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HILLE AND NEHARI TYPE OSCILLATION CRITERIA FOR HIGHER ORDER DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we consider the higher order dynamic equation of the form

$$(a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta})^{\Delta}+p(t)x(t)=0, \quad t\geqslant t_0>0,$$

where n is an arbitrary positive integer with $n\geqslant 3$, t is defined on an arbitrary time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$. By Riccari transformation technique and comparison theorem, some Hille and Nehari type oscillation criteria are established. The main results are illustrated by examples.

1. Introduction

In this paper we consider the higher order dynamic equation of the form

$$(a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta})^{\Delta}+p(t)x(t)=0, \quad t\geqslant t_0>0, \tag{1.1}$$

where t is defined on an arbitrary time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$, n is an arbitrary positive integer with $n\geqslant 3$, $a(\cdot)\in C^2_{rd}(\mathbb{T},\mathbb{R}^+)$, $r(\cdot)\in C^1_{rd}(\mathbb{T},\mathbb{R}^+)$ and $p(\cdot)\in C_{rd}(\mathbb{T},\mathbb{R}^+)$, where notation C^n_{rd} mains the set of nth-order delta differentiable rd-continuous functions. Otherwise, assume that $a(t)$ and $r(t)$ satisfy the condition

$$\int_{t_0}^{\infty}\frac{\Delta t}{a(t)}=\int_{t_0}^{\infty}\frac{\Delta t}{r(t)}=\infty. \tag{1.2}$$

Recent two decades, time-scale calculus theory has received a lot of attention, which was introduced by Hilger [1], in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals. When $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$, it means the classical theories of differential or difference. Furthermore it includes many other interesting time scales, e.g., when $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t:t\in\mathbb{N}_0\}$, where $q>1$, it represents the so-called q-difference theory. Time-scale calculus has a plenty of applications, e.g., the population dynamic models, and for the detailed applications, we refer the reader to see [2].

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In continuous case, the research for the oscillation of nth order ordinary differential equations with the research of third order differential equations, and certain results have been known for a long time. In 1911, Birkhoff [3] pioneered the study of separation and comparison theorems for equations of order exceeding two with his paper on third order equations. Ten years later Reynolds [4] extended some of Birkhoff's results to equations of arbitrary order n, and can be seen the original work for the oscillation theory of higher order equations.

In 1921 Reynolds [4] obtained separation and comparison theorems for the nth order equation

$$u^{(n)} + \sum_{i=2}^{n} a_i(x) u^{(n-1)} = 0, \qquad \alpha \leqslant x \leqslant \beta, \tag{1.3}$$

where a_i $(i = 2, 3, \ldots, n)$ is a real-valued continuous function of class $\mathbb{L}^{(n-i)}[\alpha, \beta]$. And he get some oscillation criteria with comparison theorem on the interval $\alpha \leqslant x \leqslant \beta$ for equation (1.3).

In 1962 Kiguradze [5] obtained the theorem below for the differential equation

$$u^{(n)} + c(x)u = 0, \qquad n \geqslant 2,\ x \in [0, \infty), \tag{1.4}$$

where c is continuous on $[0, \infty)$. He obtained the theorem that:

THEOREM A. *Let Q be an absolutely continuous function in $[0, \infty)$ with the properties $Q(x) > 0$, $Q'(x) \geqslant 0$ (where it exists), and*

$$\int_0^\infty [xQ(x)]^{-1} dx < \infty.$$

Then

(1) If

$$\int_0^\infty x^{n-1} |c(x)| dx < \infty,$$

equation (1.4) *is nonoscillatory;*

(2) If

$$\int_0^\infty x^{n-1} c(x) Q^{-1}(x) dx = \infty, \qquad \text{and } c(x) \geqslant 0,$$

then equation (1.4) *has an oscillatory solution and every nonoscillatory solution tends monotonically to zero as $x \to \infty$;*

(3) If

$$\int_0^\infty x^{n-1} |c(x)| Q^{-1}(x) dx = \infty, \qquad \text{and } c(x) \leqslant 0,$$

there exists a fundamental set of $[3 + (-1)^n]/2$ nonosdilatory solutions and $n - [3 + (-1)^n]/2$ oscillatory solutions, $n \geqslant 3$.

Glazman gave the following conditions, any one of which is sufficient for (1.4) to be oscillatory (when $n = 2m$) [6, 7]:

(1) $(-1)^m \int_0^\infty c(x) dx = -\infty$ (where no assumption is made on the sign of $c(x)$);

(2) $(-1)^m c(x) \leqslant 0$ for large x and

$$\limsup_{x\to\infty} x^{2m-1}\int_x^\infty |x(t)|dt > A_m^2,$$

where

$$A_m^{-1} = \frac{(2m-1)^{1/2}}{(m-1)!}\sum_{k=1}^{m}\frac{(-1)^{k-1}}{2m-k}\binom{m-1}{k-1};$$

(3) $f(x) \equiv (-1)^m c(x) + \alpha_m^2 x^{-2m} \leqslant 0$ and

$$\limsup_{r\to\infty} \log r\int_r^\infty x^{2m-1}|f(x)|dx = \infty.$$

Anan'eva and Balaganskii [8] under the assumptions that $c(x) > 0$ and replaced condition (2) of Theorem A when $Q(x) = x$ with

$$\int_0^\infty x^{n-2}c(x)dx = \infty,$$

then, for even n every nontrivial solution of (1.4) is oscillatory, and for odd n every solution which is not oscillatory has the property that $u^{(k)}(x) \to 0$ as $x \to \infty$ $(k = 0,1,\ldots,n-1)$. Anan'eva and Balaganskiĭ give an example to show that this theorem is false under the weaker assumption

$$\int_0^\infty x^{n-1}c(x)dx = \infty.$$

A classical result of Kneser [9] gives the same conclusions under the stronger assumption $\lim_{x\to\infty} c(x) > 0$.

In 2007, Erbe et al. [10] considered a third-order dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0, \tag{1.5}$$

where p is a positive real-valued rd-continuous function defined on a time scale $\mathbb{T}$. They established Hille and Nehari type oscillation criteria for dynamic equation on time scales like that: under the condition

$$\int_{t_0}^\infty\int_z^\infty\int_u^\infty p(s)\Delta s\Delta u\Delta z = \infty, \tag{1.6}$$

every solution x of (1.5) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ if

$$\liminf_{t\to\infty} t\int_t^\infty \frac{h_2(s,t_0)}{\sigma(s)}p(s)\Delta s > \frac{1}{4}, \tag{1.7}$$

or

$$\liminf_{t\to\infty} \frac{1}{t}\int_t^\infty \sigma(s)h_2(\sigma(s),t_0)p(s)\Delta s > \frac{l^*}{1+l^*} \tag{1.8}$$

where $h_2(t,s)$ is the Taylor monomial of degree 2, $l^* := \limsup_{t\to\infty} \frac{\sigma(t)}{t}$. And they also provided some continuous and discrete examples for their results.

In 2011, Saker [11] investigated a third-order functional dynamic equation

$$(p(t)[(r(t)x^{\Delta}(t))^{\Delta}]^{\gamma})^{\Delta} + q(t)f(x(\tau(t))) = 0 \quad \text{for} \quad t \geqslant t_0, \tag{1.9}$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is the quotient of odd positive integers, p, r, τ and q are positive rd-continuous functions defined on the time scale $\mathbb{T}$. In this paper, some Hille and Nehari type oscillation criteria for (1.9) have been established: when $p^{\Delta}(t) \geqslant 0$, if $x(t)$ is a solution of (1.9) and assume that

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_{\tau}^{t} \frac{s^{\gamma+1}}{p(s)} Q(s)\Delta s > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{1.10}$$

where

$$P(t,T) := \int_T^t \left(\frac{1}{p(\tau)}\right)^{\frac{1}{\gamma}} \Delta\tau > 0, \;\; R(\tau,t) := \int_t^{\tau(t)} \left(\frac{1}{r(s)}\right) \Delta s,$$
$$Q(t) := Kq(t)\left(\frac{p^{\frac{1}{\gamma}} R(\tau,t) P(t,T)}{p^{\frac{1}{\gamma}} P(t,T) + \sigma(t) - t}\right)^{\gamma}$$

and $l := \liminf_{t\to\infty} \frac{t}{\sigma(t)}$, then $x(t)$ is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

In [12], Agarwal and Li et al. studied a third order delay dynamic equation

$$(a(rx^{\Delta})^{\Delta})^{\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{1.11}$$

and presented new Hille and Nehari type asymptotic criteria for (1.11). For one, under the condition

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_z^{\infty} \frac{1}{a(u)} \int_u^{\infty} p(s)\Delta s \Delta u \Delta z = \infty, \tag{1.12}$$

They obtained that if

$$\liminf_{t\to\infty} \int_{t_0}^{t} \frac{\Delta s}{a(s)} \int_t^{\infty} \frac{\int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{v} (\Delta u/a(u))}{r(v)}}{\int_{t_1}^{\sigma(s)} (\Delta u/a(u))} p(s)\Delta s > \frac{1}{4}, \tag{1.13}$$

then every solution x of (1.11) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

To the best of our knowledge, there are a few papers that consider the higher order dynamic equations with Hille and Nehari type oscillation criteria. So in this paper, we will establish Hille and Nehari type oscillation criteria for higher order dynamic equation (1.1). The results extend Erbe et al. [10] and Agarwal et al. [12]'s work which established for the third-order dynamic equations to a kind of higher order dynamic equation. Using Riccari transformation technique and comparison theorem, some new Hille and Nehari type oscillation conditions are obtained. And the new ones have many difference towards Erbe et al. and Agarwal et al.'s oscillation criteria which established for the third-order dynamic equations.

The paper is organized as follows. In the next section, we give some preliminary notations and lemmas, including the Taylor monomial and well-known Kiguradze's lemma about higher order derivatives on time scales. In Section 3, firstly, we give some new Hille and Nehari type oscillation conditions for equation (1.1) under the assumptions:

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{t_0}^{z} \frac{1}{a(u)} \int_{u}^{\infty} p(s)\Delta s \Delta u \Delta z = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{z}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} p(s)\Delta s \Delta u \Delta z = \infty.$$

Secondly, using comparison theorem, we establish some other oscillation criteria for (1.1) considering the case:

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{t_0}^{z} \frac{1}{a(u)} \int_{u}^{\infty} p(s)\Delta s \Delta u \Delta z < \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{z}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} p(s)\Delta s \Delta u \Delta z < \infty.$$

In the last section, we present some examples to illustrate our results.

2. Some preliminary lemmas

For completeness, we recall the following concepts related to the notion of time scales. On any time scale, we defined the forward and backward jump operators by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \inf\{s \in \mathbb{T} : s < t\}$, where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup\mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$.

Next, we introduce the definitions of differential and integral on time-scale calculus. For a function $f : \mathbb{T} \to \mathbb{R}$, and is continuous at t. If t is right-scattered, the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

If t is right-dense, the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator.

We also use the following product and quotient rules for the derivative of the product $f(t)g(t)$ and the quotient $f(t)/g(t)$ of two delta-differentiable functions f and g:

$$(f(t)g(t))^{\Delta} = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$
$$\Big(\frac{f(t)}{g(t)}\Big)^{\Delta} = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))} \quad \text{if } gg^{\sigma} \neq 0.$$

For $b,c \in \mathbb{T}$ and a delta-differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\Delta t = f(c) - f(b).$$

The integration by parts formula reads

$$\int_b^c f^{\Delta}(t)g(t)\Delta t = f(c)g(c) - f(b)g(b) - \int_b^c f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined by

$$\int_b^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_b^t f(s)\Delta s.$$

For more details on the calculus on time scales, see for example [2, 13].

Next, we introduce the generalization of Taylor's formula defined on time scales:

DEFINITION 1. (see [2, Section 1.6]) The *Taylor monomials on time-scale calculus* recursively as follows:

$$h_n(t,s) = \begin{cases} 1, & n = 0, \\ \int_s^t h_{n-1}(\tau,s)\Delta\tau, & n \in \mathbb{N}_0 := \{\mathbb{N} \setminus \{0\}\}, \end{cases} \quad \text{for } s,t \in \mathbb{T}.$$

It is clear that $h_1(t,s) = t - s$ for any time scales, but simple formulas in general do not hold for $n \geqslant 2$.

DEFINITION 2. (see [2, Lemma 2]) An alternative definition of $h_n(t,s)$ is:

$$h_n(t,s) = \begin{cases} 1, & n = 0, \\ \int_s^t h_{n-1}(t,\sigma(\tau))\Delta\tau, & n \in \mathbb{N}_0 := \{\mathbb{N} \setminus \{0\}\}, \end{cases} \quad \text{for } s,t \in \mathbb{T}.$$

LEMMA 1. (Taylor's formula [2, Theorem 1.113]) *If we suppose that $n \in \mathbb{N}$, $s,t \in \mathbb{T}$ and $f \in C_{rd}^n(\mathbb{T},\mathbb{R})$, where $\mathbb{T}$ is an arbitrary time scale, then*

$$f(t) = \sum_{k=0}^{n-1} h_k(t,s) f^{\Delta^k}(s) + \int_s^t h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau$$

where $h_n(t,s)$ is defined as above.

In the sequel, we present the dynamic generalization for well-known Kiguradze's lemma on time scales.

LEMMA 2. (Kiguradze's Lemma [14, Theorem 5]) *Let suppose that* $n \in \mathbb{N}$, $f \in C^n_{rd}(\mathbb{T}, \mathbb{R}^+)$ *and* $\sup \mathbb{T} = \infty$. *Suppose that* f *is either positive or negative on* $[t_0, \infty)_{\mathbb{T}}$ *and* f^{Δ^n} *is not identically zero and is either nonnegative or nonpositive on* $[t_0, \infty)_{\mathbb{T}}$ *for some* $t_0 \in \mathbb{T}$. *Then, there exist* $t_1 \in [t_0, \infty)_{\mathbb{T}}$, $m \in [0, n]_{\mathbb{Z}}$ *with* $m+n$ *even for* $f^{\Delta^n}(t) \geqslant 0$, *or* $m+n$ *odd for* $f^{\Delta^n}(t) \leqslant 0$ *such that:*

(i) $f^{\Delta^j}(t) > 0$ *holds for all* $t \in [t_1, \infty)_{\mathbb{T}}$ *and all* $j \in [0, m)_{\mathbb{Z}}$;

(ii) $(-1)^{m+j} f^{\Delta^j}(t) > 0$ *holds for all* $t \in [t_1, \infty)_{\mathbb{T}}$ *and all* $j \in [m, n)_{\mathbb{Z}}$.

3. Hille and Nehari type oscillation criteria for higher order dynamic equations

First, we assume that

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{t_0}^{z} \frac{1}{a(u)} \int_{u}^{\infty} p(s) \Delta s \Delta u \Delta z = \infty, \tag{3.1}$$

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{z}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} p(s) \Delta s \Delta u \Delta z = \infty \tag{3.2}$$

hold.

LEMMA 3. *If* x *is an eventually positive solution of equation* (1.1). *Then there exists* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$(a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta})^{\Delta} < 0, \quad (r(t)x^{\Delta^{n-2}}(t))^{\Delta} > 0$$

for $t \in [t_1, \infty)_{\mathbb{T}}$.

Proof. Suppose that $x(t)$ is a positive solution of (1.1) on $[T, \infty)_{\mathbb{T}}$. From (1.1) we get that

$$(a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta})^{\Delta} = -p(t)x(t) < 0,$$

Thus $a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$ and has one sign eventually. We claim that $a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta} > 0$ eventually. Assume not, then there is a $t_1 \in [T, \infty)_{\mathbb{T}}$ such that

$$a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Then we can choose a positive constant c and for $t \in [t_2, \infty)_{\mathbb{T}} \subset [t_1, \infty)_{\mathbb{T}}$ such that

$$a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta} \leqslant -c < 0.$$

Dividing by $a(t)$ and integrating from t_2 to t, we obtain

$$r(t)x^{\Delta^{n-2}}(t) \leqslant r(t_2)x^{\Delta^{n-2}}(t_2) - c\int_{t_2}^{t} \frac{\Delta s}{a(s)}.$$

Letting $t \to \infty$, from (1.2) we have $r(t)x^{\Delta^{n-2}}(t) \to -\infty$. Thus, we can find a $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that for $t \in [t_3,\infty)_{\mathbb{T}}$

$$r(t)x^{\Delta^{n-2}}(t) \leqslant r(t_3)x^{\Delta^{n-2}}(t_3) < 0.$$

Dividing by $r(t)$ and integrating from t_3 to t, we obtain

$$x^{\Delta^{n-3}}(t) - x^{\Delta^{n-3}}(t_3) \leqslant r(t_3)x^{\Delta^{n-2}}(t_3)\int_{t_3}^{t}\frac{\Delta s}{r(s)},$$

which implies that $x^{\Delta^{n-3}}(t) \to -\infty$ as $t \to \infty$ by (1.2). It means that $x^{\Delta^{i}}(t) \to -\infty$ as $t \to \infty$ for each $i \leqslant n-3$. So $x(t) \to -\infty$ as $t \to \infty$, and this contradicts $x(t) > 0$. Hence we have

$$(r(t)x^{\Delta^{n-2}}(t))^{\Delta} > 0, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.3}$$

The proof is complete. □

LEMMA 4. *Suppose that* (3.1) *and* (3.2) *hold. If x is an eventually positive solution of* (1.1)*. Then there exist only two cases for $t \in [t_1,\infty)_{\mathbb{T}} \subseteq [t_0,\infty)_{\mathbb{T}}$ where t_1 is sufficiently large:*

Case 1. $x^{\Delta^{i}}(t) > 0$ *for each* $i \in [0,n-2]_{\mathbb{N}}$*;*

Case 2. $(-1)^{i}x^{\Delta^{i}}(t) > 0$*, for each* $i \in [0,n-2]_{\mathbb{N}}$.

Proof. Suppose that $x(t)$ is a positive solution of (1.1). From Lemma 3 we know there exists t_1 such that $(r(t)x^{\Delta^{n-2}}(t))^{\Delta} > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Then $r(t)x^{\Delta^{n-2}}(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$ and thus $x^{\Delta^{n-2}}(t)$ is eventually of one sign. We discuss in two cases.

Case i. There exists sufficiently large t_2 such that $x^{\Delta^{n-2}}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Since x is eventually positive, by Kiguradze's lemma there exist $t_3 \in [t_2,\infty)_{\mathbb{T}}$, $m \in [0,n-2]_{\mathbb{Z}}$ with $m+n$ even for $f^{\Delta^{n}}(t) \geqslant 0$, or $m+n$ odd for $f^{\Delta^{n}}(t) \leqslant 0$ such that

(I) $f^{\Delta^{j}}(t) > 0$ holds for all $t \in [t_3,\infty)_{\mathbb{T}}$ and all $j \in [0,m)_{\mathbb{Z}}$;

(II) $(-1)^{m+j}f^{\Delta^{j}}(t) > 0$ holds for all $t \in [t_3,\infty)_{\mathbb{T}}$ and all $j \in [m,n-2)_{\mathbb{Z}}$.

We claim $m = n-2$ or $m = 0$. Assume not, it means $m \in [1,n-3]_{\mathbb{Z}}$ and each order derivative of x satisfies:

$$x^{\Delta^{n-2}}(t) > 0,\ x^{\Delta^{n-3}}(t) < 0,\ldots,x^{\Delta^{m}}(t) > 0,\ x^{\Delta^{m-1}}(t) > 0,\ldots,x^{\Delta}(t) > 0,\ x(t) > 0$$

for $t \in [t_3,\infty)_{\mathbb{T}}$. Now, integrating both sides of (1.1) from t to u and letting $u \to \infty$ we get

$$\lim_{u\to\infty} a(u)(r(u)x^{\Delta^{n-2}}(u))^{\Delta} - a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta} = -\int_{t}^{\infty} p(s)x(s)\Delta s.$$

Due to $x(t)$ is strictly increasing on $[t_3,\infty)_{\mathbb{T}}$, setting $b = x(t_3)$, also from (3.3) the above equation can become

$$a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta} \geqslant b\int_{t}^{\infty} p(s)\Delta s, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

Dividing by $a(t)$ and integrating from t_3 to t, we obtain

$$r(t)x^{\Delta^{n-2}}(t) - r(t_3)x^{\Delta^{n-2}}(t_3) \geqslant b\int_{t_3}^{t}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u.$$

Since $x^{\Delta^{n-2}}(t) > 0$ on $[t_3,\infty)_{\mathbb{T}}$ we get

$$r(t)x^{\Delta^{n-2}}(t) \geqslant b\int_{t_3}^{t}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u.$$

Dividing by $r(t)$ and integrating from t_3 to t, we obtain

$$x^{\Delta^{n-3}}(t) - x^{\Delta^{n-3}}(t_3) \geqslant b\int_{t_3}^{t}\frac{1}{r(z)}\int_{t_3}^{z}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u\Delta z.$$

From Kiguradze's lemma, $n-2+m$ is even, which implies $m \neq n-3$. So $x^{\Delta^{n-3}}(t) < 0$ on $[t_1,\infty)_{\mathbb{T}}$ and the above inequality becomes

$$-x^{\Delta^{n-3}}(t_3) \geqslant b\int_{t_3}^{t}\frac{1}{r(z)}\int_{t_3}^{z}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u\Delta z.$$

Letting $t \to \infty$, the last inequality contradicts (3.1). So $m = n-2$ or $m = 0$, which implies Case 1 or Case 2 holds.

Case ii. There exists sufficiently large t_2 such that $x^{\Delta^{n-2}}(t) < 0$ on $[t_2,\infty)_{\mathbb{T}}$. Also from Kiguradze's lemma can get a $m \in [0,n-2]_{\mathbb{Z}}$ with $m+n$ even for $f^{\Delta^n}(t) \geqslant 0$, or $m+n$ odd for $f^{\Delta^n}(t) \leqslant 0$ such that

(I) $f^{\Delta^j}(t) > 0$ holds for all $t \in [t_3,\infty)_{\mathbb{T}}$ and all $j \in [0,m)_{\mathbb{Z}}$;

(II) $(-1)^{m+j}f^{\Delta^j}(t) > 0$ holds for all $t \in [t_3,\infty)_{\mathbb{T}}$ and all $j \in [m,n-2)_{\mathbb{Z}}$.

If $m \neq 0,\ n-2$, each order derivative of x must contain the following form:

$$x^{\Delta^{n-2}}(t) < 0,\ x^{\Delta^{n-3}}(t) > 0,\ldots,x^{\Delta^m}(t) > 0,\ x^{\Delta^{m-1}}(t) > 0,\ldots,x^{\Delta}(t) > 0,\ x(t) > 0$$

for $t \in [t_3,\infty)_{\mathbb{T}}$. Integrating (1.1) from t to u and letting $u \to \infty$, from (3.3) we get

$$a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta} \geqslant b\int_{t}^{\infty}p(s)\Delta s, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

Dividing by $a(t)$ and integrating from t to v and letting $v \to \infty$ we get

$$\lim_{v\to\infty} r(v)x^{\Delta^{n-2}}(v) - r(t)x^{\Delta^{n-2}}(t) \geqslant b\int_{t}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u.$$

Since $x^{\Delta^{n-2}}(t) < 0$, we obtain that

$$-r(t)x^{\Delta^{n-2}}(t) \geqslant b\int_{t}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u.$$

Dividing by $r(t)$ and integrating from t_1 to t, we obtain

$$x^{\Delta^{n-3}}(t_1) - x^{\Delta^{n-3}}(t) \geqslant b\int_{t_1}^{t}\frac{1}{r(z)}\int_{z}^{\infty}\frac{1}{a(u)}\int_{u}^{\infty}p(s)\Delta s\Delta u\Delta z.$$

For $x^{\Delta^{n-3}}(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$ it becomes

$$x^{\Delta^{n-3}}(t_1) \geqslant b\int_{t_1}^{t} \frac{1}{r(z)}\int_{z}^{\infty} \frac{1}{a(u)}\int_{u}^{\infty} p(s)\Delta s\Delta u\Delta z.$$

Letting $t \to \infty$, the last inequality contradicts (3.2). So $m = n-2$ or $m = 0$, which implies Case 1 or Case 2 holds.The proof is complete. □

REMARK 1. Indeed, we have not limited the parity of n, and in Lemma 4, if we know that $x^{\Delta^{n-2}}(t) > 0$ and n is even, then only Case 1 occurs. Also if $x^{\Delta^{n-2}}(t) < 0$ and n is odd, still only Case 1 occurs.

LEMMA 5. *Assume that* (3.1) *and* (3.2) *hold, and x is a positive solution of* (1.1) *which satisfies Case 2 of Lemma* 4. *Then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Since x satisfies Case 2 of Lemma 4, obviously $x^{\Delta}(t) < 0$. So $x(t)$ is strictly decreasing and has finite limit on $t \in [t_1,\infty)_{\mathbb{T}}$. We claim that $\lim_{t\to\infty} x(t) = 0$. Assume not, there exists a positive constant c such that for $t \in [t_1,\infty)_{\mathbb{T}}$,

$$\lim_{t\to\infty} x(t) = c > 0.$$

Hence there exits $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that for $t \in [t_2,\infty)_{\mathbb{T}}$, $x(t) > c/2$. Since we do not know the sign of $x^{\Delta^{n-2}}(t)$. First we assume $x^{\Delta^{n-2}}(t) > 0$, as the proof of Lemma 4, we can also get that

$$-x^{\Delta^{n-3}}(t_2) \geqslant \frac{c}{2}\int_{t_1}^{t} \frac{1}{r(z)}\int_{t_1}^{z} \frac{1}{a(u)}\int_{u}^{\infty} p(s)\Delta s\Delta u\Delta z$$

for $t \in [t_2,\infty)_{\mathbb{T}}$. Letting $t \to \infty$, this contradicts (3.1). If $x^{\Delta^{n-2}}(t) < 0$, using a similar method we can get a contradiction to (3.2). The proof is complete. □

LEMMA 6. *Assume that x satisfies Case 1 of Lemma* 4. *Then*

$$x(t) \geqslant \frac{r(t)x^{\Delta^{n-2}}(t)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\int_{t_1}^{t} \frac{h_{n-3}(t,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}{r(\tau)}\Delta\tau \tag{3.4}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$ and $r(t)x^{\Delta^{n-2}}(t)/\int_{t_1}^{t}\frac{\Delta s}{a(s)}$ is nonincreasing eventually.

Proof. Since x satisfies Case 1 of Lemma 4. There exists t_1 such that

$$(a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta})^{\Delta} < 0,\ \ (r(t)x^{\Delta^{n-2}}(t))^{\Delta} > 0,\ \ x^{\Delta^{i}} > 0,\ \ i \in [0,n-2]_{\mathbb{N}},\ t \in [t_1,\infty)_{\mathbb{T}}.$$

So $a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}$ is strictly decreasing and positive on $[t_1,\infty)_{\mathbb{T}}$, and we can deduce that

$$\begin{aligned} r(t)x^{\Delta^{n-2}}(t)-r(t_1)x^{\Delta^{n-2}}(t_1) &= \int_{t_1}^{t}(r(s)x^{\Delta^{n-2}}(s))^{\Delta}\Delta s \\ &= \int_{t_1}^{t}\frac{a(s)(r(s)x^{\Delta^{n-2}}(s))^{\Delta}}{a(s)}\Delta s \\ &\geqslant a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}\int_{t_1}^{t}\frac{\Delta s}{a(s)}. \end{aligned}$$

Thus

$$r(t)x^{\Delta^{n-2}}(t)\geqslant a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}\int_{t_1}^{t}\frac{\Delta s}{a(s)}. \tag{3.5}$$

By rules for derivative of quotient on time scales, we know that

$$\left(\frac{r(t)x^{\Delta^{n-2}}(t)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\right)^{\Delta}=\frac{(r(t)x^{\Delta^{n-2}}(t))^{\Delta}\int_{t_1}^{t}\frac{\Delta s}{a(s)}-\frac{1}{a(t)}r(t)x^{\Delta^{n-2}}(t)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}.$$

Substituting (3.5) in this equality we have

$$\left(\frac{r(t)x^{\Delta^{n-2}}(t)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\right)^{\Delta}\leqslant 0. \tag{3.6}$$

So $r(t)x^{\Delta^{n-2}}(t)/\int_{t_1}^{t}\frac{\Delta s}{a(s)}$ is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$.

Next we expand $x(t)$ by Taylor's formula from Lemma 1 and since $x^{\Delta^i}>0$ for $i\in[0,n-2]_{\mathbb{N}}$ and the nonincreasing property of $r(t)x^{\Delta^{n-2}}(t)/\int_{t_1}^{t}\frac{\Delta s}{a(s)}$, we get

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-3}h_k(t,t_1)x^{\Delta^k}(s)+\int_{t_1}^{t}h_{n-3}(t,\sigma(\tau))x^{\Delta^{n-2}}(\tau)\Delta\tau \\ &\geqslant \int_{t_1}^{t}h_{n-3}(t,\sigma(\tau))x^{\Delta^{n-2}}(\tau)\Delta\tau \\ &= \int_{t_1}^{t}h_{n-3}(t,\sigma(\tau))\frac{r(\tau)\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}x^{\Delta^{n-2}}(\tau)}{r(\tau)\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}\Delta\tau \\ &\geqslant \frac{r(t)x^{\Delta^{n-2}}(t)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}\int_{t_1}^{t}\frac{h_{n-3}(t,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}{r(\tau)}\Delta\tau. \end{aligned} \tag{3.7}$$

This completes the proof. □

Now we give some oscillation criteria for (1.1) based on the previous lemmas.

THEOREM 1. *Suppose that* (3.1) *and* (3.2) *hold and let* x *be a solution of* (1.1). *If*

$$p_* := \liminf_{t\to\infty}\int_{t_1}^{t}\frac{\Delta s}{a(s)}\int_{t}^{\infty}\frac{\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}}p(s)\Delta s > \frac{1}{4}, \tag{3.8}$$

then x *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. For the contrary, suppose that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \in [T,\infty)_{\mathbb{T}}$. From Lemma 4, x satisfies Case 1 or Case 2. If x satisfies Case 2, by Lemma 5, $\lim_{t\to\infty} x(t) = 0$.

Otherwise, x satisfies Case 1 of Lemma 4, which means there exists a sufficiently large $t_1 \in \mathbb{T}$ such that for $t \in [t_1,\infty)_{\mathbb{T}}$

$$x^{\Delta^i}(t) > 0 \ \text{ for } \ i \in [0,n-2]_{\mathbb{N}}.$$

Define the Riccati transformation by

$$w(t) := \frac{a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}}{r(t)x^{\Delta^{n-2}}(t)}. \tag{3.9}$$

Taking the derivative of $w(t)$ we get that

$$\begin{aligned} w^{\Delta}(t) &= \left(\frac{a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}}{r(t)x^{\Delta^{n-2}}(t)}\right)^{\Delta} \\ &= \frac{(a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta})^{\Delta}r(t)x^{\Delta^{n-2}}(t) - a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}(r(t)x^{\Delta^{n-2}}(t))^{\Delta}}{r(t)x^{\Delta^{n-2}}(t)r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))} \\ &= -\frac{x(t)}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))}p(t) - \frac{(r(t)x^{\Delta^{n-2}}(t))^{\Delta}}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))}w(t). \end{aligned} \tag{3.10}$$

Since $a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}$ is decreasing, we have that

$$\frac{(r(t)x^{\Delta^{n-2}}(t))^{\Delta}}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))} = \frac{a(t)[(r(t)x^{\Delta^{n-2}}(t))^{\Delta}]}{a(t)[r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))]} \geqslant \frac{a^{\sigma}(t)(r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t)))^{\Delta}}{a(t)r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))} = \frac{w^{\sigma}(t)}{a(t)}.$$

Thus we obtain

$$w^{\Delta}(t) \leqslant -\frac{x(t)}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))}p(t) - \frac{w(t)w^{\sigma}(t)}{a(t)} \tag{3.11}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. Then from Lemma 6, we have

$$\frac{x(t)}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))} = \frac{x(t)}{r(t)x^{\Delta^{n-2}}(t)}\frac{r(t)x^{\Delta^{n-2}}(t)}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))}$$

$$\geqslant \frac{\int_{t_1}^{t} \frac{h_{n-3}(t,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} \frac{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}$$

$$= \frac{\int_{t_1}^{t} \frac{h_{n-3}(t,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}.$$

And (3.11) becomes

$$w^{\Delta}(t) + \frac{\int_{t_1}^{t} \frac{h_{n-3}(t,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}} p(t) + \frac{w(t)w^{\sigma}(t)}{a(t)} \leqslant 0. \tag{3.12}$$

Next we claim that $\lim_{t\to\infty} w(t) = 0$ and $w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \leqslant 1$ for $t \in [t_1,\infty)_{\mathbb{T}}$. From (3.12) we can get that

$$w^{\Delta}(t) \leqslant -\frac{w(t)w^{\sigma}(t)}{a(t)}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$, and so

$$\left(-\frac{1}{w}\right)^{\Delta}(t) = \frac{w^{\Delta}(t)}{w(t)w^{\sigma}(t)} \leqslant -\frac{1}{a(t)}.$$

Integrating both sides from t_1 to t we have

$$\int_{t_1}^{t}\left(-\frac{1}{w}\right)^{\Delta}(s)\Delta s \leqslant -\int_{t_1}^{t}\frac{\Delta s}{a(s)}.$$

That is

$$-\frac{1}{w(t)} + \frac{1}{w(t_1)} \leqslant -\int_{t_1}^{t}\frac{\Delta s}{a(s)},$$

Since $w(t) > 0$, we have that

$$\frac{1}{w(t)} \geqslant \int_{t_1}^{t}\frac{\Delta s}{a(s)}$$

From the condition (1.2), it is easy to see $\lim_{t\to\infty} w(t) = 0$ and $w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \leqslant 1$ on $t \in [t_1,\infty)_{\mathbb{T}}$.

Then we define r_* by

$$r_* := \liminf_{t\to\infty} w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}. \tag{3.13}$$

It is clear to see $0 \leqslant r_* \leqslant 1$. Now we claim that

$$r_* \geqslant p_* + r_*^2,$$

where p_* is defined as in (3.8). Integrating (3.12) from t to ∞, and from $\lim_{t\to\infty} w(t) = 0$ we have that

$$w(t) \geqslant \int_t^\infty \frac{\int_{t_1}^s \frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^\tau \frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s + \int_t^\infty \frac{w^\sigma(s)w(s)}{a(s)}\Delta s. \tag{3.14}$$

Multiplying (3.14) by $\int_{t_1}^t \frac{\Delta s}{a(s)}$, we obtain

$$\begin{aligned} w(t)\int_{t_1}^t \frac{\Delta s}{a(s)} \geqslant &\int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{\int_{t_1}^s \frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^\tau \frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s \\ &+ \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{w^\sigma(s)w(s)}{a(s)}\Delta s \\ = &\int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{\int_{t_1}^s \frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^\tau \frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s \\ &+ \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{w(s)\int_{t_1}^s \frac{\Delta u}{a(u)} w^\sigma(s)\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}}{a(s)\int_{t_1}^s \frac{\Delta u}{a(u)} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}}\Delta s. \end{aligned} \tag{3.15}$$

Now for any $\varepsilon > 0$, from the definition of r_*, there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that for all $t \in [t_2,\infty)_{\mathbb{T}}$

$$w(t)\int_{t_1}^t \frac{\Delta s}{a(s)} \geqslant r_* - \varepsilon.$$

Taking this into (3.15) we get

$$\begin{aligned} &w(t)\int_{t_1}^t \frac{\Delta s}{a(s)} \\ &\geqslant \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{\int_{t_1}^s \frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^\tau \frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s \\ &\quad + (r_* - \varepsilon)^2 \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{1}{a(s)\int_{t_1}^s \frac{\Delta u}{a(u)} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}}\Delta s \\ &= \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{\int_{t_1}^s \frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^\tau \frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s \\ &\quad + (r_* - \varepsilon)^2 \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \left(\frac{1}{\int_{t_1}^s \frac{\Delta u}{a(u)}}\right)^\Delta \Delta s \\ &= \int_{t_1}^t \frac{\Delta s}{a(s)} \int_t^\infty \frac{\int_{t_1}^s \frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^\tau \frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s + (r_* - \varepsilon)^2 \end{aligned} \tag{3.16}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. Therefore, taking the limit inferior of both sides of (3.16) gives

$$r_* \geqslant p_* + (r_* - \varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$r_* \geqslant p_* + r_*^2.$$

Then we obtain that

$$p_* \leqslant r_* - r_*^2 = \frac{1}{4} - (r_* - \frac{1}{2})^2 \leqslant \frac{1}{4},$$

This contradicts (3.8). The proof is complete. □

REMARK 2. If $n = 3$, this result becomes Theorem 2.8 in [12]. If $n = 3$ and $a \equiv 1$, $r \equiv 1$, this result is Theorem 2 in [10]. So our research extends Erbe et al. [10] and Agarwal et al. [12]'s work.

THEOREM 2. *Assume that* (3.1) *and* (3.2) *hold and let* x *be a solution of* (1.1). *Define* $w(t)$ *as in Theorem 3.1, and*

$$R_* := \limsup_{t\to\infty} w(t) \int_{t_1}^{t} \frac{\Delta s}{a(s)}, \quad l^* := \limsup_{t\to\infty} \frac{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{a(s)}}{\int_{t_1}^{t} \frac{\Delta s}{a(s)}}.$$

If

$$q_* := \liminf_{t\to\infty} \frac{\int_{t_2}^{t} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)} \int_{t_1}^{s} \frac{h_{n-3}(s,\sigma(\tau)) \int_{t_1}^{\tau} \frac{\Delta v}{a(v)}}{r(\tau)} \Delta\tau\, p(s) \Delta s}{\int_{t_1}^{t} \frac{\Delta s}{a(s)}} > \frac{l^*}{1+l^*}, \tag{3.17}$$

then x *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. For the contrary, suppose that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Then if x satisfies Case 2 of Lemma 4, by Lemma 5, $\lim_{t\to\infty} x(t) = 0$. Next we consider the nonoscillatory positive solution x which satisfies Case 1 of Lemma 4, then from the proof of Theorem 3.1 we get that (3.10) holds. Since

$$\begin{aligned}
\frac{(r(t)x^{\Delta^{n-2}}(t))^{\Delta}}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))} w(t) &= \frac{(r(t)x^{\Delta^{n-2}}(t))^{\Delta}}{r(t)x^{\Delta^{n-2}}(t)} \frac{r(t)x^{\Delta^{n-2}}(t)}{r^{\sigma}(t)x^{\Delta^{n-2}}(\sigma(t))} w(t) \\
&= \frac{w^2(t)}{a(t)} \frac{r(t)x^{\Delta^{n-2}}(t)}{r(t)x^{\Delta^{n-2}}(t) + \mu(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}} \\
&= \frac{w^2(t)}{a(t)} \frac{1}{1+\mu(t)\frac{w(t)}{a(t)}} \\
&= \frac{w^2(t)}{a(t)+\mu(t)w(t)}.
\end{aligned}$$

We get another Ricatti inequality of the form

$$w^{\Delta}(t)+\frac{\int_{t_1}^{t}\frac{h_{n-3}(t,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}p(t)+\frac{w^2(t)}{a(t)+\mu(t)w(t)}\leqslant 0. \tag{3.18}$$

Multiplying (3.18) by $\left(\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}\right)^2$ and integrating the resulting inequality from $t_2\in(t_1,\infty)_{\mathbb{T}}$ to t, we see that

$$\int_{t_2}^{t}\Big(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\Big)^2 w^{\Delta}(s)\Delta s+\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau p(s)\Delta s \\ +\int_{t_2}^{t}\Big(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\Big)^2\frac{w^2(s)}{a(s)+\mu(s)w(s)}\Delta s\leqslant 0. \tag{3.19}$$

Integrating by part for (3.19) yields

$$\Big(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\Big)^2 w(t)-\Big(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\Big)^2 w(t_2)-\int_{t_2}^{t}\bigg(\Big(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\Big)^2\bigg)^{\Delta}w(s)\Delta s \\ +\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau p(s)\Delta s \\ +\int_{t_2}^{t}\Big(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\Big)^2\frac{w^2(s)}{a(s)+\mu(s)w(s)}\Delta s\leqslant 0.$$

Since

$$\bigg(\Big(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\Big)^2\bigg)^{\Delta}=\frac{1}{a(s)}\Big[\int_{t_1}^{s}\frac{\Delta u}{a(u)}+\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\Big]=\frac{1}{a(s)}\Big[2\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}-\frac{\mu(s)}{a(s)}\Big],$$

taking this into the above inequality, after rearranging we get

$$\Big(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\Big)^2 w(t)\leqslant\Big(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\Big)^2 w(t_2) \\ -\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau p(s)\Delta s+\int_{t_2}^{t}H(s,w(s))\Delta s, \tag{3.20}$$

where $H(s,w(s)):=\frac{1}{a(s)}\Big[2\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}-\frac{\mu(s)}{a(s)}\Big]w(s)-\Big(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\Big)^2\frac{w^2(s)}{a(s)+\mu(s)w(s)}$. In [12, Lemma 2.7], the author proved that $H(s,w(s))\leqslant 1/a(s)$, when $w(s)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$, and we do not repeat here. So we get

$$\int_{t_2}^{t}H(s,w(s))\Delta s\leqslant\int_{t_2}^{t}\frac{\Delta s}{a(s)}.$$

Substituting this inequality into (3.20) and dividing by $\int_{t_1}^{t}\frac{\Delta s}{a(s)}$, we have

$$w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \leqslant \frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} - \frac{\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau p(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} + \frac{\int_{t_2}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}. \tag{3.21}$$

If we take the superior limits of both sides of (3.21), we get

$$R_* \leqslant 1 - q_*. \tag{3.22}$$

Next we give another Ricatti inequality. Using Lemma 6 we get that

$$\frac{(r(t)x^{\Delta^{n-2}}(t))^\Delta}{r^\sigma(t)x^{\Delta^{n-2}}(\sigma(t))} = \frac{(r(t)x^{\Delta^{n-2}}(t))^\Delta}{r(t)x^{\Delta^{n-2}}(t)}\frac{r(t)x^{\Delta^{n-2}}(t)}{r^\sigma(t)x^{\Delta^{n-2}}(\sigma(t))} = \frac{w(t)}{a(t)}\frac{r(t)x^{\Delta^{n-2}}(t)}{r^\sigma(t)x^{\Delta^{n-2}}(\sigma(t))} \geqslant \frac{w(t)}{a(t)}\frac{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}.$$

Thus from (3.10) we obtain

$$w^\Delta(t) + \frac{\int_{t_1}^{t}\frac{h_{n-3}(t,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta s}{a(s)}}{r(\tau)}\Delta\tau}{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}p(t) + \frac{w^2(t)}{a(t)}\frac{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}} \leqslant 0. \tag{3.23}$$

If $\varepsilon > 0$ is given arbitrary, then there exists $t_2 \in (t_1,\infty)_{\mathbb{T}}$ such that

$$r_* - \varepsilon \leqslant w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \leqslant R_* + \varepsilon \quad \text{for} \quad t \in [t_2,\infty)_{\mathbb{T}}$$

where r_* is defined as in Theorem 3.1. And

$$\frac{\int_{t_1}^{\sigma(t)}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} \leqslant l^* + \varepsilon \quad \text{for} \quad t \in [t_2,\infty)_{\mathbb{T}}.$$

Using (3.23) and a similar proceeding operating on (3.18), we have

$$\left(\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)^2 w(t) \leqslant \left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2) + \int_{t_2}^{t}\frac{\left[\int_{t_1}^{s}\frac{\Delta u}{a(u)} + \int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]w(s)}{a(s)}\Delta s - \int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau p(s)\Delta s - \int_{t_2}^{t}\left(\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right)^2\frac{w^2(s)}{a(s)}\frac{\int_{t_1}^{s}\frac{\Delta u}{a(u)}}{\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}}\Delta s,$$

Then

$$w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \leqslant \frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} + \frac{\int_{t_2}^{t}\frac{1}{a(s)}\left[\int_{t_1}^{s}\frac{\Delta u}{a(u)}+\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]w(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-\frac{\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau p(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-\frac{\int_{t_2}^{t}\left(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\right)^2\frac{w^2(s)}{a(s)}\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}},$$

That is

$$w(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)} \leqslant \frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} + \frac{\int_{t_2}^{t}\frac{\frac{1}{a(s)}\left[\int_{t_1}^{s}\frac{\Delta u}{a(u)}+\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\right]\left[w(s)\int_{t_1}^{s}\frac{\Delta s}{a(s)}\right]}{\int_{t_1}^{s}\frac{\Delta s}{a(s)}}\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-\frac{\int_{t_2}^{t}\int_{t_1}^{\sigma(s)}\frac{\Delta u}{a(u)}\int_{t_1}^{s}\frac{h_{n-3}(s,\sigma(\tau))\int_{t_1}^{\tau}\frac{\Delta v}{a(v)}}{r(\tau)}\Delta\tau p(s)\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-\frac{\int_{t_2}^{t}\left(\int_{t_1}^{s}\frac{\Delta u}{a(u)}\right)^2\frac{w^2(s)}{a(s)}\Delta s}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$\leqslant \frac{\left(\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}\right)^2 w(t_2)}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}} + (R_*+\varepsilon)(1+l^*+\varepsilon)\frac{\int_{t_2}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}$$
$$-q_* - (r_*-\varepsilon)^2\frac{\int_{t_2}^{t}\frac{\Delta s}{a(s)}}{\int_{t_1}^{t}\frac{\Delta s}{a(s)}}.$$

Taking the superior limits of both sides of this inequality, since $\varepsilon > 0$ is arbitrary, we get

$$R_* \leqslant R_*(1+l^*) - r_*^2 - q_*,$$

that is

$$q_* \leqslant R_* l^* - r_*^2.$$

Now substituting (3.22) into this inequality we have that

$$q_* \leqslant l^* - l^* q_*,$$

$$q_* \leqslant \frac{l^*}{1+l^*},$$

which contradicts condition (3.17). The proof is complete. □

REMARK 3. When $n = 3$ and $a \equiv 1$, $r \equiv 1$, this result is Theorem 3 in [10]. So this result also extends Erbe et al. [10]'s work.

REMARK 4. Assertions of Theorems 3.1 and 3.2 are that any non-trivial solution of given equation is either oscillatory or vanishes at infinity. This properties for ODE are usually called "Property A" and "Property B" and were introduced in earlier 60s of 20th century by Kondratiev and Kiguradze. More detail about "Property A" and "Property B" can be found in the book [15].

Next, we consider the case:

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{t_0}^{z} \frac{1}{a(u)} \int_{u}^{\infty} p(s)\Delta s\Delta u\Delta z < \infty, \tag{3.24}$$

$$\int_{t_0}^{\infty} \frac{1}{r(z)} \int_{z}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} p(s)\Delta s\Delta u\Delta z < \infty \tag{3.25}$$

hold. For convenience we define that

$$q(t) := \int_{t}^{\infty} \frac{1}{r(u)} \int_{u}^{\infty} \frac{1}{a(s)} \int_{s}^{\infty} \frac{p(\tau)h_m(\tau,t_1)}{h_1(\tau,t_1)}\Delta\tau\Delta s\Delta u;$$

$$q^*(t) := \int_{t}^{\infty} \frac{1}{r(u)h_1(u,t_1)} \int_{t_1}^{u} \frac{h_1(s,t_1)}{a(s)} \int_{s}^{\infty} \frac{p(\tau)h_m(\tau,t_1)}{h_1(\tau,t_1)}\Delta\tau\Delta s\Delta u,$$

and

$$Q_m(t) := \int_{t}^{\infty} \int_{u_{n-m-5}}^{\infty} \cdots \int_{u_1}^{\infty} q(u)\Delta u \cdots \Delta u_{n-m-6}\Delta u_{n-m-5};$$

$$Q_m^*(t) := \int_{t}^{\infty} \int_{u_{n-m-5}}^{\infty} \cdots \int_{u_1}^{\infty} q^*(u)\Delta u \cdots \Delta u_{n-m-6}\Delta u_{n-m-5},$$

where $h_m(\tau,t_1)$ is the Taylor monomial defined in preliminary section, especially for $h_1(\tau,t_1) = \tau - t_1$.

LEMMA 7. (see [16, Lemma 2.2]) *If the inequality*

$$x^{\Delta\Delta} + Q(t)x \leqslant 0, \tag{3.26}$$

where Q is a positive real-valued, rd-continuous function on $\mathbb{T}$, has an eventually positive solution, then the equation

$$x^{\Delta\Delta} + Q(t)x = 0 \tag{3.27}$$

also has an eventually positive solution.

THEOREM 3. *Suppose that there is a positive integer $k \in [0, n-2)$ such that*

$$\int_{t_0}^{\infty} \int_{u_{n-k+1}}^{\infty} \cdots \int_{u_4}^{\infty} \frac{1}{r(u_3)} \int_{u_3}^{\infty} \frac{1}{a(u_2)} \int_{u_2}^{\infty} p(u_1)\Delta u_1\Delta u_2 \cdots \Delta u_{n-k+1} = \infty, \tag{3.28}$$

and

$$\int_{t_0}^{\infty}\int_{u_{n-k+1}}^{\infty}\cdots\int_{u_4}^{\infty}\frac{1}{r(u_3)}\int_{t_1}^{u_3}\frac{1}{a(u_2)}\int_{u_2}^{\infty}p(u_1)\Delta u_1\Delta u_2\cdots\Delta u_{n-k+1}=\infty \tag{3.29}$$

hold. If the second order dynamic equations

$$y^{\Delta\Delta}(t)+Q_m(t)y(t)=0, \tag{3.30}$$

and

$$y^{\Delta\Delta}(t)+Q_m^*(t)y(t)=0 \tag{3.31}$$

are oscillatory for every $m\in\{k,k+1,\ldots,n-3\}$. *Then every solution* x *of* (1.1) *is oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Proof. For the contrary, we assume that x is an eventually positive solution of (1.1), and without loss of generality let $x(t)>0$ on $t\in[T,\infty)_{\mathbb{T}}$. By Lemma 3 we have

$$(a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta})^{\Delta}<0,\ \ (r(t)x^{\Delta^{n-2}}(t))^{\Delta}>0,$$

for $t\in[T,\infty)_{\mathbb{T}}$ and $r(t)x^{\Delta^{n-2}}(t)$ is strictly increasing and has eventually one sign, then $x^{\Delta^{n-2}}(t)$ has eventually one sign and satisfies Kiguradze's lemma. By Kiguradze's lemma there exist $t_1>T$ and $m\in[0,n-2]_{\mathbb{N}}$ with $n-2+m$ even for $x^{\Delta^{n-2}}(t)>0$, or $n-2+m$ odd for $x^{\Delta^{n-2}}(t)<0$ such that for $t\in[t_1,\infty)_{\mathbb{T}}$

$$\begin{aligned}(-1)^{j+m}x^{\Delta^j}>0 &\quad \text{for } j\in[m,n-2)_{\mathbb{N}},\\ x^{\Delta^j}>0 &\quad \text{for } j\in[0,m)_{\mathbb{N}}.\end{aligned}$$

First we prove that if $\lim_{t\to\infty}x(t)\neq 0$, then $m\geqslant k$. Assume not, $m<k$, which implies $(-1)^{j+m}x^{\Delta^j}>0$ for $j\in[k-1,n-2)_{\mathbb{N}}$, $t\in[t_1,\infty)_{\mathbb{T}}$. Due to we cannot ensure wether $x^{\Delta^{n-2}}(t)$ is eventually positive or eventually negative, we also discuss from two cases.

Case i. $x^{\Delta^{n-2}}(t)<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Integrating equation (1.1) from t to v and letting $v\to\infty$ for $t\in[t_1,\infty)_{\mathbb{T}}$, we get that

$$a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}-\lim_{v\to\infty}a(v)(r(v)x^{\Delta^{n-2}}(v))^{\Delta}=\int_t^{\infty}p(s)x(s)\Delta s. \tag{3.32}$$

Note that $a(t)(r(t)x^{\Delta^{n-2}}(t))^{\Delta}$ is positive on $[t_1,\infty)_{\mathbb{T}}$, we have

$$(r(t)x^{\Delta^{n-2}}(t))^{\Delta}\geqslant\frac{1}{a(t)}\int_t^{\infty}p(s)x(s)\Delta s\quad \text{for}\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.33}$$

Integrating from t to v and letting $v\to\infty$ again, and since $r(t)x^{\Delta^{n-2}}(t)<0$ on $[t_1,\infty)_{\mathbb{T}}$, we obtain

$$-x^{\Delta^{n-2}}(t)\geqslant\frac{1}{r(t)}\int_t^{\infty}\frac{1}{a(u)}\int_u^{\infty}p(s)x(s)\Delta s\Delta u. \tag{3.34}$$

Since $(-1)^{j+m}x^{\Delta^j} > 0$ for $j \in [k-1, n-2)_{\mathbb{N}}$, repeating above process $n-k-2$ times, we have

$$x^{\Delta^k}(t) \geqslant \int_t^\infty \int_{u_{n-k}}^\infty \cdots \int_{u_4}^\infty \frac{1}{r(u_3)} \int_{u_3}^\infty \frac{1}{a(u_2)} \int_{u_2}^\infty p(u_1)x(u_1)\Delta u_1 \Delta u_2 \cdots \Delta u_{n-k}, \quad (3.35)$$

or

$$-x^{\Delta^k}(t) \geqslant \int_t^\infty \int_{u_{n-k}}^\infty \cdots \int_{u_4}^\infty \frac{1}{r(u_3)} \int_{u_3}^\infty \frac{1}{a(u_2)} \int_{u_2}^\infty p(u_1)x(u_1)\Delta u_1 \Delta u_2 \cdots \Delta u_{n-k}, \quad (3.36)$$

Whether is (3.35) or (3.36) is determined by the parity of $n-k$. According to the assumption $\lim_{t\to\infty} x(t) \neq 0$, we set $\lim_{t\to\infty} x(t) = d > 0$. Thus we can find a $t_2 > t_1$ such that $x(t) > d/2$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Then, integrating above equations from t_1 to t, we get

$$-x^{\Delta^{k-1}}(t_2) \geqslant \frac{d}{2} \int_{t_2}^\infty \int_{u_{n-k+1}}^\infty \cdots \int_{u_4}^\infty \frac{1}{r(u_3)} \int_{u_3}^\infty \frac{1}{a(u_2)} \int_{u_2}^\infty p(u_1)\Delta u_1 \Delta u_2 \cdots \Delta u_{n-k+1}, \quad (3.37)$$

or

$$x^{\Delta^{k-1}}(t_2) \geqslant \frac{2}{d} \int_{t_2}^\infty \int_{u_{n-k+1}}^\infty \cdots \int_{u_4}^\infty \frac{1}{r(u_3)} \int_{u_3}^\infty \frac{1}{a(u_2)} \int_{u_2}^\infty p(u_1)\Delta u_1 \Delta u_2 \cdots \Delta u_{n-k+1}, \quad (3.38)$$

Both contradict (3.28).

Case ii. $x^{\Delta^{n-2}}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Integrating (3.33) from t_2 to t, where $x(t) > d/2$ for $t \in [t_2, \infty)_{\mathbb{T}}$ still holds, considering $r(t)x^{\Delta^{n-2}}(t) > 0$ we have

$$x^{\Delta^{n-2}}(t) \geqslant \frac{1}{r(t)} \int_{t_2}^t \frac{1}{a(u)} \int_u^\infty p(s)x(s)\Delta s \Delta u. \quad (3.39)$$

The remaining part is similar to the proof of Case i, and we can finally get a contradiction to (3.29).

Next, we prove that when $\lim_{t\to\infty} x(t) \neq 0$, for each $m \in \{k, k+1, \ldots, n-3\}$ equations (3.30) and (3.31) are oscillatory makes the equation (1.1) oscillate. Also discuss in two cases:

Case i'. $x^{\Delta^{n-2}}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From Kiguradze's lemma, we have $x^{\Delta^m}(t) > 0$ and $x^{\Delta^{m+1}}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Thus we have

$$\begin{aligned} x^{\Delta^{m-1}}(t) - x^{\Delta^{m-1}}(t_1) &= \int_{t_1}^t x^{\Delta^m}(s)\Delta s \\ &\geqslant x^{\Delta^m}(t) \int_{t_1}^t \Delta s \\ &= x^{\Delta^m}(t)(t-t_1). \end{aligned} \quad (3.40)$$

On the other hand,

$$\left(\frac{x^{\Delta^{m-1}}(t)}{t-t_1}\right)^\Delta = \frac{x^{\Delta^m}(t)(t-t_1) - x^{\Delta^{m-1}}(t)}{(t-t_1)(\sigma(t)-t_1)}. \quad (3.41)$$

Substituting (3.40) into (3.41), we get $\left(x^{\Delta^{m-1}}(t)/t-t_1\right)^{\Delta}<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$, which means $x^{\Delta^{m-1}}(t)/t-t_1$ is strictly decreasing. Thus

$$\begin{aligned} x^{\Delta^{m-2}}(t)-x^{\Delta^{m-2}}(t_1) &= \int_{t_1}^{t} x^{\Delta^{m-1}}(s)\Delta s \\ &\geqslant \frac{x^{\Delta^{m-1}}(t)}{t-t_1}\int_{t_1}^{t}(s-t_1)\Delta s \\ &= \frac{h_2(t,t_1)}{t-t_1}x^{\Delta^{m-1}}(t). \end{aligned} \tag{3.42}$$

And

$$\begin{aligned} x^{\Delta^{m-3}}(t)-x^{\Delta^{m-3}}(t_1) &= \int_{t_1}^{t} x^{\Delta^{m-2}}(s)\Delta s \\ &\geqslant \int_{t_1}^{t} \frac{h_2(s,t_1)}{s-t_1}x^{\Delta^{m-1}}(s)\Delta s \\ &\geqslant \frac{x^{\Delta^{m-1}}(t)}{t-t_1}\int_{t_1}^{t} h_2(s,t_1)\Delta s \\ &= \frac{h_3(t,t_1)}{t-t_1}x^{\Delta^{m-1}}(t). \end{aligned} \tag{3.43}$$

Thus by the recursive method, we can get the unequal relation between $x(t)$ and $x^{\Delta^{m-1}}(t)$ as

$$x(t)\geqslant \frac{h_m(t,t_1)}{t-t_1}x^{\Delta^{m-1}}(t). \tag{3.44}$$

From the proof of Case i above, inequality (3.36) can be rewritten as

$$-x^{\Delta^{m+1}}(t)\geqslant \int_t^{\infty}\int_{u_{n-m-1}}^{\infty}\cdots\int_{u_4}^{\infty}\frac{1}{r(u_3)}\int_{u_3}^{\infty}\frac{1}{a(u_2)}\int_{u_2}^{\infty}p(u_1)x(u_1)\Delta u_1\Delta u_2\cdots\Delta u_{n-m-1}, \tag{3.45}$$

when we replace k with $m+1$. Substituting (3.44) into (3.45) and from $x^{\Delta^{m-1}}(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} x^{\Delta^{m-1}}(t)\int_t^{\infty}\int_{u_{n-m-1}}^{\infty}\cdots\int_{u_4}^{\infty}\frac{1}{r(u_3)}\int_{u_3}^{\infty}\frac{1}{a(u_2)}\int_{u_2}^{\infty}\frac{p(u_1)h_m(u_1,t_1)}{h_1(u_1,t_1)}\Delta u_1\Delta u_2\cdots\Delta u_{n-m-1} \\ +x^{\Delta^{m+1}}(t)\leqslant 0, \end{aligned} \tag{3.46}$$

Set $y(t)=x^{\Delta^{m-1}}(t)$. Thus $y(t)$ is a positive solution of the inequality

$$y^{\Delta\Delta}(t)+Q_m(t)y(t)\leqslant 0. \tag{3.47}$$

From Lemma 7, the dynamic equation

$$y^{\Delta\Delta}(t)+Q_m(t)y(t)=0 \tag{3.48}$$

also has a positive solution, which is a contradiction to the hypothesis.

Case ii'. $x^{\Delta^{n-2}}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From Case ii (3.39) and (3.44) we have

$$\begin{aligned} x^{\Delta^{n-2}}(t) &\geqslant \frac{1}{r(t)} \int_{t_1}^{t} \frac{1}{a(u)} \int_{u}^{\infty} p(s)x(s)\Delta s\Delta u \\ &\geqslant \frac{1}{r(t)} \int_{t_1}^{t} \frac{1}{a(u)} \int_{u}^{\infty} p(s)\frac{h_m(s,t_1)}{s-t_1} x^{\Delta^{m-1}}(s)\Delta s\Delta u \\ &\geqslant \frac{1}{r(t)} \int_{t_1}^{t} \frac{x^{\Delta^{m-1}}(u)}{a(u)} \int_{u}^{\infty} p(s)\frac{h_m(s,t_1)}{s-t_1}\Delta s\Delta u \\ &\geqslant \frac{1}{r(t)} \frac{x^{\Delta^{m-1}}(t)}{t-t_1} \int_{t_1}^{t} \frac{u-t_1}{a(u)} \int_{u}^{\infty} p(s)\frac{h_m(s,t_1)}{s-t_1}\Delta s\Delta u \end{aligned} \tag{3.49}$$

Integrating (3.49) from t to v, letting $v \to \infty$, and repeating this process $n-m-3$ times, we have

$$-x^{\Delta^{m+1}}(t) \geqslant \int_{t}^{\infty} \int_{u_{n-m-1}}^{\infty} \cdots \int_{u_4}^{\infty} \frac{x^{\Delta^{m-1}}(u_3)}{r(u_3)h_1(u_3,t_1)} \int_{t_1}^{u_3} \frac{h_1(u_2,t_1)}{a(u_2)} \int_{u_2}^{\infty} \frac{p(u_1)h_m(u_1,t_1)}{h_1(u_1,t_1)}\Delta u_1 \cdots \Delta u_{n-m-1}. \tag{3.50}$$

Thus

$$-x^{\Delta^{m+1}}(t) \geqslant x^{\Delta^{m-1}}(t) \cdot \int_{t}^{\infty} \int_{u_{n-m-1}}^{\infty} \cdots \int_{u_4}^{\infty} \frac{1}{r(u_3)h_1(u_3,t_1)} \int_{t_1}^{u_3} \frac{h_1(u_2,t_1)}{a(u_2)} \int_{u_2}^{\infty} \frac{p(u_1)h_m(u_1,t_1)}{h_1(u_1,t_1)}\Delta u_1 \cdots \Delta u_{n-m-1}, \tag{3.51}$$

Setting $y(t) = x^{\Delta^{m-1}}(t)$ it becomes

$$y^{\Delta\Delta}(t) + Q_m^*(t)y(t) \leqslant 0.$$

Then, by Lemma 7, dynamic equation

$$y^{\Delta\Delta}(t) + Q_m^*(t)y(t) = 0$$

also has a positive solution, which is a contradiction to the hypothesis. The proof is complete. □

REMARK 5. For the extreme case

$$\int_{t_0}^{\infty} \int_{u_{n+1}}^{\infty} \cdots \int_{u_4}^{\infty} \frac{1}{r(u_3)} \int_{u_3}^{\infty} \frac{1}{a(u_2)} \int_{u_2}^{\infty} p(u_1)\Delta u_1 \Delta u_2 \cdots \Delta u_{n+1} < \infty, \tag{3.52}$$

$$\int_{t_0}^{\infty} \int_{u_{n+1}}^{\infty} \cdots \int_{u_4}^{\infty} \frac{1}{r(u_3)} \int_{t_1}^{u_3} \frac{1}{a(u_2)} \int_{u_2}^{\infty} p(u_1)\Delta u_1 \Delta u_2 \cdots \Delta u_{n+1} < \infty \tag{3.53}$$

holds, the existence of nonoscillatory solution of (1.1) can be discussed, but in this paper we omit it.

4. Examples

In this section, we will show the application of our main results.

EXAMPLE 1. Consider the higher-order differential equation

$$x^{(4)} + \frac{6}{t^4} x(t) = 0, \qquad t \geqslant 1. \tag{4.1}$$

Here $r(t) = 1,\ a(t) = 1,\ p(t) = 6/t^4$. It is clear that the conditions (3.1) and (3.2) hold. To apply Theorem 1 it remains to prove that (3.8) is satisfied. In our case the condition reads

$$\begin{aligned}
&\liminf_{t\to\infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_{t}^{\infty} \frac{\int_{t_1}^{s} \frac{h_{n-3}(s,\sigma(\tau)) \int_{t_1}^{\tau} \frac{\Delta v}{a(v)}}{r(\tau)} \Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s \\
&= \liminf_{t\to\infty} \int_{t_1}^{t} ds \int_{t}^{\infty} \frac{\int_{t_1}^{s} (s-\tau)\left(\int_{t_1}^{\tau} dv\right) d\tau}{\int_{t_1}^{s} du} \cdot \frac{6}{s^4} ds \\
&= \liminf_{t\to\infty} (t-t_1) \int_{t}^{\infty} \frac{s^3 - 3t_1 s^2 + 3t_1^2 s - t_1^3}{(s-t_1)s^4} ds \\
&\geqslant \liminf_{t\to\infty} (t-t_1) \int_{t}^{\infty} \frac{s^3 - 3t_1 s^2 + 3t_1^2 s - t_1^3}{s^5} ds \\
&= 1 > \frac{1}{4}.
\end{aligned}$$

Then, from Theorem 1, we get that all solutions of (4.1) are oscillatory or converge to zero. In fact, one can easily see that the basis of solution space of (4.1) is given by

$$\{t^{-1},\ t^2 \cos\sqrt{2}\log t,\ t^2 \sin\sqrt{2}\log t\}.$$

EXAMPLE 2. Let $\mathbb{T} = q^{\mathbb{N}_0},\ q > 1$ and consider the higher-order q-difference equation

$$x^{\Delta^4}(t) + \frac{\alpha}{t h_2(t,0)} x(t) = 0, \tag{4.2}$$

Here $r(t) = 1,\ a(t) = 1,\ p(t) = \alpha / t h_2(t,0)$. It is clear that the conditions (3.1) and (3.2) hold. In our case the condition reads

$$\begin{aligned}
p_* :&= \liminf_{t\to\infty} \int_{t_1}^{t} \frac{\Delta s}{a(s)} \int_{t}^{\infty} \frac{\int_{t_1}^{s} \frac{h_{n-3}(s,\sigma(\tau)) \int_{t_1}^{\tau} \frac{\Delta v}{a(v)}}{r(\tau)} \Delta\tau}{\int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)}} p(s)\Delta s \\
&= \liminf_{t\to\infty} \int_{t_1}^{t} d_q s \int_{t}^{\infty} \frac{\int_{t_1}^{s} (s-\sigma(\tau))\left(\int_{t_1}^{\tau} d_q v\right) d_q\tau}{\int_{t_1}^{\sigma(s)} d_q u} \cdot \frac{\alpha}{s h_2(s,0)} d_q s \\
&\geqslant \liminf_{t\to\infty} \int_{t_1}^{t} d_q s \int_{t}^{\infty} \frac{\int_{t_1}^{s} (s-\sigma(\tau)) d_q\tau}{\int_{t_1}^{\sigma(s)} d_q u} \cdot \frac{\alpha}{s h_2(s,0)} d_q s
\end{aligned}$$

$$\geqslant \liminf_{t\to\infty} \int_{t_1}^{t} d_q s \int_{t}^{\infty} \frac{h_2(s,0)}{\int_{t_1}^{\sigma(s)} d_q u} \cdot \frac{\alpha}{\int_{t_1}^{s} d_q u h_2(s,0)} d_q s$$
$$= \liminf_{t\to\infty} \int_{t_1}^{t} d_q s \int_{t}^{\infty} \Big(- \frac{\alpha}{\int_{t_1}^{s} d_q v} \Big)^{\Delta} d_q s$$
$$= \alpha.$$

Hence, if $\alpha > 1/4$, then from Theorem 1, all solutions of (4.2) are oscillatory or converge to zero. If $\alpha \leqslant 1/4$, since

$$q_* := \liminf_{t\to\infty} \frac{\int_{t_2}^{t} \int_{t_1}^{\sigma(s)} \frac{\Delta u}{a(u)} \int_{t_1}^{s} \frac{h_{n-3}(s,\sigma(\tau)) \int_{t_1}^{\tau} \frac{\Delta v}{a(v)}}{r(\tau)} \Delta\tau p(s)\Delta s}{\int_{t_1}^{t} \frac{\Delta s}{a(s)}}$$
$$= \liminf_{t\to\infty} \frac{\int_{t_2}^{t} \int_{t_1}^{\sigma(s)} d_q v \big(\int_{t_1}^{s} (s-\sigma(\tau)) \int_{t_1}^{\tau} d_q v d_q \tau \big) \frac{\alpha}{s h_2(s,0)} d_q s}{\int_{t_1}^{t} d_q s}$$
$$\geqslant \liminf_{t\to\infty} \frac{\int_{t_2}^{t} \int_{t_1}^{\sigma(s)} d_q v h_2(s,0) \frac{\alpha}{s h_2(s,0)} d_q s}{\int_{t_1}^{t} d_q s}$$
$$\geqslant \liminf_{t\to\infty} \frac{q\alpha \int_{t_1}^{t} d_q v - \int_{t_2}^{t} \big(\frac{\alpha}{q} - \frac{\alpha}{qs} \int_{t_1}^{s} d_q v \big) d_q s}{\int_{t_1}^{t} d_q s}$$
$$\geqslant q\alpha.$$

Also note that $l^* = q$. We see that if $q_* > q/1+q$, that is if $\alpha > 1/1+q$, from Theorem 2, all solutions of (4.2) are oscillatory or converge to zero.

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REFERENCES

[1] S. HILGER, *Analysis on measure chains–a unified approach to countinuous and discrete calculus*, Results Math. **18** (1990), 18–56.
[2] M. BOHNER AND A. PETERSON, *Dynamic Equation on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
[3] G. D. BIRKHOFF, *On solutions of ordinary linear homogeneous differential equations of the third order*, Ann. of Math. **12** (1911), 103–127.
[4] C. N. REYNOLDS, JR., *On the zeros of solutions of homogeneous linear differential equations*, Trans. Amer. Math. Soc. **22** (1921), 220–229.
[5] I. T. KIGURADZE, *Oscillatory properties of solutions of certain ordinary differential equations*, Soviet Math. Dokl **3** (1962), 649–652.
[6] I. M. GLAZMAN, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators (Israel Program for Scientific Translations)*. Davey, New York, 1965.
[7] I. M. GLAZMAN, *Oscillation theorems for differential equations of high orders and the spectrum of the respective differential operators*, Dokl. Akad. Nauk SSSR [N.S.] **118** (1958), 423–426.

[8] G. V. ANAN'EVA AND V. I. BALAGANSKII, *Oscillation of the solutions of certain differential equations of high order*, Uspehi Mat. Nauk 14, No. 1 (85) (1959), 135–140.

[9] A. KNESER, *Untersuchungen uber die reelen Nullstellen der Integrale linearer Differentialgleichungen*, Math. Ann. **42** (1893), 409-435; J. Reine Angew. Math. **116** (1896), 178–212.

[10] L. ERBE, A. PETERSON AND S. H. SAKER, *Hille and Nehari type criteria for third-order dynamic equations*, J. Math. Anal. Appl. **329** (2007), 112–131.

[11] S. H. SAKER, *Oscillation of third-order functional dynamic equations on time scales*, Sci. China Math. **54** (2011), 2597–2614.

[12] R. P. AGARWAL, M. BOHNER, T. LI AND C. ZHANG, *Hille and Nehari type criteria for third-order delay dynamic equations*, J. Difference Equ. Appl. **19** (2013), 1563–1579.

[13] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equation on Time Scales*, Birkhäuser, Boston, 2003.

[14] R. P. AGARWAL, M. BOHNER, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), 3–22.

[15] T. CHANTURIA, I. KIGURADZE, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publisher, Dordrecht-Boston-London, 1993.

[16] S. R. GRACE, R. P. AGARWAL, AND A. ZAFER, *Oscillation of higher order nonlinear dynamic equations on time scales*, Adv. Difference Equ. **2012** (2012), Article ID 67, 1–18.

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