficients of the different powers of φ to zero, we find eleven solutions. For the first three solutions, we use the following Lemma:

LEMMA. If (6.105) is expressed as

$$\frac{d}{d\xi}\varphi(\xi) = r(\varphi + n)\sqrt{(\varphi - p + q\sqrt{D})(\varphi - p - q\sqrt{D})}, \tag{6.106}$$

then its solution is given by

$$\varphi = -\frac{ne^{-2rK\xi} + 4(q^2D - pn - p^2)e^{-rK\xi} + 4nq^2D}{e^{-2rK\xi} + 4(n+p)e^{-rK\xi} + 4q^2D}, \tag{6.107}$$

where

$$K = \sqrt{(n+p)^2 - q^2D}. \tag{6.108}$$

For the Solution 1, we have

$$n_0 + n_1\varphi + n_2\varphi^2 + n_3\varphi^3 + n_4\varphi^4 = 16s^2a_1^2\left(\varphi + \frac{a_0}{a_1}\right)^2 \times$$
$$\left(\varphi - \frac{32a_0a_1s^2 - n_3}{32s^2a_1^2} + \frac{\sqrt{D}}{32s^2a_1^2}\right)\left(\varphi - \frac{32a_0a_1s^2 - n_3}{32s^2a_1^2} - \frac{\sqrt{D}}{32s^2a_1^2}\right),$$

where

$$D = (64a_0a_1s^2 - n_3)^2 - (16msa_1)^2.$$

Equation (6.105) then gives by integration the relation (6.107), where

$$p = \frac{32a_0a_1s^2 - n_3}{32s^2a_1^2} \quad \text{and} \quad q = \frac{1}{32s^2a_1^2}. \tag{6.109}$$

For the Solution 2, we have

$$n_0 + n_1\varphi + n_2\varphi^2 + n_3\varphi^3 + n_4\varphi^4$$

$$
\begin{aligned}
&= \frac{144s^2a_0^2n_3^2}{(96s^2a_0^2-4m^2+n_2)^2} \\
&\quad \times \left(\varphi + \frac{96s^2a_0^2-4m^2+n_2}{3n_3}\right)^2 \\
&\quad \times \left(\varphi - \frac{(96s^2a_0^2-4m^2+n_2)(4m^2-n_2)}{288s^2a_0^2n_3} + \frac{(96s^2a_0^2-4m^2+n_2)\sqrt{D}}{288s^2a_0^2n_3}\right) \\
&\quad \times \left(\varphi - \frac{(96s^2a_0^2-4m^2+n_2)(4m^2-n_2)}{288s^2a_0^2n_3} - \frac{(96s^2a_0^2-4m^2+n_2)\sqrt{D}}{288s^2a_0^2n_3}\right),
\end{aligned}
$$

where

$$D = (96s^2a_0^2+4m^2-n_2)^2 - (48msa_0)^2$$

(6.105) then gives by integration the relation (6.107), where

$$p = \frac{(96s^2a_0^2-4m^2+n_2)(4m^2-n_2)}{288s^2a_0^2n_3}, \quad q = \frac{96s^2a_0^2-4m^2+n_2}{288s^2a_0^2n_3}. \tag{6.110}$$

For the Solution 3, we have

$$
\begin{aligned}
&n_0 + n_1\varphi + n_2\varphi^2 + n_3\varphi^3 + n_4\varphi^4 \\
&\quad = \frac{16s^2a_0^2(32s^2a_0^2-4m^2-n_2)^2}{n_1^2} \\
&\qquad \times \left(\varphi + \frac{n_1}{4m^2+n_2-32s^2a_0^2}\right)^2 \\
&\qquad \times \left(\varphi - \frac{n_1(n_2-4m^2)}{96s^2a_0^2(32s^2a_0^2-4m^2-n_2)} + \frac{n_1\sqrt{D}}{96s^2a_0^2(32s^2a_0^2-4m^2-n_2)}\right) \\
&\qquad \times \left(\varphi - \frac{n_1(n_2-4m^2)}{96s^2a_0^2(32s^2a_0^2-4m^2-n_2)} - \frac{n_1\sqrt{D}}{96s^2a_0^2(32s^2a_0^2-4m^2-n_2)}\right),
\end{aligned}
$$

where

$$D = (96s^2a_0^2+4m^2-n_2)^2 - (48msa_0)^2.$$

Then (6.105) gives by integration the relation (6.107), where

$$p = \frac{n_1(n_2-4m^2)}{96s^2a_0^2(32s^2a_0^2-4m^2-n_2)}, \; q = \frac{n_1}{96s^2a_0^2(32s^2a_0^2-4m^2-n_2)}. \tag{6.111}$$

The other eight solutions are not considered here, since they lead to very complicated expressions. They will be published elsewhere in electronic form.

A.VII. Seventh Method. We substitute

$$A(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \tag{6.112}$$

where a_0 and a_1 are $\xi-$ dependent quantities, $a_0 = a_0(\xi)$ and $a_1 = a_1(\xi)$, while $G = G(\xi)$. Upon substituting (6.112) into (6.101) and equating to zero the coefficients of the different powers of G, we obtain a system of ordinary differential equations, from which we find $a_1 = \pm\frac{1}{s}$ and that $a_0(\xi)$ satisfies the equation

$$a_0 a_0'' - \frac{3}{2}(a_0')^2 + 2m^2 a_0^2 - 8s^2 a_0^4 = 0. \tag{6.113}$$

We also find that the ratio G'''/G'' is given by

$$\frac{G'''}{G''} = F(m,s), \tag{6.114}$$

where

$$F(m,s) = \frac{3\left(\frac{a_0'}{a_0}\right)^2 - 9\left(\frac{a_0'}{a_1}\right) - 4m^2 + 32s^2a_0^2 \pm \frac{a_0'}{a_0 a_1}\sqrt{D}}{\left(\frac{a_0'}{a_0} - 3\frac{a_0}{a_1}\right) \pm \frac{\sqrt{D}}{3a_1}} \tag{6.115}$$

with

$$D = \frac{9}{16s^2}\left(\frac{a_0'}{a_0}\right)^2 + 75a_0^2 - 36a_1a_0' - \frac{3m^2}{4s^2}. \tag{6.116}$$

Equation (6.113) is essentially equation (6.1) and admits **all** the solutions equation (6.1) admits. When a solution is substituted in (6.115), we can integrate in principle equation (6.114). We then can evaluate the function $G(\xi)$ and then the ratio G'/G. The function $A(\xi)$ is determined from (6.112).

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REFERENCES

[1] P. C. Abbott, E. J. Parkes and B. R. Duffy, *The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations*, Available online at http//physics.uwa.edu.au/pub/Mathematica/Solitons .

[2] M. A. Abdou, *Adomian decomposition method for solving the telegraph equation in charged particle transport*, J. Quant. Spectro. Rad. Trans., 95 (2005), 407–414.

[3] M. A. Abdou, *Exact solutions for nonlinear evolution equations via the extended projective Riccati equation expansion method*, Electr. J. Theor. Phys., 4 (2007), 17–30.

[4] M. A. Abdou, *The extended F-expansion method and its applications for a class of nonlinear evolution equations*, Chaos, Solitons and Fractals, 31 (2007), 95–104.

[5] M. A. Abdou, *On the variational iteration method*, Phys. Lett. A, 366 (2007), 61–68.

[6] M. A. Abdou, *A generalized auxiliary equation method and its applications*, J. Nonl. Dyn., 52 (2008), 95–102.

[7] M. A. Abdou, *An improved generalized F-expansion method and its applications*, J. Comp. Appl. Math., 214 (2008), 202–208.

[8] M. A. Abdou, *Generalized solitary and periodic solutions for nonlinear partial differential equations by the exp-function method*, J. Nonl. Dyn., 52 (2008), 1–9.

[9] M. A. Abdou and A. A. Soliman, *New applications of Variational Iteration Method*, Physica D, 211 (2005), 1–8.

[10] M. A. Abdou and A. Elhanbaly, *Construction of periodic and solitary wave solutions by the extended Jacobi elliptic function expansion method*, Comm. Nonl. Sci. Num. Sim., 12 (2007), 1229–1241.

[11] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *The Inverse Scattering Transform. Fourier Analysis for Nonlinear Problems*, Stud. Appl. Math., 53 (1974), 249–315.

[12] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform*, Cambridge University Press, Cambridge, 1991.

[13] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, 1981.

[14] G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press, San Diego, (1986).

[15] S. Antoniou, *The Riccati equation method with variable expansion coefficients. I. Solving the Burgers equation*, submitted for publication.

[16] S. Antoniou, *The Riccati equation method with variable expansion coefficients. II. Solving the KdV equation*, submitted for publication.

[17] I. Aslan, *Application of the exp-function method to nonlinear lattice differential equations for multi-wave and rational solutions*, Math. Meth. Appl. Sci., 60 (2011), 1707–1710.

[18] D. Baldwin, Ü. Göktas, W. Hereman, L. Hong, R.S. Martino and J.C. Miller, *Symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs*, J. Symb. Comp., 37 (2004), 669–705.

[19] A. Bekir and A. Boz, *Exact Solutions for Nonlinear Evolution Equations using Exp-Function Method*, Phys. Lett. A, 372 (2008), 1619–1625.

[20] E. D. Belokolos, A. Bobenko, V. Z. Enolskii, A. R. Its and V. Matveev, *Algebro-Geometric Approach to Nonlinear Integral Equations*, Springer-Verlag, 1994.

[21] G. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, 1989.

[22] A. Borhanifar and A. Z. Moghanlu, *Application of the $(G'/G)-$ expansion method for the Zhiber-Sabat equation and other related equations*, Math. Comp. Mod., 54 (2011), 2109–2116.

[23] H. T. Chen and H. Q. Zhang, *Improved Jacobian elliptic function method and its applications*, Chaos, Solitons and Fractals, 15 (2003), 585–591.

[24] Y. Chen and Q. Wang, *Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic function solutions to (1+1)-dimensional dispersive long wave equation*, Chaos, Solitons and Fractals, 24 (2005), 745–757.

[25] A. E. Ebaid, *Generalization of He's Exp-Function Method and New Exact Solutions for Burgers Equation*, Z. Naturforsch, 64a (2009), 604–608.

[26] S. A. El-Wakil, E. M. Abulwafa, A. Elhanbaly and M. A. Abdou, *The extended homogeneous balance method and its applications*, Chaos, Solitons and Fractals, 33 (2007), 1512–1522.

[27] S. A. El-Wakil and M. A. Abdou, *New exact travelling wave solutions using modified extended tanh-function method*, Chaos, Solitons and Fractals, 31 (2007), 840–852.

[28] S. A. El-Wakil, M. A. Abdou and A. Hendi, *New periodic wave solutions via exp-function method*, Phys. Lett., A 372, (2008), 830–840.

[29] S. A. El-Wakil and M. A. Abdou, *New applications of the homotopy analysis method*, Zeitschrift für Naturforschung, A (2008).

[30] E. Fan, *Two new applications of the homogeneous balance method*, Phys. Lett., A 265, (2000), 353–357.

[31] E. Fan, *Extended tanh-function method and its applications to nonlinear equations*, Phys. Lett., A 277, (2000), 212–218.

[32] E. Fan and Y. C. Hon, *Applications of extended tanh-method to "special types" of nonlinear equations*, Appl. Math. Comp., 141 (2003), 351–358.

[33] E. Fan and H. Zhang, *Applications of the Jacobi elliptic function method to special-type nonlinear equations*, Phys. Lett., A 305, (2002), 383–392.

[34] J. Feng, W. Li and Q. Wan, *Using $(G'/G)-$ expansion method to seek traveling wave solution of Kolmogorov-Petrovskii-Piskunov equation*, Appl. Math. Comp., 217 (2011), 5860–5865.

[35] Z. S. Feng, *The first integral method to study the Burgers-Korteweg de Vries equation*, J. Phys. A: Math Gen., A 302 (2002), 343–349.

[36] P. Gray and S. Scott, *Chemical Oscillations and Instabilities*, Clarendon Press, Oxford, 1990.

[37] P. Griffiths and W. E. Sciesser, *Traveling Wave Analysis of Partial Differential Equations*, Academic Press, 2012.

[38] J. H. He, *A variational iteration method-a kind of nonlinear analytical technique: Some examples*, Int. J. Nonl. Mech., 34 (1999), 699–708.

[39] J. H. He and M. A. Abdou, *New periodic solutions for nonlinear evolution equations using exp-function method*, Chaos, Solitons and Fractals, 34 (2007), 1421–1429.

[40] J. H. He and X. H. Wu, *Exp-Function method for nonlinear wave equations*, Chaos, Solitons and Fractals, 30 (2006), 700–708.

[41] W. Hereman and W. Malfliet, *The tanh method: A Tool to Solve Nonlinear Partial Differential Equations with Symbolic Software*, Available online, Colorado School of Mines.

[42] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, 2004.

[43] R. Hirota, *Exact Solution of the KdV Equation for Multiple Collisions of Solitons*, Phys. Rev. Lett., 27 (1971), 1192–1194.

[44] P. E. Hydon, *Symmetry Methods for Differential Equations*, Cambridge University Press, 2000.

[45] M. Inc and M. Ergüt, *Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method*, Appl. Math. E-Notes, 5 (2005), 89–96.

[46] A. Jabbari, H. Kheiri and A. Bekir, *Exact solutions of the coupled Higgs equation and the Maccari system using He's semi-inverse method and expansion method*, Comp. Math. Appl., 62 (2011), 2177–2186.

[47] A. J. M. Jawad, M. D. Petkovich and A. Biswas, *Modified simple equation method for nonlinear evolution equations*, Appl. Math. Comp., 217 (2010), 869–877.

[48] Y. Keskin and G. Oturanc, *Reduced Differential Transform Method for Partial Differential Equations*, Int. J. Nonl. Sci. Num. Sim., 10 (2009), 741–749.

[49] H Kheiri, N. Alipour and R. Dehghani, *Homotopy analysis and Homotopy Padé methods for the modified Burgers-Korteweg-de Vries and the Newell-Whitehead equations*, Mathematical Sciences, 5 (2011), 33–50.

[50] N. A. Kudryashov, *Exact Solutions of the Generalized Kuramoto- Sivashinsky Equation*, Phys. Lett., A 147 (1990), 287–291.

[51] N. A. Kudryashov, *Simplest equation method to look for exact solutions of nonlinear differential equations*, arXiv:nlin/0406007v1, 4 Jun 2004.

[52] N. A. Kudryashov, *Nonlinear differential equations with exact solutions expressed via the Weierstrass function*, arXiv:nlin/0312035v1,16 Dec., 2003.

[53] S. Liao, *Homotopy Analysis Method in Nonlinear Differential Equations*, Springer, 2012.

[54] S. K. Liu, Z. Fu, S. Liu and Q. Zhao, *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Lett., A 289 (2001), 69–74.

[55] S. K. Liu, Z. Fu, S. Liu and Q. Zhao, *Expansion about the Jacobi Elliptic Function and its applications to Nonlinear Wave Equations*, Acta Phys. Sinica, 50 (2001), 2068–2072.

[56] D. Lu and Q. Shi, *New Jacobi elliptic functions solutions for the combined KdV-mKdV Equation*, Int. J. Nonl. Sci., 10 (2010), 320–325.

[57] B. Q. Lu, B. Z. Xiu, Z. L. Pang and X. F. Jiang, *Exact traveling wave solution of one class of nonlinear diffusion equations*, Phys. Lett., A175 (1993), 113–115.

[58] W. Malfliet, *Solitary wave solutions of nonlinear wave equations*, Am. J. Phys., 60 (1992), 650–654.

[59] W. Malfliet, *The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations*, J. Comp. Appl. Math., 164-165 (2004), 529–541.

[60] W. Malfliet and W. Hereman, *The tanh method: I. Exact solutions of nonlinear evolution and wave equations*, Phys. Scripta, 54 (1996), 563–568.

[61] W. Malfliet and W. Hereman, *The tanh method: II. Perturbation technique for conservative systems*, Phys. Scripta, 54 (1996), 569–575.

[62] A. Malik, F. Chand, H. Kumar and S. C. Mishra, *Exact solutions of some physical models using the* (G'/G) *- expansion method*, Pramana, 78 (2011), 513–529.

[63] B. A. Malomed, *The Newell-Whitehead-Segel equation for traveling waves*, arXiv:patt-sol/9605001v1.

[64] J. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, 1989.

[65] H. Naher, F. Abdullah and M. A. Akbar, *The exp-function method for new exact solutions of the nonlinear partial differential equations*, Int. J. Phys. Sci., 6 (2011), 6706–6716.

[66] H. Naher, F. A. Abdullah and M. A. Akbar, *New travelling wave solutions of the higher dimensional nonlinear partial differential equation by the exp-function method*, J. Appl. Math., (2012).

[67] H. Naher, F. A. Abdullah and M. A. Akbar, *The* (G'/G) *- expansion method for abundant traveling wave solutions of Caudrey-Dodd-Gibbon equation*, Math. Prob. Eng., (2011).

[68] A. C. Newell and J. A. Whitehead, *Finite bandwidth, finite amplitude convection*, J. Fluid Mech., 38 (1969), 279–303.

[69] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Plenum, NY 1984.

[70] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, vol.107, Springer Verlag, N.Y. 1993.

[71] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[72] T. Ozis and I. Aslan, *Application of the $(G'/G)-$ expansion method to Kawahara type equations using symbolic computation*, Appl. Math. Comp., 216 (2010), 2360–2365.

[73] E. J. Parkes and B. R. Duffy, *An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations*, Comp. Phys. Comm., 98 (1996), 288–300.

[74] E. J. Parkes, B. R. Duffy and P. C. Abbott, *The Jacobi elliptic function method for finding periodic-wave solutions to nonlinear evolution equations*, Phys. Lett., A 295 (2002), 280–286.

[75] E. J. Parkes, E. J. Zhu, B. R. Duffy and H. C. Huang, *Sech-polynomial traveling solitary-wave solutions of odd-order generalized KdV equations*, Phys. Lett., A 248 (1998), 219–224.

[76] K. R. Raslan, *The first integral method for solving some important nonlinear partial differential equations*, Nonl. Dyn., (2007).

[77] C. Rogers and W. F. Shadwick, *Bäcklund Transformations*, Academic Press, New York, 1982.

[78] R. G. Rojas, R. G. Elias and M. G. Clerc, *Dynamics of an interface connecting a stripe pattern and a uniform state: amended Newell- Whitehead-Segel equation*, Int. J. Bifurcation and Chaos, 19 (2009), 2801–2812.

[79] W. Rui, S. Xie, Y. Long and B. He, *Integral Bifurcation Method and its Applications for solving the modified Equal Width Wave equation and its variants*, Rostock Math. Kolloq., 62 (2007), 87–106.

[80] A. H. Salas, and C. A. Gomez, *Application of the Cole-Hopf transformation for finding exact solutions to several forms of the seventh-order KdV equation*, Math. Prob. Eng., (2010).

[81] L. A. Segel, *Distant side-walls cause slow amplitudes modulation of cellular convection*, J. Fluid Mech., 38 (1969), 203–224.

[82] A. A. Soliman and H. A. Abdou, *New exact solutions of nonlinear variants of the RLW, the phi-four and Boussinesq equations based on modified extended direct algebraic method*, Int. J. Nonl. Sci., 7 (2009), 274–282.

[83] H. Stephani, *Differential Equations: Their Solutions Using Symmetries*, Cambridge University Press, 1989.

[84] N. Taghizadeh, M. Akbari and A. Ghelichzadeh, *Exact solution of Burgers equations by homotopy perturbation method and reduced differential transformation method*, Austr. J. Basic Appl. Sci., 5 (2011), 580–589.

[85] N. K. Vitanov, *Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity*, Comm. Nonl. Sci. Num. Sim., 15 (2010), 2050–2060.

[86] M. L. Wang, Y. B. Zhou and Z. B. Li, *Application of a homogeneous balance method to exact solutions of nonlinear equations in Mathematical Physics*, Phys. Lett., A 216 (1996), 67–75.

[87] M. L. Wang and X. Z. Li, *Applications of F-Expansion to periodic wave solutions for a new Hamiltonian amplitude equation*, Chaos, Solitons and Fractals, 24 (2005), 1257–1268.

[88] M. Wang, X. Li and J. Zhang, *The (G'/G) - expansion method and travelling wave solutions of nonlinear evolution equations in Mathematical Physics*, Phys. Lett., A 372 (2008), 417–421.

[89] A.M. Wazwaz, *A reliable modification of Adomian's decomposition Method*, Appl. Math.Comp., 92 (1998), 1–7.

[90] A. M. Wazwaz, *The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations*, Appl. Math. Comp., 188 (2007), 1467–1475.

[91] A. M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Springer-Verlag, Berlin Heidelberg, 2009.

[92] J. Weiss, M. Tabor and G. Carnevale, *The Painlevé Property for Partial Differential Equations*, J. Math. Phys., 24 (1982), 522–526.

[93] J. Weiss, M. Tabor and G. Carnevale, *The Painlevé Property*, J. Math. Phys., 24 (1983), 1405-

[94] G. Whitham, *Linear and Nonlinear Waves*, Wiley, NY, 1974.

[95] K. Yahya, J. Biafar, H. Azari and P. R. Fard, *Homotopy Perturbation Method for Image Restoration and Denoising*, Available online.

[96] O. Yu. Yefimova, *The modified simplest equation method to look for exact solutions of nonlinear partial differential equations*, arXiv:1011.4606v1 [nlin.SI], 20 Nov., 2010.

[97] E. M. E. Zayed, *Traveling wave solutions for higher dimensional nonlinear evolution equations using the (G'/G)-expansion method*, J. Appl. Math. Inform., 28 (2010), 383–395.

[98] E. M. E. Zayed, *A note on the modified simple equation method applied to Sharma-Tasso-Olver equation*, Appl. Math. Comp., 218 (2011), 3962–3964.

[99] E. M. E. Zayed and K. A. Gepreel, *The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics*, J. Math. Phys., 50 (2009), 013502.

[100] J. Zhang, *Exact and explicit solitary wave solutions to some nonlinear equations*, Int. J. Theor. Phys., 35 (1996), 1793–1798.

[101] X. L. Zhang and H. Q. Zhang, *A new generalized Riccati equation rational expansion method to a class of nonlinear evolution equations with nonlinear terms of any order*, Appl. Math. Comp., 186 (2007), 705–714.

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