Differential
Equations
& Applications
Volume 7, Number 3 (2015), 347–361
doi:10.7153/dea-07-20

ON THE SOLUTIONS OF SOME SEMILINEAR INTEGRO–DIFFERENTIAL INCLUSIONS

AURELIAN CERNEA

(*Communicated by Rodrigo L. Pouso*)

Abstract. We consider Cauchy problems associated to first and second order semilinear integro-differential inclusions in separable Banach spaces and we establish some Filippov type existence existence results.

1. Introduction

This paper is concerned with the following semilinear differential inclusions

$$x' \in Ax + F(t,x,V(x)(t)), \quad x(0) = x_0, \tag{1.1}$$

$$x'' \in Ax + F(t,x,V(x)(t)), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{1.2}$$

where X is a real separable Banach space, $\mathscr{P}(X)$ is the family of all subsets of X, $I = [0,T]$, $F(.,.,.) : I \times X^2 \to \mathscr{P}(X)$, $x_0, y_0 \in X$ and $V : C(I,X) \to C(I,X)$ is a non-linear Volterra integral operator. In (1.1) A is the infinitesimal generator of a strongly continuous semigroup $\{G(t);\ t \geqslant 0\}$ of bounded linear operators on X and in (1.2) A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t);\ t \geqslant 0\}$ on X.

We consider both classes of semilinear inclusions because it is well known that the study of mild solutions of second order semilinear differential equations defined by the infinitesimal generator of a strongly continuous cosine family of operators is similar to the study of mild solutions of first order semilinear differential equations defined by the infinitesimal generator of a strongly continuous semigroup of operators.

In the case when F does not depend on the last variable, i.e., without Volterra integral operators, existence results and qualitative properties of the mild solutions of problem (1.1) may be found in [5, 6, 7, 9, 11] etc. and for problem (1.2) in [2, 3, 5] etc..

The present paper is motivated by a recent paper of Tatar [12] where it is considered problem (1.2) with F single valued and with $V(x)(t) = t^\gamma D^\beta x(t)$ with $\beta \in (1,2)$, $\gamma \geqslant 0$, D^β is the Riemann-Liouville fractional derivative of order β and where several existence results are provided.

Mathematics subject classification (2010): 34A60, 34A08.
Keywords and phrases: semilinear differential inclusion; mild solution; decomposable set.
This research is supported by the CNCS grant PN-II-ID-PCE-2011-3-0198.

Paper DEA-07-20

In the present paper we extend the study in [12] to the more general problem (1.2) and our aim is twofold. On one hand, we show that Filippov's ideas ([9]) can be suitably adapted in order to obtain the existence of mild solutions of problems (1.1) and (1.2). We recall that for a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([9]) consists in proving the existence of o solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. On the other hand, we prove the existence of mild solutions continuously depending on a parameter for problems (1.1) and (1.2). This result may be seen as a continuous variant of Filippov's theorem. The key tool in the proof of this theorem is a result of Bressan and Colombo ([4]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel, in Section 3 we obtain our Filippov type existence results and in Section 4 we treat the parameterized situation.

2. Preliminaries

In what follows $I=[0,T]$, X is a real separable Banach space with norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.):I\to X$ endowed with the norm $|x(.)|_C=\sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.):I\to X$ endowed with the norm $|x(.)|_1=\int_0^T|x(t)|\mathrm{d}t$. By $B(X)$ we mean the Banach space of bounded linear operators from X into X.

Denote by $\mathscr{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I and by $\mathscr{B}(X)$ the family of all Borel subsets of X. If $A\subset I$ then $\chi_A(.):I\to\{0,1\}$ denotes the characteristic function of A. For any subset $A\subset X$ we denote by $\mathrm{cl}(A)$ the closure of A.

In the sequel $V:C(I,X)\to C(I,X)$ is a nonlinear Volterra integral operator defined by $V(x)(t)=\int_0^t k(t,s,x(s))ds$ where $k(.,.,.):I\times X\times X\to X$ is a given function and $F(.,.,.):I\times X\times X\to\mathscr{P}(X)$ is a set-valued map.

Let $\{G(t)\}_{t\geqslant 0}\subset B(X)$ be a strongly continuous semigroup of bounded linear operators from X to X having the infinitesimal generator A which defines differential inclusion (1.1)

It is well known that, in general, the Cauchy problem

$$x'=Ax+f(t),\quad x(0)=x_0 \tag{2.1}$$

may not have a classical solution and that a way to overcome this difficulty is to look for continuous solutions of the integral equation

$$x(t)=G(t)x_0+\int_0^t G(t-u)f(u)du. \tag{2.2}$$

This is why the concept of the mild solution is convenient for solving (2.1).

A continuous mapping $x(.) \in C(I,X)$ is called a *mild solution* of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t,x(t),V(x)(t)) \quad a.e.\ (I) \tag{2.3}$$

and (2.2) is satisfied, i.e., $f(.)$ is a (Bochner) integrable selection of the set-valued map $F(.,x(.),V(x)(.))$ and $x(.)$ is the mild solution of the initial value problem (2.1).

We shall call $(x(.),f(.))$ a *trajectory-selection pair* of (1.1) if $f(.)$ verifies (2.3) and $x(.)$ is a mild solution of (2.1).

We shall use the following notations for the solution set of (1.1)

$$\mathscr{S}_1(x_0) = \{x(.);\quad x(.) \text{ is a mild solution of } (1.1)\}.$$

We recall that a family $\{C(t);\, t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following conditions are satisfied:

(i) $C(0) = I$, where I is the identity operator in X,

(ii) $C(t+s)+C(t-s) = 2C(t)C(s)\ \ \forall t,s \in \mathbb{R}$,

(iii) the map $t \to C(t)x$ is strongly continuous $\forall x \in X$.

The strongly continuous sine family $\{S(t);\, t \in \mathbb{R}\}$ associated to a strongly continuous cosine family $\{C(t);\ t \in \mathbb{R}\}$ is defined by $S(t)x := \int_0^t C(s)x ds$, $x \in X, t \in \mathbb{R}$.

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t);\ t \in \mathbb{R}\}$ is defined by $Ax = (\frac{d^2}{dt^2})C(t)x|_{t=0}$.

Fore more details on strongly continuous cosine and sine family of operators we refer to [8, 13].

In what follows A is infinitesimal generator of a cosine family $\{C(t);\ t \in \mathbb{R}\}$ which defines Cauchy problem (1.2).

A continuous mapping $x(.) \in C(I,X)$ is called a *mild solution* of problem (1.2) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that (2.3) is satisfied and

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)f(u)du \quad \forall t \in I,$$

i.e., $f(.)$ is a (Bochner) integrable selection of the set-valued map $F(.,x(.),$ $V(x)(.))$ and $x(.)$ is the mild solution of the Cauchy problem

$$x'' = Ax + f(t) \quad x(0) = x_0, \quad x'(0) = y_0.$$

We make the following notation

$$\mathscr{S}_2(x_0,y_0) = \{x(.);\quad x(.) \text{ is a mild solution of } (1.2)\}.$$

Finally, we recall several preliminary results we shall use in the sequel.

LEMMA 2.1. *Let X be a separable Banach space, let $H : I \to \mathscr{P}(X)$ be a measurable set-valued map with nonempty closed values and $g,h : I \to X, L : I \to (0,\infty)$ measurable functions. Then one has.*

i) The function $t \to d(h(t),H(t)$ is measurable.

ii) If $H(t) \cap (g(t)+L(t)B) \neq \emptyset$ a.e. (I) then the set-valued map $t \to H(t) \cap (g(t)+L(t)B)$ has a measurable selection.

Its proof may be found in [1].

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot),v(\cdot) \in D$ and any subset $A \in \mathscr{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$. We denote by $\mathscr{D}(I,X)$ the family of all decomposable closed subsets of $L^1(I,X)$.

Next (S,d) is a separable metric space; we recall that a set-valued map $G(\cdot) : S \to \mathscr{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed. The proof of the next two lemmas may be found in [4].

LEMMA 2.2. *Let $F^*(.,.) : I \times S \to \mathscr{P}(X)$ be a closed-valued $\mathscr{L}(I) \otimes \mathscr{B}(S)$ measurable set-valued map such that $F^*(t,.)$ is l.s.c. for any $t \in I$.*

Then the set-valued map $G(.) : S \to \mathscr{D}(I,X)$ defined by

$$G(s) = \{v \in L^1(I,X); \quad v(t) \in F^*(t,s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p(.) : S \to L^1(I,X)$ such that

$$d(0,F^*(t,s)) \leqslant p(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

LEMMA 2.3. *Let $G(.) : S \to \mathscr{D}(I,X)$ be a l.s.c. set-valued map with closed decomposable values and let $\phi(.) : S \to L^1(I,X)$, $\psi(.) : S \to L^1(I,\mathbb{R})$ be continuous such that the set-valued map $H(.) : S \to \mathscr{D}(I,X)$ defined by*

$$H(s) = cl\{v \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then H has a continuous selection, i.e. there exists a continuous mapping $h : S \to L^1(I,X)$ such that $h(s) \in H(s) \quad \forall s \in S$.

3. A Filippov type result

In order to establish our existence result for problem (1.1) we need the following hypotheses.

Hypothesis 1. i) *$F(.,.,.) : I \times X \times X \to \mathscr{P}(X)$ has nonempty closed values and is $\mathscr{L}(I) \otimes \mathscr{B}(X \times X)$ measurable.*

ii) *There exists $L(.) \in L^1(I,R_+)$ such that, for almost all $t \in I, F(t,.,.)$ is L(t)-Lipschitz in the sense that for almost $t \in I$*

$$d(F(t,x_1,y_1),F(t,x_2,y_2)) \leqslant L(t)(|x_1 - x_2| + |y_1 - y_2|)\ \forall\, x_1,x_2,y_1,y_2 \in X,$$

where d(A,B) is the Hausdorff distance

$$d(A,B) = \max\{d^*(A,B),d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\}.$$

iii) *$k(.,.,.) : I \times X \times X \to X$ satisfy: $\forall x \in X$, $(t,s) \to k(t,s,x)$ is measurable and $|k(t,s,x) - k(t,s,y)| \leqslant L(t)|x - y|$ a.e.$(t,s) \in I \times I$, $\forall x,y \in X$.*

We shall use next the following notations

$$m(t)=\int_0^t L(u)du, \quad \alpha(x)=\frac{(x+1)^2-1}{2}, \quad x\in\mathbb{R}.$$

In what follows $u_0\in X$, $g(.)\in L^1(I,X)$ and $y(.)\in C(I,X)$ is a mild solution of the Cauchy problem

$$y'=Ay+g(t) \quad y(0)=u_0,$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{G(t);t\geqslant 0\}$ of bounded linear operators on X .

Let $M_1\geqslant 1$ be such that $|G(t)|\leqslant M_1 \quad \forall t\in I$.

Hypothesis 2. i) *Hypothesis 1 is satisfied.*

ii) *A is the infinitesimal generator of a strongly continuous semigroup $\{G(t);\ t\geqslant 0\}$ of bounded linear operators on X.*

iii) *The function $t\to p(t):=d(g(t),F(t,y(t),V(y)(t))$ is integrable on I.*

THEOREM 3.1. *Consider $\delta\geqslant 0$ and assume that Hypothesis 2 is satisfied. Then for any $x_0\in X$ with $|x_0-u_0|\leqslant\delta$ and any $\varepsilon>0$ there exists $(x(.),f(.))$ a trajectory-selection pair of (1.1) such that*

$$|x(t)-y(t)|\leqslant\xi(t) \quad \forall\, t\in I,$$

$$|f(t)-g(t)|\leqslant L(t)(\xi(t)+\int_0^t L(u)\xi(u)du)+\gamma(t)+\varepsilon \quad a.e.\ (I),$$

where

$$\xi(t)=\delta e^{M_1\alpha(m(t))}+\int_0^t p(u)e^{M_1\alpha(m(t)-m(u))}du+M_1t\varepsilon.$$

Proof. Let $\varepsilon>0$ and set $x_0(t)\equiv y(t)$, $f_0(t)\equiv g(t)$, $t\in I$,

$$p_n(t)=\int_0^t p(u)\frac{(\alpha(m(t)-m(u))^{n-1}}{(n-1)!}du+\frac{(\alpha(m(t))^{n-1}}{(n-1)!}|x_0-u_0|, n\geqslant 1.$$

We claim that is enough to construct the sequences $x_n(.)\in C(I,X)$, $f_n(.)\in L^1(I,X)$, $n\geqslant 1$ with the following properties

$$x_n(t)=G(t)x_0+\int_0^t G(t-s)f_n(s)ds, \quad \forall t\in I, \tag{3.1}$$

$$|x_1(t)-x_0(t)|\leqslant M_1(\delta+\int_0^t p(u)du+\varepsilon t)=:p_0(t) \quad \forall t\in I, \tag{3.2}$$

$$|f_1(t)-f_0(t)|\leqslant p(t)+\varepsilon \quad a.e.\ (I), \tag{3.3}$$

$$f_n(t)\in F(t,x_{n-1}(t),V(x_{n-1})(t)) \quad a.e.\ (I),\ n\geqslant 1, \tag{3.4}$$

$$|f_{n+1}(t)-f_n(t)|\leqslant L(t)(|x_n(t)-x_{n-1}(t)|+\int_0^t L(u)|x_n(u)-x_{n-1}(u)|du)\ a.e., \tag{3.5}$$

$$|x_n(t)-x_{n-1}(t)|\leqslant M_1^{n-1}p_n(t)\quad \forall t\in I. \tag{3.6}$$

Indeed, from (3.6) $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,X)$. Thus, from (3.5) for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Moreover, from (3.2) and the last inequality we have

$$|x_n(t)-y(t)|\leqslant \sum_{i=0}^{n-1}|x_{i+1}(t)-x_i(t)|\leqslant \sum_{i=0}^{n-1}M_1^i p_{i+1}(t)\leqslant \xi(t) \tag{3.7}$$

On the other hand, from (3.3), (3.5) and (3.6) we obtain for almost all $t\in I$

$$\begin{aligned}&|f_n(t)-g(t)|\\ &\quad\leqslant \textstyle\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-g(t)|\\ &\quad\leqslant L(t)(\xi(t)+\int_0^t L(u)\xi(u)du)+\gamma(t)+\varepsilon.\end{aligned} \tag{3.8}$$

Let $x(.)\in C(I,X)$ be the limit of the Cauchy sequence $x_n(.)$. From (3.8) the sequence $f_n(.)$ is integrably bounded and we have already proved that for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Take $f(.)\in L^1(I,X)$ with $f(t)=\lim_{n\to\infty}f_n(t)$.

Passing to the limit in (3.1) and using Lebesgue's dominated convergence theorem we get (2.2). Finally, passing to the limit in (3.7) and (3.8) we obtained the desired estimations.

It remains to construct the sequences $x_n(.),f_n(.)$ with the properties in (3.1)-(3.6). The construction will be done by induction.

The set-valued map $t\to F(t,y(t),V(y)(t))$ is measurable with closed values and

$$F(t,y(t),V(y)(t))\cap\{g(t)+(p(t)+\varepsilon)B\}\neq\emptyset\quad a.e.\ (I).$$

From Lemma 2.1 we find $f_1(.)$ a measurable selection of the set-valued map $H_1(t):=F(t,y(t),V(y)(t))\cap\{g(t)+(p(t)+\varepsilon)B\}$. Obviously, $f_1(.)$ satisfy (3.3). Define $x_1(.)$ as in (3.1) with $n=1$. Therefore, we have

$$\begin{aligned}|x_1(t)-y(t)|&\leqslant |G(t)(x_0-u_0)|+|\int_0^t G(t-s)(f_1(s)-g(s))ds|\\ &\leqslant M_1\delta+M\int_0^t(p(s)+\varepsilon)ds=p_0(t).\end{aligned}$$

Assume that for some $N\geqslant 1$ we already constructed $x_n(.)\in C(I,X)$ and $f_n(.)\in L^1(I,X), n=1,2,...N$ satisfying (3.1)-(3.6). We define the set-valued map

$$\begin{aligned}H_{N+1}(t):=F(t,x_N(t),V(x_N)(t))\cap\Big\{&f_N(t)+L(t)(|x_N(t)-x_{N-1}(t)|\\ &+\int_0^t L(u)|x_N(u)-x_{N-1}(u)|duB\Big\},\quad t\in I.\end{aligned}$$

The set-valued map $t\to F(t,x_N(t),V(x_N)(t))$ is measurable and from the lipschitzianity of $F(t,.,.)$ we have that for almost all $t\in I$ $H_{N+1}(t)\neq\emptyset$. We apply Lemma

2.1 and find a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.),V(x_N)(.))$ such that for almost $t\in I$

$$|f_{N+1}(t)-f_N(t)|\leqslant L(t)\left(|x_N(t)-x_{N-1}(t)|+\int_0^t L(u)|x_N(u)-x_{N-1}(u)|du\right).$$

We define $x_{N+1}(.)$ as in (3.1) with $n=N+1$ and we get

$$\begin{aligned}|x_{N+1}(t)-x_N(t)| &\leqslant M_1\int_0^t |f_{N+1}(u)-f_N(u)|du\\ &\leqslant M_1\int_0^t L(u)\bigg(|x_N(u)-x_{N-1}(u)|\\ &\qquad+\int_0^u L(s)|x_N(s)-x_{N-1}(s)|ds\bigg)du\\ &\leqslant M_1\int_0^t L(u)\left(M_1^{N-1}p_N(u)+\int_0^u L(s)M_1^{N-1}p_N(r)dr\right)du.\end{aligned}$$

We shall prove next that

$$\int_0^t L(u)\left(p_n(u)+\int_0^u L(r)p_n(r)dr\right)du\leqslant p_{n+1}(t) \tag{3.9}$$

and therefore (3.6) holds true with $n=N+1$ which completes the proof.

One has

$$\begin{aligned}&\int_0^t L(u)(p_n(u)+\int_0^u L(r)p_n(r)dr)du\\ &\quad=\int_0^t (1+m(t)-m(u))L(u)p_n(u)du\\ &\quad=\int_0^t (1+m(t)-m(u))L(u)\frac{(\alpha(m(u))^{n-1}}{(n-1)!}|x_0-u_0|du\\ &\qquad+\int_0^t (1+m(t)-m(u))L(u)\left(\int_0^u p(r)\frac{(\alpha(m(t)-m(r))^{n-1}}{(n-1)!}dr\right)du\\ &\quad\leqslant |x_0-u_0|\int_0^t (1+m(t)-m(u))L(u)\frac{(\alpha(m(u)))^{n-1}}{(n-1)!}\\ &\qquad+\int_0^t\left(\int_r^t \frac{(\alpha(m(u)-m(r)))^{n-1}}{(n-1)!}(1+m(t)-m(u))L(u)\right)p(r)drdu.\end{aligned}$$

According to the definition of $\alpha(.)$ we have

$$\begin{aligned}&\int_0^t (1+m(t)-m(u))L(u)\frac{\alpha(m(u)))^{n-1}}{(n-1)!}du\\ &\quad=\int_0^t (2+m(t))L(u)\frac{(\alpha(m(u))^{n-1}}{(n-1)!}du\frac{(\alpha(m(t))^{n}}{n!}\end{aligned}$$

$$\leqslant (m(t)+2)\frac{(m(t)/2+1)^{n-1}}{(n-1)!}\int_0^t (m(u))^{n-1}L(u)du - \frac{(\alpha(m(t)))^n}{n!}$$
$$= \frac{(\alpha(m(t)))^n}{n!}.$$

As above we deduce that

$$\int_r^t \frac{(\alpha(m(u)-m(r)))^{n-1}}{(n-1)!}(1+m(t)-m(u))L(u)du \leqslant \frac{(\alpha(m(t)-m(r)))^n}{n!}$$

and inequality (3.9) is proved.

REMARK 3.1. In the particular case when F does not depend on the last variable, problem (1.1) reduces to the semilinear differential inclusion

$$x' \in Ax + F(t,x), \quad x(0) = x_0, \tag{3.10}$$

and the corresponding Filippov type theorem for the mild solutions of problem (3.10) may be found in [10].

EXAMPLE 3.1. Consider the following Cauchy problem

$$x' \in ax + [0, t\sin(\int_0^t x(s)ds)], \quad x(0) = x_0,$$

where $a, x_0 \in \mathbb{R}$ and $I = [0,1]$. We take

$$V(x)(t) = \int_0^t x(s)ds, \; F(t,z) = [0, t\sin z],$$
$$G(t) = e^{at} \text{ and } g = y = 0.$$

We have

$$\sup\{|u|; u \in F(t,z)\} \leqslant |t| \leqslant 1 \text{ and } d_H(F(x,z_1), F(x,z_2)) \leqslant |z_1 - z_2|, \; \forall z_1, z_2 \in \mathbb{R}.$$

Therefore, we apply Theorem 3.1 in order to obtain the existence of a solution for the problem considered.

Next we consider $u_0, v_0 \in X$, $g(.) \in L^1(I,X)$ and $y(.) \in C(I,X)$ is a mild solution of the Cauchy problem

$$y'' = Ay + g(t) \quad y(0) = u_0, \quad y'(0) = v_0,$$

where A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t);\, t \in \mathbb{R}\}$ on X.

Let $M_2 \geqslant 0$ be such that $|C(t)| \leqslant M_2 \;\; \forall\, t \in I$. Note that $|S(t)| \leqslant M_2 t \;\; \forall\, t \in I$.

Hypothesis 3. i) *Hypothesis 1 is satisfied.*

ii) *A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t);\, t \in \mathbb{R}\}$ on X.*

iii) *The function $t \to p(t) := d(g(t), F(t, y(t), V(y)(t))$ is integrable on I.*

THEOREM 3.2. *Consider $\delta \geqslant 0$ and assume that Hypothesis 3 is satisfied. Then for any $x_0, y_0 \in X$ with $M_2(|x_0 - u_0| + T|y_0 - v_0|) \leqslant \delta$ and any $\varepsilon > 0$ there exists $(x(.), f(.))$ a trajectory-selection pair of (1.2) such that*

$$|x(t) - y(t)| \leqslant \xi(t) \quad \forall\, t \in I,$$

$$|f(t) - g(t)| \leqslant L(t)(\xi(t) + \int_0^t L(u)\xi(u)du) + \gamma(t) + \varepsilon \quad a.e.\ (I),$$

where

$$\xi(t) = \delta e^{M_2 T\alpha(m(t))} + \int_0^t p(u)e^{M_2 T\alpha(m(t)-m(u))}du + M_2 T t\varepsilon.$$

Proof. The proof is similar to the one of Theorem 3.1.

EXAMPLE 3.2. Consider the following Cauchy problem

$$x'' \in -a^2 x + \left[0, t\sin(\int_0^t x(s)ds)\right], \quad x(0) = x_0, \quad x'(0) = y_0,$$

where $a, x_0, y_0 \in \mathbb{R}$, $a \neq 0$ and $I = [0,1]$. We take

$$V(x)(t) = \int_0^t x(s)ds,\ F(t,z) = [0, t\sin z],$$
$$C(t) = \cos(at),\ S(t) = \frac{\sin(at)}{a} \text{ and } g = y = 0.$$

As in Example 3.1 the assumptions of Theorem 3.2 are satisfied; hence we deduce the existence of a solution for the problem considered.

4. Continuous family of solutions

In order to establish our continuous version of Filippov theorem for problem (1.1) we need the following hypotheses.

Hypothesis 4. i) *A is the infinitesimal generator of a strongly continuous semigroup $\{G(t);\ t \geqslant 0\}$ of bounded linear operators on X.*

ii) *S is a separable metric space and $a(.) : S \to X$, $c(.) : S \to (0,\infty)$ are continuous mappings.*

(ii) *There exists the continuous mappings $g(.) : S \to L^1(I,X)$, $p(.) : S \to \mathbb{R}$, $y(.) : S \to C(I,X)$ such that*

$$(y(s))'(t) = Ay(s)(t) + g(s)(t) \quad \forall s \in S, t \in I$$

and

$$d(g(s)(t), F(t, y(s)(t), V(y(s))(t)) \leqslant p(s)(t) \quad a.e.\ (I),\ \forall\, s \in S.$$

THEOREM 4.1. *Assume that Hypotheses 1 and 4 are satisfied.*

Then there exist the continuous mappings $x(.) : S \to C(I,X)$, $f(.) : S \to L^1(I,X)$ *such that for any* $s \in S$, $(x(s)(.), f(s)(.))$ *is a trajectory-selection pair of*

$$x' \in Ax + F(t,x,V(x)(t)), \quad x(0) = a(s)$$

and

$$|x(s)(t) - y(s)(t)| \leqslant \xi(s)(t) \quad \forall (t,s) \in I \times S, \tag{4.1}$$

$$|f(s)(t) - g(s)(t)| \leqslant L(t)(\xi(s,t) + \int_0^t L(u)\xi(s,u)du) + p(s)(t) + c(s) \; a.e.\,(I), \tag{4.2}$$

$\forall s \in S$, *where*

$$\xi(s,t) = M_1 e^{M_1\alpha(m(t))}[tc(s) + |a(s) - y(s)(0)|] + \int_0^t p(s)(u) e^{M_1\alpha(m(t)-m(u))} du.$$

Proof. We denote

$$\varepsilon_n(s) = c(s)\frac{n+1}{n+2},\; n = 0,1,\ldots,\; d(s) = M_1|a(s) - y(s)(0)|,$$

and

$$p_n(s)(t) = M_1^n \int_0^t p(s)(u)\frac{(m(t)-m(u))^{n-1}}{(n-1)!}du + M_1^{n-1}\frac{(m(t))^{n-1}}{(n-1)!}(M_1 t\varepsilon_n(s) + d(s)), \quad n \geqslant 1.$$

Set also $x_0(s)(t) = y(s)(t)$, $f_0(s)(t) = g(s)(t)$, $\forall s \in S$.

We consider the set-valued maps $G_0(.), H_0(.)$ defined, respectively, by

$$G_0(s) = \{v \in L^1(I,X); \quad v(t) \in F(t,y(s)(t),V(y(s))(t)) \quad a.e.\,(I)\},$$

$$H_0(s) = \mathrm{cl}\{v \in G_0(s); \quad |v(t) - g(s)(t)| < p(s)(t) + \varepsilon_0(s)\}.$$

Since

$$d(g(s)(t), F(t,y(s)(t),V(y(s))(t)) \leqslant p(s)(t) < p(s)(t) + \varepsilon_0(s),$$

according with Lemma 2.1, the set $H_0(s)$ is not empty.

Set $F_0^*(t,s) = F(t,y(s)(t),V(y(s))(t))$ and note that

$$d(0, F_0^*(t,s)) \leqslant |g(s)(t)| + p(s)(t) = p^*(s)(t)$$

and $p^*(.) : S \to L^1(I,X)$ is continuous.

Applying now Lemmas 2.2 and 2.3 we obtain the existence of a continuous selection f_0 of H_0, i.e. such that

$$f_0(s)(t) \in F(t,y(s)(t),V(y(s))(t)) \quad a.e.\,(I),\; \forall s \in S,$$

$$|f_0(s)(t)-g(s)(t)|\leqslant p_0(s)(t)=p(s)(t)+\varepsilon_0(s)\quad \forall s\in S,\ t\in I.$$

We define

$$x_1(s)(t)=G(t)a(s)+\int_0^t G(t-u)f_0(s)(u)du$$

and one has

$$\begin{aligned}|x_1(s)(t)-x_0(s)(t)| &\leqslant M_1|a(s)-y(s)(0)|+M_1\int_0^t|f_0(s)(u)-g(s)(u)|du\\ &\leqslant d(s)+M_1\int_0^t(p(s)(u)+\varepsilon_0(s))du=p_1(s)(t).\end{aligned}$$

We shall construct two sequences of approximations $f_n(.):S\to L^1(I,X)$, $x_n(.):S\to C(I,X)$ with the following properties:

a) $f_n(.):S\to L^1(I,X)$, $x_n(.):S\to C(I,X)$ are continuous.

b) $f_n(s)(t)\in F(t,x_n(s)(t),V(x_n(s))(t))$, a.e. (I), $s\in S$.

c) $|f_n(s)(t)-f_{n-1}(s)(t)|\leqslant L(t)(p_n(s)(t)+\int_0^t L(u)p_n(s)(u)du)$, a.e. (I), $s\in S$.

d) $x_{n+1}(s)(t)=G(t)a(s)+\int_0^t G(t-u)f_n(s)(u)du$, $\forall t\in I, s\in S$.

Suppose we have already constructed $f_i(.), x_i(.)$, $i=1,\dots,n$ satisfying a)-c) and define $x_{n+1}(.)$ as in d). As in the proof of inequality (3.9) we have

$$\int_0^t L(u)(p_n(s)(u)+\int_0^u L(r)p_n(s)(r)dr)du\leqslant p_{n+1}(s)(t)-\frac{c(s)(\alpha(m(t)))^n t}{(n+2)(n+3)n!}. \quad (4.3)$$

From c) and d) one has

$$\begin{aligned}&|x_{n+1}(s)(t)-x_n(s)(t)|\\ &\quad\leqslant M_1\textstyle\int_0^t|f_n(s)(u)-f_{n-1}(s)(u)|du\\ &\quad\leqslant M_1\textstyle\int_0^t L(u)(p_n(s)(u)+\int_0^u L(r)p_n(s)(r)dr)du<p_{n+1}(s)(t).\end{aligned} \quad (4.4)$$

Consider the following set-valued maps, for any $s\in S$,

$$G_{n+1}(s)=\left\{v\in L^1(I,X);\quad v(t)\in F(t,x_{n+1}(s)(t),V(x_{n+1}(s))(t))\quad a.e.(I)\right\},$$

and

$$\begin{aligned}H_{n+1}(s)=\mathrm{cl}\Big\{v\in G_{n+1}(s);\quad &|v(t)-f_n(s)(t)|<L(t)(p_n(s)(t)\\ &+\int_0^t L(u)p_n(s)(u)du)\quad a.e.(I)\Big\}.\end{aligned}$$

To prove that $H_{n+1}(s)$ is nonempty we note first that the real function

$$t\to r_n(s)(t)=c(s)\frac{(MT)^{n+1}tL(t)(m(t))^n}{(n+2)(n+3)n!}$$

is measurable and strictly positive for any s. From (4.3) we get

$$\begin{aligned} & d(f_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \\ & \quad \leqslant L(t)\left(|x_n(s)(t) - x_{n+1}(s)(t)| + \int_0^t L(u)|x_n(s)(u) - x_{n+1}(s)(u)|du\right) \\ & \quad \leqslant L(t)\left(p_n(s)(t) + \int_0^t L(u)p_n(s)(u)du\right) - r_n(s)(t) \end{aligned}$$

and therefore according to Lemma 2.1 there exists $v(.) \in L^1(I,X)$ such that $v(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t))$ a.e. (I) and

$$|v(t) - f_n(s)(t)| < d(f_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) + r_n(s)(t)$$

and hence $H_{n+1}(s)$ is not empty.

Set $F^*_{n+1}(t,s) = F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t))$ and note that we may write

$$\begin{aligned} d(0, F^*_{n+1}(t,s)) & \leqslant |f_n(s)(t)| + L(t)(p_{n+1}(s)(t) + \int_0^t L(u)p_{n+1}(s)(u)du) \\ & = p^*_{n+1}(s)(t) \quad a.e.\,(I) \end{aligned}$$

and $p^*_{n+1}(.) : S \to L^1(I,X)$ is continuous.

By Lemmas 2.2 and 2.3 there exists a continuous map $f_{n+1}(.) : S \to L^1(I,X)$ such that for any $s \in S$

$$f_{n+1}(s)(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \quad a.e.\,(I),$$

$$|f_{n+1}(s)(t) - f_n(s)(t)| \leqslant L(t)(p_{n+1}(s)(t) + \int_0^t L(u)p_{n+1}(s)(u)du) \quad a.e.\,(I).$$

From (4.4) and d) we obtain

$$\begin{aligned} & |x_{n+1}(s)(.) - x_n(s)(.)|_C \\ & \quad \leqslant M_1|f_{n+1}(s)(.) - f_n(s)(.)|_1 \\ & \quad \leqslant \tfrac{(M_1\alpha(m(T)))^n}{n!}(M_1|p(s)(.)|_1 + M_1Tc(s) + d(s)). \end{aligned} \tag{4.5}$$

Therefore $f_n(s)(.)$, $x_n(s)(.)$ are Cauchy sequences in the Banach space $L^1(I,X)$ and $C(I,X)$, respectively. Let $f(.) : S \to L^1(I,X)$, $x(.) : S \to C(I,X)$ be their limits. The function $s \to M_1|p(s)(.)|_1 + M_1Tc(s) + d(s)$ is continuous, hence locally bounded. Therefore (4.5) implies that for every $s' \in S$ the sequence $f_n(s')(.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighborhood of s. Hence, $s \to f(s)(.)$ is continuous from S into $L^1(I,X)$.

From (4.5), as before, $x_n(s)(.)$ is Cauchy in $C(I,X)$ locally uniformly with respect to s. So, $s \to x(s)(.)$ is continuous from S into $C(I,X)$. On the other hand, since $x_n(s)(.)$ converges uniformly to $x(s)(.)$ and

$$d(f_n(s)(t), F(t, x(s)(t), V(x(s))(t))$$

$$\leqslant L(t)(|x_n(s)(t)-x(s)(t)|+\int_0^t L(u)|x_n(s)(u)-x(s)(u)|du) \quad a.e.\ (I),\ \forall s\in S$$

passing to the limit along a subsequence of $f_n(.)$ converging pointwise to $f(.)$ we obtain

$$f(s)(t)\in F(t,x(s)(t),V(x(s))(t)) \quad a.e.\ (I),\ \forall s\in S.$$

Passing to the limit in d) we obtain

$$x(s)(t)=G(t)a(s)+\int_0^t G(t-u)f(s)(u)du.$$

By adding inequalities c) for all n and using the fact that $\sum_{i\geqslant 1} p_i(s)(t)\leqslant \xi(s)(t)$ we obtain

$$\begin{aligned} &|f_{n+1}(s)(t)-g(s)(t)|\\ &\quad\leqslant \textstyle\sum_{l=0}^n |f_{l+1}(s)(u)-f_l(s)(u)|+|f_0(s)(t)-g(s)(t)|\\ &\quad\leqslant \textstyle\sum_{l=0}^n L(t)p_{l+1}(s)(t)+p(s)(t)+\varepsilon_0(s)\\ &\quad\leqslant L(t)\xi(s)(t)+p(s)(t)+c(s). \end{aligned} \tag{4.6}$$

Similarly, by adding (4.4) we get

$$|x_{n+1}(s)(t)-y(s)(t)|\leqslant \sum_{l=0}^n p_l(s)(t)\leqslant \xi(s)(t). \tag{4.7}$$

By passing to the limit in (4.6) and (4.7) we obtain (4.1) and (4.2), respectively.

Theorem 4.2 allows to obtain the next corollary which is a general result concerning continuous selections of the solution set of problem (1.1).

Hypothesis 5. *Hypothesis 1 is satisfied and there exists* $p_0(.)\in L^1(I,\mathbb{R}_+)$ *such that* $d(0,F(t,0,V(0)(t)))\leqslant p_0(t)$ *a.e.* (I).

THEOREM 4.2. *Assume that Hypothesis 5 is satisfied.*
Then there exists a function $x(.,.): I\times X\to X$ *such that*
a) $x(.,\xi)\in \mathscr{S}_1(\xi),\ \forall \xi\in X$.
b) $\xi\to x(.,\xi)$ *is continuous from* X *into* $C(I,X)$.

Proof. We take $S=X$, $a(\xi)=\xi$, $\forall \xi\in X$, $c(.):X\to(0,\infty)$ an arbitrary continuous function, $g(.)=0$, $y(.)=0$, $p(\xi)(t)=p_0(t)\ \forall \xi\in X$, $t\in I$ and we apply Theorem 4.1 in order to obtain the conclusion of the theorem.

REMARK 4.1. In the particular case when F does not depend on the last variable, problem (1.1) reduces to the semilinear differential inclusion (3.10) and a corresponding continuous selection of the solution set of problem (3.10) is obtained in [11].

Finally, we consider problem (1.2).

Hypothesis 6. i) *A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t);\ t\in\mathbb{R}\}$ on X..*

ii) *S is a separable metric space and $a(.),b(.):S\to X,\ c(.):S\to(0,\infty)$ are continuous mappings.*

(ii) *There exists the continuous mappings $g(.):S\to L^1(I,X),\ p(.):S\to\mathbb{R},\ y(.):S\to C(I,X)$ such that*

$$(y(s))''(t)=Ay(s)(t)+g(s)(t)\quad \forall s\in S,t\in I$$

and

$$d(g(s)(t),F(t,y(s),V(y(s))(t))\leqslant p(s)(t)\quad a.e.\ (I),\ \forall\ s\in S.$$

THEOREM 4.3. *Assume that Hypotheses 1 and 6 are satisfied.*

Then there exist the continuous mappings $x(.):S\to C(I,X),\ f(.):S\to L^1(I,X)$ such that for any $s\in S,\ (x(s)(.),f(s)(.))$ is a trajectory-selection pair of

$$x''\in Ax+F(t,x),\quad x(0)=a(s),\quad x'(0)=b(s)$$

and

$$|x(s)(t)-y(s)(t)|\leqslant\xi(s)(t)\quad\forall(t,s)\in I\times S,$$

$$|f(s)(t)-g(s)(t)|\leqslant L(t)\xi(s)(t)+p(s)(t)+c(s)\quad a.e.\ (I),\ \forall s\in S,$$

where

$$\xi(s,t)=M_2e^{M_2T\alpha(m(t))}[tc(s)+|a(s)-y(s)(0)|+T|b(s)-(y(s))'(0)|]\\ +\int_0^t p(s)(u)e^{M_2T\alpha(m(t)-m(u))}du.$$

Proof. The proof is similar to the one of Theorem 4.1.

THEOREM 4.4. *Assume that Hypothesis 5 is satisfied.*

Then there exists a function $x(.,.):I\times X^2\to X$ such that

a) $x(.,(\xi,\eta))\in\mathscr{S}_2(\xi,\eta),\ \forall(\xi,\eta)\in X^2$.

b) $(\xi,\eta)\to x(.,(\xi,\eta))$ is continuous from X^2 into $C(I,X)$.

Proof. We take

$$S=X\times X,\ a(\xi,\eta)=\xi,\ b(\xi,\eta)=\eta,\ \forall(\xi,\eta)\in X\times X,$$

$c(.):X\times X\to(0,\infty)$ an arbitrary continuous function, $g(.)=0,\ y(.)=0,\ p(\xi,\eta)(t)=p_0(t)\ \forall(\xi,\eta)\in X\times X,\ t\in I$ and we apply Theorem 4.3 in order to obtain the conclusion of the theorem.

REFERENCES

[1] J.P. AUBIN AND H. FRANKOWSKA, *Set-valued Analysis*, Birkhauser, Basel, 1990.

[2] M. BENCHOHRA, *On second-order differential inclusions in Banach spaces*, New Zealand J. Math. **30**, (2001), 5–13.

[3] M. BENCHOHRA AND S. K. NTOUYAS, *Existence of mild solutions of second-order initial value problems for differential inclusions with nonlocal conditions*, Atti. Sem. Mat. Fis. Univ. Modena, **49**, (2001), 351–361.

[4] A. BRESSAN AND G. COLOMBO, *Extensions and selections of maps with decomposable values*, Studia Math. **90**, (1988), 69–86.

[5] A. CERNEA, *On the solution set of some classes of nonconvex nonclosed differential inclusions*, Portugaliae Math. **65**, (2008), 485–496.

[6] F. S. DE BLASI AND G. PIANIGIANI, *Evolution inclusions in non separable Banach spaces*, Comment. Math. Univ. Carolinae, **40**, (1999), 227–250.

[7] F. S. DE BLASI, G. PIANIGIANI AND V. STAICU, *Topological properties of nonconvex differential inclusions of evolution type*, Nonlinear Analysis **24**, (1995), 711–720.

[8] O. FATTORINI, *Second-order linear differential equations in Banach spaces*, Mathematical Studies **108**, North Holland, Amsterdam, 1985.

[9] A.F. FILIPPOV, *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control **5**, (1967), 609–621.

[10] H. FRANKOWSKA, *A priori estimates for operational differential inclusions*, J. Diff. Equations, **84**, (1990), 100–128.

[11] V. STAICU, *Continuous selections of solutions sets to evolutions equations*, Proc. Amer. Math. Soc. **113**, (1991), 403–414.

[12] N. TATAR, *On a second-order differential problem with fractional derivative of order greater than one*, Math. Modelling Analysis **18**, (2015), 53–65.

[13] C. C. TRAVIS AND G. F. WEBB, *Cosine families and abstract nonlinear second-order differential equations*, Acta Math. Hungarica, **32**, (1978), 75–96.

(Received April 2, 2015)

Aurelian Cernea
Faculty of Mathematics and Computer Science
University of Bucharest, Academiei 14
010014 Bucharest, Romania
e-mail: acernea@fmi.unibuc.ro