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INEQUALITIES FOR ZEROS OF SOLUTIONS TO SECOND ORDER ODE WITH ONE SINGULAR POINT

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Abstract. We consider the equation $y'' + P(z)y' + Q(z)y = 0 \ (z \in \mathbb{C})$, where

$$P(z) = \sum_{k=0}^{n_P} p_k z^{k-1} \text{ and } Q(z) = \sum_{k=0}^{n_Q} q_k z^{k-2}$$

with real coefficients $p_k, q_j \ (k = 0, \ldots, n_P; \ j = 0, \ldots, n_Q; n_P, n_Q < \infty)$.

Let $z_k(y), k = 1, 2, \ldots$ be the nontrivial zeros of a solution $y(z)$ to that equation. Estimates for the sums $\sum_{k=1}^{j} \frac{1}{|z_k(y)|} \ (j = 1, 2, \ldots)$ are derived. Applications of the obtained estimates to the counting function of the zeros of solutions are also discussed.

1. Introduction and statement of the main result

In the present paper, we investigate the complex zeros of solutions to the equation

$$y'' + P(z)y' + Q(z)y = 0 \ (z \in \mathbb{C}), \tag{1.1}$$

where

$$P(z) = \sum_{k=0}^{n_P} p_k z^{k-1} \text{ and } Q(z) = \sum_{k=0}^{n_Q} q_k z^{k-2} \tag{1.2}$$

with real coefficients $p_k, q_j \ (k = 0, \ldots, n_P; \ j = 0, \ldots, n_Q; n_P, n_Q < \infty)$. That is, equation (1.1) has one regular singular point at $z = 0$. It is possible that either $p_0 = 0$ or $q_0 = 0$. If both p_0 and q_0 are zero, then the equation does not have a singular point. It is well-known that the zeros of solutions of ODE play an essential role in mathematical physics, cf. [11]. The literature devoted to the zeros of solutions of equations without singular points is very rich. Here the main tool is the Nevanlinna theory. An excellent exposition of the Nevanlinna theory and its applications to differential equations is given in the book [13]. In connection with the recent results see the interesting papers [1, 2, 3, 14, 15, 18]. In the above cited works mainly the asymptotic distributions of zeros and counting functions of zeros are investigated. At the same time, bounds for the zeros of solutions are very important in various applications. But to the best of our

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knowledge, they have been investigated considerably less than the asymptotic distributions. In the paper [7] the author has established bounds for the sums of the zeros of solutions for the second order equations with polynomial coefficients. In the interesting paper [4], some results from [7] have been extended to the equation $u^{(m)} = P(z)u$, where P is a polynomial and $m > 2$. In [9] the main result from [7] is extended to the second order ODE with non-polynomial coefficients. In the paper [10] the results of the paper [7] have been extended to nonhomogeneous ODE. Perturbations of the zeros of solutions to second order differential equations with polynomial coefficients were investigated in the paper [9]. Certainly, we could not survey the whole subject here and refer the reader to the above listed publications and references given therein. It should be noted that, to the best of our knowledge, bounds for the sums of the complex zeros of solutions to ODE with singular points have not been obtained in the available literature. In the present paper we establish such bounds for solutions to equation (1.1). Some applications of these bounds are also discussed.

It is assumed that

$$(1-p_0)^2 > 4q_0. \tag{1.3}$$

That is, the indicial equation

$$x(x-1)+p_0x+q_0 = 0 \tag{1.4}$$

has real different roots. Below $z_k(f)$ are the zeros of a function f taken with the multiplicities are enumerated in order of increasing modulus: $|z_k(f)| \leqslant |z_{k+1}(f)|$. If $f(0) = 0$, then we enumerate the nontrivial zeros, only: $z_k(f) \neq 0$ $(k = 1,2,\ldots)$. Denote

$$\eta = (1-p_0)/2 + [(1-p_0)^2/4 - q_0]^{1/2},\ \rho = \max\{n_P, n_Q\}$$

and

$$B_0 = \sqrt{e}\left[\left(2+\frac{1}{2\eta+p_0}\right)|\eta|\sum_{j=1}^{n_P}|p_j| + \frac{1}{2\eta+p_0}\sum_{k=1}^{n_Q}|q_k|\right].$$

Now we are in a position to formulate our main result.

THEOREM 1. *Under condition (1.3), the non-trivial zeros $z_k(y)$ of any solution y of (1.1) satisfy the inequalities*

$$\sum_{k=1}^{j}\frac{1}{|z_k(y)|} \leqslant \sqrt{2}B_0^{1/\rho}\left[C_0 + \sum_{k=1}^{j}\frac{1}{(k+1)^{1/\rho}}\right] \quad (j = 1,2,\ldots),$$

where

$$C_0 := exp\left[\frac{1}{2\rho} + \left(2+\frac{1}{2\eta+p_0}\right)|\eta|\sum_{j=1}^{n_P}|p_j|\frac{(\rho-j)}{j\rho} + \frac{1}{2\eta+p_0}\sum_{k=1}^{n_Q}|q_k|\frac{(\rho-k)}{k\rho}\right].$$

The proof of this theorem is presented in the next section.

2. Proof of Theorem 1

Applying the method of Frobenius, cf. [12, Chapter 16, p. 399], we seek a solution in the form

$$y(z) = z^{\eta} v(z), \tag{2.1}$$

where

$$v(z) = \sum_{k=0}^{\infty} v_k z^k.$$

Besides, $v_0 \neq 0$ and v_k can be found by the undefined coefficients method. Since the origin is the unique regular singular point, the Taylor series of v absolutely converges for all $z \in \mathbb{C}$, [12, p. 399]. So $v(z)$ is an entire function. Substitute (2.1) into (1.1) and delete by z^{η}. Then we obtain the equation

$$v'' + \frac{2\eta}{z}v' + \frac{\eta(\eta-1)}{z^2}v + P(z)\left(\frac{\eta}{z}v + v'\right) + Q(z)v = 0.$$

Put $P_1(z) = P(z) - p_0/z$ and $Q_1(z) = Q(z) - q_0/z^2$. Then

$$v'' + \frac{2\eta}{z}v' + \frac{\eta(\eta-1)}{z^2}v + \left(P_1(z) + \frac{p_0}{z}\right)(\frac{\eta}{z}v + v') + \left(Q_1(z) + q_0\frac{\eta}{z^2}\right)v = 0.$$

Due to the indicial equation we obtain

$$v'' + (2\eta + p_0)\frac{v'}{z} + P_1(z)v' + (\frac{P_1(z)\eta}{z} + Q_1(z))v = 0,$$

or

$$v'' + \xi\frac{v'}{z} + P_1(z)v' + F(z)\frac{v}{z} = 0, \tag{2.2}$$

where $F(z) = P_1(z)\eta + Q_1(z)z$ is a polynomial, and

$$\xi := 2\eta + p_0 = 1 + [(1-p_0)^2 - 4q_0]^{1/2} > 1.$$

Furthermore, for a fixed $t \in [0, 2\pi)$ with $z = re^{it}$ we can write

$$\frac{d^2v(z)}{dr^2} + \xi\frac{dv(z)}{rdr} + e^{it}P_1(z)\frac{dv(z)}{dr} + e^{it}\frac{F(z)v(z)}{r} = 0.$$

Take into account that

$$exp\,[-\xi\int_s^r \frac{dr_1}{r_1}] = (s/r)^{\xi}.$$

Then for a $z_0 = r_0e^{it} \in \mathbb{C}$ with $c_1 = \frac{dv(z_0)}{dr}$ we arrive at the equation

$$\frac{dv(z)}{dr} + c_1(r_0/r)^{\xi} + \int_{r_0}^r (s/r)^{\xi}P_1(se^{it})\frac{dv(se^{it})}{ds}ds + \int_{r_0}^r (s/r)^{\xi}F(se^{it})v(se^{it})\frac{ds}{s} = 0.$$

Hence, letting $r_0 \to 0$, we get

$$\frac{dv(z)}{dr} + \int_0^r (s/r)^{\xi} P_1(se^{it}) \frac{dv(se^{it})}{ds} ds + \int_0^r (s/r)^{\xi} F(se^{it}) v(se^{it}) \frac{ds}{s} = 0. \qquad (2.3)$$

Integrating this equation, we obtain

$$v(z) - c_0 + J_1(z) + J_2(z) = 0 \quad (c_0 = v(0)), \qquad (2.4)$$

where

$$J_1(z) := \int_0^r \int_0^{\tau} (s/\tau)^{\xi} P_1(se^{it}) \frac{dv(se^{it})}{ds} ds\, d\tau,$$

and

$$J_2(z) := \int_0^r \int_0^{\tau} (s/\tau)^{\xi} F(se^{it}) v(se^{it}) \frac{ds}{s}\, d\tau.$$

Let

$$J_3(\tau) := \int_0^{\tau} s^{\xi} P_1(se^{it}) \frac{dv(se^{it})}{ds} ds.$$

Integrating by parts, we can write

$$J_3(\tau) = \tau^{\xi} P_1(\tau e^{it}) v(\tau e^{it}) - \int_0^{\tau} (s^{\xi} P_1(se^{it}))' v(se^{it}) ds.$$

Take into account that

$$|P_1(z)| \leqslant \hat{P}_1(r) := \sum_{k=1}^{n_P} |p_k| r^{k-1}, |zQ_1(z)| \leqslant \hat{Q}_1(r) := \sum_{k=1}^{n_Q} |q_k| r^{k-1},$$

and $|(s^{\xi} P_1(se^{it}))'| \leqslant (s^{\xi} \hat{P}_1(s))', s \geqslant 0$. Put $M_f(r) = \sup_{|z| \leqslant r} |f(z)|$ for a function $f(z)$. Then with $w(\tau) = M_v(\tau)$ we have

$$\left| \int_0^{\tau} (s^{\xi} P_1(se^{it}))' v(se^{it}) ds \right| \leqslant w(\tau) \hat{P}_1(\tau) \tau^{\xi} \quad (\tau > 0)$$

and, consequently, $|J_3(\tau)| \leqslant 2w(\tau)\hat{P}_1(\tau)\tau^{\xi}$. Therefore,

$$|J_1(z)| = |\int_0^r J_3(\tau) \tau^{-\xi} d\tau| \leqslant 2 \int_0^r \hat{P}_1(\tau) w(\tau) d\tau. \qquad (2.5)$$

Furthermore, $|F(z)| \leqslant \hat{P}_1(r)|\eta| + \hat{Q}_1(r) r$ and

$$\begin{aligned} |J_2(z)| &\leqslant \int_0^r w(\tau)(\hat{P}_1(\tau)|\eta| + \hat{Q}_1(\tau)\tau)\tau^{-\xi} \int_0^{\tau} s^{\xi-1} ds\, d\tau \\ &\leqslant \frac{1}{\xi} \int_0^r w(\tau)(\hat{P}_1(\tau)|\eta| + \hat{Q}_1(\tau)) d\tau. \end{aligned}$$

Thus, due to (2.4) and (2.5),

$$w(r) \leqslant |c_0| + \int_0^r w(\tau)[(2 + \frac{1}{\xi})\hat{P}_1(\tau)|\eta| + \frac{1}{\xi}\hat{Q}_1(\tau)\tau] d\tau.$$

Now the Gronwall lemma implies

$$w(r) \leqslant |c_0| exp \Big[\int_0^r [(2+\frac{1}{\xi})\hat{P}_1(\tau)|\eta| + \frac{1}{\xi}\hat{Q}_1(\tau)\tau] d\tau \Big].$$

We thus have proved

LEMMA 1. *Any solution y of (1.1) can be defined by*

$$y(z) = z^{\eta} v(z), \tag{2.6}$$

where $v(z)$ is an entire function satisfying equation (2.2) and the inequality

$$M_v(r) \leqslant |v(0)| exp \, [\int_0^r [(2+\frac{1}{\xi})\hat{P}_1(\tau)|\eta| + \frac{1}{\xi}\hat{Q}_1(\tau)] d\tau.$$

Furthermore, put

$$\begin{aligned} W_0(r) &:= \int_0^r [(2+\frac{1}{\xi})\hat{P}_1(\tau)|\eta| + \frac{1}{\xi}\hat{Q}_1(\tau)\tau] d\tau \\ &= |\eta|(2+\frac{1}{\xi}) \sum_{k=1}^{n_P} |p_k| \frac{r^k}{k} + \frac{1}{\xi} \sum_{k=1}^{n_Q} |q_k| \frac{r^k}{k}. \end{aligned}$$

Recall the Young inequality $ab \leqslant a^t/t + b^s/s$ $(a,b>0;\ 1/s+1/t=1;\ t>1)$. By that inequality, with $s=\rho/k$, we have

$$r^k \leqslant \frac{1}{t} + \frac{r^{ks}}{s} = 1 - \frac{k}{\rho} + r^{\rho}\frac{k}{\rho}.$$

Thus,

$$\begin{aligned} W_0(r) &\leqslant |\eta|(2+\frac{1}{\xi}) \sum_{k=1}^{n_P} |p_k|(1/k - 1/\rho + r^{\rho}/\rho) + \frac{1}{\xi} \sum_{k=1}^{n_Q} |q_k|(1/k - 1/\rho + r^{\rho}/\rho) \\ &= W_1 + B_1 r^{\rho}, \end{aligned}$$

where

$$W_1 := |\eta|(2+\frac{1}{\xi}) \sum_{k=1}^{n_P} |p_k|(1/k - 1/\rho) + \frac{1}{\xi} \sum_{k=1}^{n_Q} |q_k|(1/k - 1/\rho)$$

and

$$B_1 := |\eta|(2+\frac{1}{\xi})\frac{1}{\rho} \sum_{k=1}^{n_P} |p_k| + \frac{1}{\rho\xi} \sum_{k=1}^{n_Q} |q_k|.$$

Consequently, we arrive at

COROLLARY 1. *A solution v of equation (2.2) satisfies the inequality*

$$M_v(r) \leqslant |v(0)| e^{W_1} exp[B_1 r^{\rho}]. \tag{2.7}$$

We need the following result proved in [10].

LEMMA 2. *Let $f(z)$ be an entire function satisfying $f(0)=1$ and*

$$M_f(r) \leqslant D_f\, exp\left[B_f r^{\rho_f}\right] \quad (D_f, B_f = const > 0;\ \rho_f \geqslant 1, r > 0). \tag{2.8}$$

Then its zeros satisfy the inequalities

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|} \leqslant \sqrt{2}(\sqrt{e}B_f\rho_f)^{1/\rho_f}\left[D_f e^{1/(2\rho_f)} + \sum_{k=1}^{j}\frac{1}{(k+1)^{1/\rho_f}}\right] \quad (j=1,2,\ldots).$$

Proof of Theorem 1: Due to Corollary 1 and the previous lemma we have

$$\sum_{k=1}^{j} \frac{1}{|z_k(v)|} \leqslant \sqrt{2}(\sqrt{e}B_1\rho)^{1/\rho}\left[e^{W_1+1/(2\rho)} + \sum_{k=1}^{j}\frac{1}{(k+1)^{1/\rho}}\right] \quad (j=1,2,\ldots).$$

But

$$\sqrt{e}B_1\rho = \sqrt{e}\left[|\eta|(2+\frac{1}{\xi})\sum_{k=1}^{n_P}|p_k| + \frac{1}{\xi}\sum_{k=1}^{n_Q}|q_k|\right] = B_0$$

and

$$W_1 + 1/(2\rho) = (2+\frac{1}{\xi})|\eta|\sum_{k=1}^{n_P}|p_k|(1/k - 1/\rho) + \frac{1}{\xi}\sum_{k=1}^{n_Q}|q_k|(1/k-1/\rho) + 1/(2\rho)$$
$$= \ln\, C_0.$$

Now (2.6) implies the required result. □

3. Applications of Theorem 1

Again $y(z)$ is a solution of (1.1). Since the nontrivial zeros of $y(z)$ satisfy the inequality $|z_k(y)| \leqslant |z_{k+1}(y)|$, Theorem 1 implies

$$\frac{j}{|z_j(y)|} \leqslant \beta(B_0,\rho)\left[C_0 + \sum_{k=1}^{j}\frac{1}{(k+1)^{1/\rho}}\right] \quad (j=1,2,\ldots),$$

where $\beta(B_0,\rho) = \sqrt{2}B_0^{1/\rho}$.

Take into account that

$$\sum_{k=1}^{j}(k+1)^{-1/\rho} \leqslant \int_1^{j+1}\frac{dx}{x^{1/\rho}} = \frac{(1+j)^{1-1/\rho}-1}{1-1/\rho} \quad (\rho > 1) \tag{3.1}$$

and denote by $\nu(f,a)$ $(a>0)$ the counting function of the nontrivial zeros of f in the disc $|z| \leqslant a$. We thus get

COROLLARY 2. *Let condition (1.3) hold. Then with* $\rho > 1$ *and the notation*

$$\theta_j(B_0,\rho) := \frac{j}{\beta(B_0,\rho)\left(C_0 + \frac{(1+j)^{1-1/\rho}-1}{1-1/\rho}\right)},$$

the inequality $|z_j(y)| \geqslant \theta_j(B_0,\rho)$ *holds and thus* $\nu(B_0,a) \leqslant j$ *for any positive* $a \leqslant \theta_j(B_0,\rho)$ $(j = 1,2,...)$.

The just obtained results asserts that in the disc $|z| < \theta_1(B_0,\rho)$, y does not have nontrivial zeros. Furthermore, put $\vartheta_1 = \beta(B_0,\rho)(C_0 + 2^{-1/\rho})$ and $\vartheta_k = \beta(B_0,\rho)(k+1)^{-1/\rho}$ $(k = 2,3,...)$. Then Theorem 1 and [6, Lemma 1.2.1] yield

COROLLARY 3. *Let* $\psi(t)$ $(0 \leqslant t < \infty)$ *be a continuous convex scalar-valued function, such that* $\psi(0) = 0$ *and condition (1.3) hold. Then*

$$\sum_{k=1}^{j} \psi(|z_k(y)|^{-1}) \leqslant \sum_{k=1}^{j} \psi(\vartheta_k) \ \ (j = 1,2,...).$$

In particular, for any $p \geqslant 1$ *and* $j = 2,3,...$, *we have*

$$\sum_{k=1}^{j} \frac{1}{|z_k(y)|^p} \leqslant \sum_{k=1}^{j} \vartheta_k^p \text{ and therefore } \sum_{k=1}^{\infty} \frac{1}{|z_k(y)|^p} < \infty, \text{ provided } p > \rho.$$

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