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POSITIVE SOLUTIONS FOR A SINGULAR THIRD ORDER BOUNDARY VALUE PROBLEM

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Abstract. The existence of positive solutions is shown for the third order boundary value problem, $u''' = f(x,u), 0 < x < 1,\ u(0) = u(1) = u''(1) = 0,$ where $f(x,y)$ is singular at $x = 0$, $x = 1$, $y = 0$, and may be singular at $y = \infty$. The method involves application of a fixed point theorem for operators that are decreasing with respect to a cone.

1. Introduction

Singular boundary value problems for ordinary differential equations, often times of the second order and involving semi-infinite intervals, for which there are positive solutions are often used to model applications, such as, glacial advance and transport of coal slurries down conveyor belts as examples of nonNewtonian fluid theory in studies of pseudoplastic fluids [9], for problems involving draining flows [1, 5] and semipositone and positone problems [2], and as models in boundary layer applications, Emden-Fowler boundary value problems, and reaction-diffusion applications [6, 7, 8, 18]. In addition, there is a large literature for semi-linear boundary value problems for bounded domains Ω in any space of dimension $N > 1$, for second order differential operators (such as the Laplacian $-\Delta$) with nonlinearities $f(x,u)$ which are singular both in u (when u goes to 0) and in x (when $d(x) = d(x,\Omega)$ goes to zero); see, for example [17] and the references therein.

Moreover, much theoretical interest has been given to singular boundary value problems for ordinary differential equations. For several of these studies, see [4, 14, 15, 21, 22, 23, 25, 26, 27]. In this paper, our methods involve applying a fixed point theorem by Gatica, Oliker and Waltman [11] for operators that are decreasing with respect to a cone. This method has been used to obtain positive solutions for other singular boundary value problems by Eloe and Henderson [10], Henderson and Yin [16], Maroun [19, 20] and Singh [24]. Fundamental to our obtaining positive solutions of (1)-(2) is a positivity result by Graef and Yang [12, 13].

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In this paper, we establish the existence of positive solutions for the singular third order boundary value problem,

$$u''' = f(x,u), \quad 0 < x < 1, \tag{1}$$

$$u(0) = u(1) = u''(1) = 0, \tag{2}$$

where $f(x,y)$ is singular at $x = 0, 1,\ y = 0$, and may be singular at $y = \infty$.

We assume the following conditions on f:

(H1) $f(x,y) : (0,1) \times (0,\infty) \to (0,\infty)$ is continuous, and $f(x,y)$ is decreasing in y, for every x.

(H2) $\lim_{y \to 0^+} f(x,y) = +\infty$ and $\lim_{y \to +\infty} f(x,y) = 0$ uniformly on compact subsets of $(0,1)$.

We note that the function $f(x,y) := \frac{1}{\sqrt[3]{x(1-x)y}}$ satisfies (H1) and (H2).

We will convert the problem (1)-(2) into an integral equation problem, from which we define a sequence of decreasing integral operators associated with a sequence of perturbed integral equations. Applications of a Gatica, Oliker and Waltman fixed point theorem yield a sequence of fixed points of the integral operators. A solution of (1)-(2) is then obtained from a subsequence of the fixed points.

2. Definitions, cone properties and the Gatica, Oliker and Waltman fixed point theorem

In this section, we state some definitions and properties of Banach space cones, and we state the fixed point theorem on which the paper's main result depends.

Let $(B, ||\cdot||)$ be a real Banach space. A nonempty closed $K \subset B$ is called a *cone* if the following hold:

(i) $\alpha u + \beta v \in K$, for all $u, v \in K$, and for all $\alpha, \beta \in [0,\infty)$.

(ii) $K \cap (-K) = \{0\}$.

Given a cone K, a *partial order*, $\leqslant$, is induced on B by $x \leqslant y$, for $x, y \in B$ if, and only if, $y - x \in K$. (We sometimes will write $x \leqslant y$ (w.r.t. K).) If $x, y \in B$ with $x \leqslant y$, let $\langle x, y\rangle$ denote the *closed order interval between* x *and* y and be defined by, $\langle x, y\rangle := \{z \in B \mid x \leqslant z \leqslant y\}$. A cone K is *normal* in B provided there exists a $\delta > 0$ such that $||e_1 + e_2|| \geqslant \delta$, for all $e_1, e_2 \in K$ with $||e_1|| = ||e_2|| = 1$.

REMARK 1. If K is a normal cone in B, then closed order intervals are norm bounded.

We now state the Gatica, Oliker and Waltman [11] fixed point theorem on which the main result of this paper depends.

THEOREM 1. *Let B be a Banach space, K a normal cone, J a subset of K such that, if $x,y \in J$, $x \leqslant y$, then $\langle x,y\rangle \subseteq J$, and let $T : J \to K$ be a continuous decreasing mapping which is compact on any closed order interval contained in J. Suppose there exists $x_0 \in J$ such that T^2x_0 is defined, and furthermore, Tx_0 and T^2x_0 are order comparable to x_0.*

Then T has a fixed point in J provided that, either

(I) $Tx_0 \leqslant x_0$ and $T^2x_0 \leqslant x_0$, or $x_0 \leqslant Tx_0$ and $x_0 \leqslant T^2x_0$, or

(II) The complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is defined, and there exists $y_0 \in J$ such that $y_0 \leqslant T^n x_0$, for every n.

3. Properties of positive solutions

In setting the stage for application of Theorem 1, we consider the Banach space $(B, ||\cdot||)$ defined by

$$B := \{u : [0,1] \to \mathbf{R} \mid u \text{ is continuous}\}, \quad ||u|| := \sup_{0\leqslant x\leqslant 1} |u(x)|.$$

Also, we define a cone $K \subset B$ by

$$K := \{u \in B \mid u(x) \geqslant 0 \text{ on } [0,1]\}.$$

We observe that, if $y(x)$ is a solution of (1)-(2), then

$$y'''(x) \geqslant 0,\ y(x) \geqslant 0 \text{ and } y(x) \text{ is concave.}$$

Next, we define $g(x) : [0,1] \to [0, \frac{3}{4}]$ by

$$g(x) := \min\{1-x, 3x\},$$

and for $\theta > 0$, we define

$$g_\theta(x) := \theta g(x).$$

Notice that

$$\max_{0\leqslant x\leqslant 1} g(x) = \frac{3}{4} \text{ and } \max_{0\leqslant x\leqslant 1} g_\theta(x) = \frac{3\theta}{4}.$$

We will assume hereafter:

(H3) $\int_0^1 f(x, g_\theta(x))dx < \infty$, for all $\theta > 0$.

Now, we note that the function $f(x,y) := \frac{1}{\sqrt[3]{x(1-x)y}}$ also satisfies (H3). In particular, for each $\theta > 0$,

$$\int_0^1 f(x, g_\theta(x))dx = \frac{1}{\sqrt[3]{\theta}}\left[\int_0^{\frac{1}{4}} \frac{1}{\sqrt[3]{3x^2(1-x)}}dx + \int_{\frac{1}{4}}^1 \frac{1}{\sqrt[3]{x(1-x)^2}}dx\right] < 4\sqrt[3]{\frac{3}{\theta}}.$$

We shall make extensive application of the following theorem due to Graef and Yang [12, 13].

THEOREM 2. *Let* $u(x) \in C^{(3)}[0,1]$. *If* $u(x)$ *satisfies the boundary conditions* (2) *is such that* $u''' \geqslant 0$ *on* $[0,1]$, *then*

$$u(x) \geqslant \min\{1-x, 3x\} \sup_{0\leqslant x\leqslant 1} |u(x)|. \tag{3}$$

So, from this theorem, for each positive solution $u(x)$ of (1)-(2), there exists a $\theta > 0$ such that

$$g_\theta(x) \leqslant u(x), \ \ 0 \leqslant x \leqslant 1.$$

In particular, with $\theta = \sup_{0\leqslant x\leqslant 1} |u(x)|$, then

$$u(x) \geqslant \min\{1-x, 3x\}\theta = g_\theta(x), \ \ 0 \leqslant x \leqslant 1.$$

Next, we let $D \subset K$ be defined by

$$D := \{v \in K \mid \text{there exists } \theta(v) > 0 \text{ such that } g_\theta(x) \leqslant v(x), \ 0 \leqslant x \leqslant 1\}.$$

We observe that, for each $v \in D$ and $\frac{1}{8} \leqslant x \leqslant \frac{5}{8}$,

$$v(x) \geqslant g_\theta(x) = \min\{1-x, 3x\}\theta \geqslant \frac{3}{8}\theta, \tag{4}$$

and for each positive solution $u(x)$ of (1)-(2),

$$u(x) \geqslant g(x) \sup_{0\leqslant x\leqslant 1} |u(x)| \geqslant \frac{3}{8} \sup_{0\leqslant x\leqslant 1} |u(x)|, \ \frac{1}{8} \leqslant x \leqslant \frac{5}{8}. \tag{5}$$

There is a Green's function, $G(x,s)$, for $y''' = 0$ satisfying (2) which will play the role of a kernel for certain compact operators meeting the requirements of Theorem 1. By direct computation,

$$G(x,s) = \frac{1}{2} \begin{cases} x(1-x) - x(1-s)^2, & 0 \leqslant x < s \leqslant 1, \\ x(1-x) - x(1-s)^2 + (x-s)^2, & 0 \leqslant s \leqslant x \leqslant 1, \end{cases}$$

and properties to which we will appeal include

(i) $G(x,s) > 0$ on $(0,1) \times (0,1)$ and continuous on $[0,1] \times [0,1]$.

(ii) $G(0,s) = 0, 0 < s \leqslant 1$, and $G(1,s) = \frac{\partial^2}{\partial x^2} G(1,s) = 0, 0 \leqslant s < 1$.

(iii) $\frac{\partial^2}{\partial x^2} G(x,s)$ is continuous as a function of x on $[0,s]$ and on $[s,1]$.

(iv) $\frac{\partial}{\partial x} G(0,s) = \frac{s(2-s)}{2} > 0$ and $\frac{\partial}{\partial x} G(1,s) = -\frac{s^2}{2} < 0$, for $0 < s < 1$.

Now we define an integral operator $T : D \to K$ by

$$(Tu)(x) := \int_0^1 G(x,s) f(s,u(s)) ds, \ \ u \in D.$$

We shall show that T is well-defined on D and decreasing and that $T: D \to D$. First, let $v, u \in D$ be given, with $v(x) \leqslant u(x)$. Then, there exists $\theta > 0$ such that $g_\theta(x) \leqslant v(x)$. By Assumptions (H1) and (H3), and (i) above,

$$0 \leqslant \int_0^1 G(x,s)f(x,u(x))dx \leqslant \int_0^1 G(x,s)f(x,v(x))dx \leqslant \int_0^1 G(x,s)f(x,g_\theta(x))dx < \infty.$$

Therefore, T is well-defined on D and T is a decreasing operator.

Next, for $v \in D$, let $w(x) := (Tv)(x) = \int_0^1 G(x,s)f(s,v(s))ds \geqslant 0,\ 0 \leqslant x \leqslant 1$. From properties of Green's functions , $w'''(x) = f(x,v(x)) > 0,\ 0 < x < 1$, and $w(0) = w(1) = w''(1) = 0$, which imply $w''(x) \leqslant 0$, or that $w(x)$ is concave. Moreover, by Theorem 2, $w = Tv \in D$. So, we also have $T: D \to D$.

REMARK 2. It is well-known that $Tu = u$ if, and only if, u is a solution of (1)-(2). Therefore, we seek solutions of (1)-(2) that belong to D. It follows from (4) and (5), in the context of our Banach space B, that for each positive solution $u(x)$ of (1)-(2),

$$u(x) \geqslant g(x)||u|| \geqslant \frac{3}{8}||u||,\ \frac{1}{8} \leqslant x \leqslant \frac{5}{8}. \tag{6}$$

4. *A priori* bounds on norms of solutions

In this section, we exhibit that solutions of (1)-(2) have positive *a priori* upper and lower bounds on their norms.

LEMMA 1. *If f satisfies* (H1) - (H3)*, then there exists an $S > 0$ such that $||u|| \leqslant S$, for any solution u of* (1)-(2) *in D.*

Proof. Assume the conclusion is false. Then there exists a sequence $\{u_m\}_{m=1}^\infty$ of solutions of (1)-(2) in D such that $u_m(x) > 0$, for all $0 < x < 1$, and

$$||u_m|| \leqslant ||u_{m+1}|| \text{ and } \lim_{m\to\infty} ||u_m|| = \infty.$$

From (5) or (6),

$$u_m(x) \geqslant \frac{3}{8}||u_m||,\ \ \frac{1}{8} \leqslant x \leqslant \frac{5}{8}.$$

So,

$$\lim_{m\to\infty} u_m(x) = \infty \text{ uniformly on } \left[\frac{1}{8}, \frac{5}{8}\right].$$

Next, let

$$M := \max\{G(x,s) \mid (x,s) \in [0,1] \times [0,1]\}.$$

From (H2), there exists $m_0 \in \mathbf{N}$ such that, for each $m \geqslant m_0$ and $\frac{1}{8} \leqslant x \leqslant \frac{5}{8}$,

$$f(x,u_m(x)) \leqslant \frac{2}{M}.$$

Let

$$\theta := ||u_{m_0}||.$$

Then, for $m \geqslant m_0$,

$$u_m(x) \geqslant g_{||u_m||}(x) \geqslant g_{||u_{m_0}||}(x) = g_\theta(x),\ 0 \leqslant x \leqslant 1.$$

So, for $m \geqslant m_0$ and $0 \leqslant x \leqslant 1$, we have

$$\begin{aligned} u_m(x) &= Tu_m(x) \\ &= \int_0^1 G(x,s)f(s,u_m(s))ds \\ &= \int_0^{\frac{1}{8}} G(x,s)f(s,u_m(s))ds + \int_{\frac{5}{8}}^1 G(x,s)f(s,u_m(s))ds \\ &\quad + \int_{\frac{1}{8}}^{\frac{5}{8}} G(x,s)f(s,u_m(s))ds \\ &\leqslant \int_0^{\frac{1}{8}} G(x,s)f(s,u_m(s))ds + \int_{\frac{5}{8}}^1 G(x,s)f(s,u_m(s))ds + \int_{\frac{1}{8}}^{\frac{5}{8}} M \cdot \frac{2}{M} ds \\ &\leqslant \int_0^{\frac{1}{8}} G(x,s)f(s,g_\theta(s))ds + \int_{\frac{5}{8}}^1 G(x,s)f(s,g_\theta(s))ds + 1 \\ &\leqslant M \int_0^1 f(s,g_\theta(s))ds + 1, \end{aligned}$$

which contradicts $\lim_{m\to\infty} ||u_m|| = \infty$. Therefore, there exists an $S > 0$ such that $||u|| \leqslant S$, for any solution $u \in D$ of (1)-(2). ☐

Now, we turn our attention to exhibiting positive *a priori* lower bounds on the solution norms.

LEMMA 2. *If f satisfies* (H1) - (H3)*, then there exists an $R > 0$ such that $||u|| \geqslant R$, for any solution u of* (1)-(2) *in D.*

Proof. Again, we assume the conclusion to the lemma is false. Then, there exists a sequence $\{u_m\}_{m=1}^\infty$ of solutions of (1)-(2) in D such that $u_m(x) > 0$, for $0 < x < 1$, and

$$||u_m|| \geqslant ||u_{m+1}|| \text{ and } \lim_{m\to\infty} ||u_m|| = 0.$$

In particular,

$$\lim_{m\to\infty} u_m(x) = 0 \text{ uniformly on } [0,1].$$

Now, define

$$\overline{m} := \min\left\{ G(x,s) \mid (x,s) \in \left[\frac{1}{8}, \frac{5}{8}\right] \times \left[\frac{1}{8}, \frac{5}{8}\right] \right\} > 0.$$

From (H2), $\lim_{y\to 0^+} f(x,y) = \infty$ uniformly on compact subsets of $(0,1)$, and so, there exists a $\delta > 0$ such that, for $\frac{1}{8} \leqslant x \leqslant \frac{5}{8}$ and $0 < y < \delta$,

$$f(x,y) > \frac{2}{\overline{m}}.$$

Also, there exists $m_0 \in \mathbf{N}$ such that, for $m \geqslant m_0$ and $0 < x < 1$,

$$0 < u_m(x) < \frac{\delta}{2}.$$

So, for $m \geqslant m_0$ and $\frac{1}{8} \leqslant x \leqslant \frac{5}{8}$, we have

$$\begin{aligned} u_m(x) &= Tu_m(x) \\ &= \int_0^1 G(x,s) f(s,u_m(s))ds \\ &\geqslant \int_{\frac{1}{8}}^{\frac{5}{8}} G(x,s) f(s,u_m(s))ds \\ &\geqslant \overline{m} \int_{\frac{1}{8}}^{\frac{5}{8}} f(s,u_m(s))ds \\ &\geqslant \overline{m} \int_{\frac{1}{8}}^{\frac{5}{8}} f(s,\frac{\delta}{2})ds \\ &\geqslant \overline{m} \int_{\frac{1}{8}}^{\frac{5}{8}} \frac{2}{\overline{m}} ds \\ &= 1. \end{aligned}$$

This contradicts $\lim_{m\to\infty} u_m(x) = 0$ uniformly on $[0,1]$. Therefore, there exists an $R > 0$ such that $R \leqslant ||u||$ for any solution $u \in D$ of (1)-(2). $\square$

In summary, there exist $0 < R < S$ such that, for each solution $u \in D$ of (1)-(2), we have

$$R \leqslant ||u|| \leqslant S.$$

5. Existence of positive solutions

In this section, we will construct a sequence of operators, $\{T_m\}_{m=1}^{\infty}$, each of which is defined on all of K. We then proceed to show, via applications of Theorem 1, that each T_m has a fixed point $\phi_m \in K$, for every m. Then, we will extract a subsequence from $\{\phi_m\}_{m=1}^{\infty}$ that converges to a fixed point of T.

THEOREM 3. *If f satisfies* (H1) - (H3)*, then* (1)-(2) *has at least one positive solution $u \in D$.*

Proof. For each $m \in \mathbf{N}$, let

$$u_m(x) := T(m) = \int_0^1 G(x,s) f(s,m)ds, \quad 0 \leqslant x \leqslant 1.$$

Since f is decreasing with respect to its second component, we have

$$0 < u_{m+1}(x) < u_m(x), \text{ for } 0 < x < 1,$$

and by (H2), $\lim_{m\to\infty} u_m(x) = 0$ uniformly on $[0,1]$.

Next, we define $f_m(x,y) : (0,1)\times[0,\infty) \to (0,\infty)$ by

$$f_m(x,y) := f(x, \max\{y, u_m(x)\}).$$

Then, f_m is continuous and f_m does not have the singularity at $y=0$ possessed by f. In addition, for $(x,y) \in (0,1)\times(0,\infty)$,

$$f_m(x,y) \leqslant f(x,y) \text{ and } f_m(x,y) \leqslant f(x,u_m(x)).$$

Now, let us define a sequence of operators, $T_m : K \to K$, for $\phi \in K$ and $0 \leqslant x \leqslant 1$, by

$$T_m\phi(x) := \int_0^1 G(x,s) f_m(s,\phi(s))ds.$$

The standard arguments yield that each T_m is a compact operator on K. Furthermore,

$$\begin{aligned} T_m(0) &= \int_0^1 G(x,s) f_m(s,0)ds \\ &= \int_0^1 G(x,s) f(s, \max\{0, u_m(s)\})ds \\ &= \int_0^1 G(x,s) f(s, u_m(s))ds \\ &\geqslant 0, \end{aligned}$$

and

$$T_m^2(0) = T_m\left(\int_0^1 G(x,s) f_m(s,0)ds\right) \geqslant 0.$$

By Theorem 1, with $J = K$ and $x_0 = 0$, T_m has a fixed point in K, for each m. That is, for each m, there exists $\phi_m \in K$ such that

$$T_m\phi_m(x) = \phi_m(x), \quad 0 \leqslant x \leqslant 1.$$

So, for each $m \geqslant 1$, ϕ_m satisfies the boundary conditions (2), and also,

$$\begin{aligned} T_m\phi_m(x) &= \int_0^1 G(x,s) f_m(s,\phi_m(s))ds \\ &\leqslant \int_0^1 G(x,s) f(s,u_m(s))ds \\ &= Tu_m(x). \end{aligned}$$

That is, for each $0 \leqslant x \leqslant 1$ and for each m, $\phi_m(x) = T_m\phi_m(x) \leqslant Tu_m(x)$.

By arguments along the lines of Lemmas 1 and 2, there exist $R > 0$ and $S > 0$ such that

$$R \leqslant ||\phi_m|| \leqslant S, \text{ for every } m.$$

Now, let $\theta := R$. Since ϕ_m belongs to K and is a fixed point of T_m, the conditions of Theorem 2 hold. So, for every m and $0 \leqslant x \leqslant 1$,

$$\phi_m(x) \geqslant g(x)||\phi_m|| \geqslant g(x) \cdot R = g_\theta(x).$$

So, the sequence $\{\phi_m\}_{m=1}^{\infty}$ is contained in the closed order interval $\langle g_\theta, S\rangle$, and therefore, the sequence is contained in D. Since T is a compact mapping, we may assume $\lim_{m\to\infty} T\phi_m$ exists; let us say that the limit is ϕ^*.

To complete the proof, it suffices to show that

$$\lim_{m\to\infty} (T\phi_m(x) - \phi_m(x)) = 0$$

uniformly on $[0,1]$. It will follow that $\phi^* \in \langle g_\theta, S\rangle$.

To that end, let $\varepsilon > 0$ be given, and choose $0 < \delta < \frac{1}{2}$ such that

$$\int_0^\delta f(s, g_\theta(s))ds + \int_{1-\delta}^1 f(s, g_\theta(s))ds < \frac{\varepsilon}{2M},$$

where as before $M := \{G(x,s) \mid (x,s) \in [0,1]\times[0,1]\}$. Then, there exists m_0 such that, for $m \geqslant m_0$ and for $\delta \leqslant x \leqslant 1-\delta$,

$$u_m(x) \leqslant g_\theta(x) \leqslant \phi_m(x).$$

So, for $m \geqslant m_0$ and for $\delta \leqslant x \leqslant 1-\delta$,

$$f_m(x, \phi_m(x)) = f(x, \max\{\phi_m(x), u_m(x)\}) = f(x, \phi_m(x)).$$

We have, for $m \geqslant m_0$ and $0 \leqslant x \leqslant 1$,

$$\begin{aligned}
|T\phi_m(x) - \phi_m(x)| &= |T\phi_m(x) - T_m\phi_m(x)| \\
&= \left| \int_0^1 G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]ds \right| \\
&= \left| \int_0^\delta G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]ds \right. \\
&\quad \left. + \int_{1-\delta}^1 G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]ds \right| \\
&\leqslant M \int_0^\delta [f(s,\phi_m(s)) + f_m(s,\phi_m(s))]ds \\
&\quad + M \int_{1-\delta}^1 [f(s,\phi_m(s)) + f_m(s,\phi_m(s))]ds \\
&\leqslant M \int_0^\delta [f(s,\phi_m(s)) + f(s,\phi_m(s))]ds
\end{aligned}$$

$$\begin{aligned}
&+M\int_{1-\delta}^{1}[f(s,\phi_m(s))+f(s,\phi_m(s))]ds\\
&=2M\left[\int_0^{\delta}f(s,\phi_m(s))ds+\int_{1-\delta}^{1}f(s,\phi_m(s))ds\right]\\
&\leqslant 2M\left[\int_0^{\delta}f(s,g_\theta(s))ds+\int_{1-\delta}^{1}f(s,g_\theta(s))ds\right]\\
&<2M\cdot\frac{\varepsilon}{2M}\\
&=\varepsilon.
\end{aligned}$$

So, for $m \geqslant m_0$,

$$||T\phi_m - \phi_m|| < \varepsilon.$$

That is, $\lim_{m\to\infty}(T\phi_m(x)-\phi_m(x)) = 0$ uniformly on $[0,1]$. Hence, for $0 \leqslant x \leqslant 1$,

$$\begin{aligned}
T\phi^*(x) &= T(\lim_{m\to\infty} T\phi_m(x))\\
&= T(\lim_{m\to\infty} \phi_m(x))\\
&= \lim_{m\to\infty} T\phi_m(x)\\
&= \phi^*(x),
\end{aligned}$$

and ϕ^* is a desired positive solution of (1)-(2) belonging to D. □

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