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CRITICAL GROWTH PROBLEMS FOR $\frac{1}{2}$–LAPLACIAN IN $\mathbb{R}$

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Abstract. We study the existence of weak solutions for fractional elliptic equations of the type,

$$(-\Delta)^{\frac{1}{2}}u+V(x)u=h(u),\ u>0 \text{ in } \mathbb{R},$$

where h is a real valued function that behaves like e^{u^2} as $u\to\infty$ and $V(x)$ is a positive, continuous unbounded function. Here $(-\Delta)^{\frac{1}{2}}$ is the fractional Laplacian operator. We show the existence of mountain-pass solution when the nonlinearity is superlinear near $t=0$. We also study the corresponding critical exponent problem for the Kirchhoff equation

$$m\left(\int_{\mathbb{R}}|(-\Delta)^{\frac{1}{4}}u|^2dx+\int_{\mathbb{R}}V(x)u^2dx\right)\left((-\Delta)^{\frac{1}{2}}u+V(x)u\right)=f(u) \text{ in } \mathbb{R},$$

where $f(u)$ behaves like e^{u^2} as $u\to\infty$ and $f(u)\sim u^{\theta}$, with $\theta>3$, as $u\to 0$.

1. Introduction

In this article, we study the existence of weak solutions for fractional elliptic equations of the type,

$$(P)\qquad (-\Delta)^{\frac{1}{2}}u+V(x)u=h(u),u>0 \text{ in } \mathbb{R},$$

where the nonlinearity $h(u)$ satisfies critical growth of exponential type which will be stated later.

We also study the corresponding critical exponent problem for the Kirchhoff equation

$$(Q)\qquad \left\{m\left(\int_{\mathbb{R}}|(-\Delta)^{\frac{1}{4}}u|^2dx+\int_{\mathbb{R}}V(x)u^2dx\right)\left((-\Delta)^{\frac{1}{2}}u+V(x)u\right)=f(u) \text{ in } \mathbb{R}\right.$$

where $m:\mathbb{R}^+\to\mathbb{R}^+$ and $f:\mathbb{R}\to\mathbb{R}^+$ are continuous functions that satisfy some suitable conditions.

The function $V(x)$ is a continuous function satisfying the following assumption:
(V) $V(x)\geqslant V_0>0$ in $\mathbb{R}$ and $V(x)\to\infty$ as $|x|\to\infty$.

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An example of function satisfying the above assumption is $V(x) = |x|^p + V_0$ with $p > 0$ and $V_0 > 0$.

Here $(-\Delta)^{\frac{1}{2}}$ is the $\frac{1}{2}$-Laplacian operator defined as

$$(-\Delta)^{\frac{1}{2}} u(x) = \frac{1}{\pi} \mathrm{P.V.} \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^2} dy \quad \text{for all } x \in \mathbb{R},$$

where P.V. denotes the Cauchy principal value (see for instance [8] and [6]). The fractional Laplacian operator has been a classical topic in Fourier analysis and nonlinear partial differential equations for a long time. Fractional operators are involved in financial mathematics, where Levy processes with jumps appear in modeling the asset prices (see [5]). Recently the fractional Laplacian has attracted many researchers. In particular, concerning nonlinear elliptic equations involving fractional operators, the issues of existence and properties of solutions (regularity, a priori bounds, asymptotic behavior, symmetry, etc.) have been discussed in detail (see for instance [7], [8], [10], [13], [15], [24], [25] and [28]). The critical exponent problems for square root of Laplacian are studied in [9], [26].

In [26], authors have studied the following Brezis-Nirenberg type critical exponent problem on bounded domains $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$:

$$(-\Delta)^{\frac{1}{2}} u = \lambda u + u^{\frac{2n}{n-1}},\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \backslash \Omega,$$

by studying its harmonic extension problem. The idea of these harmonic extensions was initially introduced and studied in the beautiful work of Caffarelli and Silvestre [8]. The critical exponent problems in the limiting case $n = 1$ and with nonlinearities with exponential growth are studied in [13]. Here the exponential type nonlinearity is motivated by fractional Moser-Trudinger embedding due to Ozawa [22].

In [19], authors considered the problem in the whole space $\mathbb{R}$:

$$(-\Delta)^{1/2} u + u = K(x) g(u) \text{ in } \mathbb{R},$$

where K is a real valued positive function and g has a critical exponential growth and is super-quadratic near 0. Here authors proved the existence of solutions by studying the corresponding harmonic extension problem under suitable conditions on K and g. In section 3, we improve this result by identifying more accurately the first critical level under which the Palais-Smale condition holds. To achieve this, we extend the sharp Trudinger-Moser inequality proved for bounded domains in [20] to the whole space case (see Theorem 2.2). For that, we use some extensions of Adams type inequalities proved in [17]. Next, using this new Trudinger-Moser inequality, we show the existence of a Palais-Smale sequence that concentrates on the boundary $\mathbb{R} \times \{0\}$ in the spirit of [4] and whose energy level is strictly below the first critical level. We highlight that the assumption (h4) (see section 3) plays an important role in proving such compactness of the exhibited Palais-Smale sequence. Furthermore, in the local setting (see [4], [3], [14]), (h4) appears to be the sharp condition on the asymptotic behaviour of nonlinearity h to ensure the existence of nontrivial solutions for critical problems in two dimensions. We show that it still holds for more general non local problems as (Q) investigated in section 4.

Elliptic problems with exponential growth nonlinearities are motivated by the Moser-Trudinger inequality [21], namely

$$\sup_{\|u\|_{H_0^1(\Omega)}\leqslant 1}\int_\Omega e^{\alpha u^2}dx<\infty, \text{ if and only if } \alpha\leqslant 4\pi,$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain. The existence of solutions for critical exponent problem was initiated and studied in [2, 4, 11]. Subsequently, these results were generalized to unbounded domains in [18, 23].

The space $H^{\frac{1}{2}}(\mathbb{R})$ is the Hilbert space with the norm defined as

$$\|u\|_{1/2}^2=\|u\|_{L^2(\mathbb{R})}^2+\int_{\mathbb{R}}|(-\Delta)^{\frac{1}{4}}u|^2dx.$$

The space $H_0^{\frac{1}{2}}(\mathbb{R})$ is the completion of $C_0^\infty(\mathbb{R})$ under $[u]=\left(\int_{\mathbb{R}}|(-\Delta)^{\frac{1}{4}}u|^2dx\right)^{\frac{1}{2}}$.

The problems of the type (P) with exponential growth nonlinearities are motivated from the fractional Trudinger-Moser inequality [20], which gives the optimal constant and improves the former results of Ozawa [22] and Kozono, Sato and Wadade [16]. Precisely, let I be a bounded interval of $\mathbb{R}$. Set $X(I):=\{u\in H^{\frac{1}{2}}(\mathbb{R}) : u\equiv 0 \text{ in } \mathbb{R}\backslash I\}$. Then,

THEOREM 1.1. *For $u\in H^{\frac{1}{2}}(I)$, $e^{\beta u^2}\in L^1(I)$ for any $\beta>0$. Moreover there exists a constant $C>0$ such that*

$$\sup_{u\in X(I),\|(-\Delta)^{\frac{1}{4}}u\|_{L^2(I)}\leqslant 1}\int_I e^{\alpha u^2}dx\leqslant C|I| \quad \text{for all } \alpha\leqslant\pi.$$

Our approach in the present paper is based on the Caffarelli-Silvestre approach to fractional Laplacians in [8]. In [8] it was shown that for any $v\in H^{\frac{1}{2}}(\mathbb{R})$, the unique function $w(x,y)$ that minimizes the weighted integral

$$\mathscr{E}_{\frac{1}{2}}(w)\overset{\text{def}}{=}\int_0^\infty\int_{\mathbb{R}}|\nabla w(x,y)|^2dxdy$$

over the set $\left\{w(x,y):\mathscr{E}_{\frac{1}{2}}(w)<\infty, w\big|_{y=0}=v\right\}$ satisfies $\int_{\mathbb{R}}|(-\Delta)^{\frac{1}{4}}v|^2=\mathscr{E}_{\frac{1}{2}}(w)$. Moreover $w(x,y)$ solves the boundary value problem

$$-\operatorname{div}(\nabla w)=0 \text{ in } \mathbb{R}\times\mathbb{R}_+,\ w\big|_{y=0}=v \quad \frac{\partial w}{\partial \nu}=(-\Delta)^{1/2}v(x), \tag{1.1}$$

where $\frac{\partial w}{\partial\nu}=-\lim\limits_{y\to 0^+}\frac{\partial w}{\partial y}(x,y)$. We denote the upper half space in $\mathbb{R}^2$ as $\mathbb{R}_+^2=\{(x,y)\in\mathbb{R}^2|\ y>0\}$. The space $X_1(\mathbb{R}_+^2)$ is defined as the completion of $C_0^\infty(\overline{\mathbb{R}_+^2})$ under the semi-norm

$$\|w\|_{X_1}=\left(\int_{\mathbb{R}_+^2}|\nabla w|^2\,dxdy\right)^{\frac{1}{2}}.$$

For a function $u \in H_0^{\frac{1}{2}}(\mathbb{R})$, the solution $w \stackrel{\text{def}}{=} E_{\frac{1}{2}}(u) \in X_1(\mathbb{R}^2_+)$ of the problem (1.1) is called harmonic extension of u. The map $E_{\frac{1}{2}} : H_0^{\frac{1}{2}}(\mathbb{R}) \to X_1(\mathbb{R}^2_+)$ is an isometry. We look for solutions in the Hilbert space E_V defined as

$$E_V \stackrel{\text{def}}{=} \left\{ w \in X_1(\mathbb{R}^2_+) : \int_{\mathbb{R}} V(x)|w(x,0)|^2 dx < \infty \right\}$$

equipped with the norm

$$\|w\| = \left(\int_{\mathbb{R}^2_+} |\nabla w|^2 dxdy + \int_{\mathbb{R}} V(x)|w(x,0)|^2 dx \right)^{\frac{1}{2}}.$$

From the assumption **(V)** and the continuous embedding of E_V into $L^q(\mathbb{R})$ for $q \in [2,\infty)$, it follows that the embedding $E_V \ni u \mapsto |u|^r \in L^1(\mathbb{R})$ is compact for all $r \in [2,\infty)$. Moreover, the minimization problem

$$\lambda_1 = \min_{w \in E_V} \left\{ \int_{\mathbb{R}^2_+} |\nabla w|^2 + \int_{\mathbb{R}} V(x)|w(x,0)|^2 dx : \int_{\mathbb{R}} |w(x,0)|^2 dx = 1 \right\}$$

admits a non-negative minimizer and $\lambda_1 \geqslant V_0 > 0$.

In this paper, first we discuss the Adimurthi [2] type existence result for the fractional Laplacian equation in (P) with nonlinearity $h(u)$ that has superlinear growth near zero and critical exponential growth near ∞. To prove our result we analyze the first critical level using the Moser functions which are dilations and truncations of fundamental solutions in $\mathbb{R}^2$ and study the compactness of Palais-Smale sequences below this level. In the second part, we discuss the Kirchhoff fractional Laplacian equation in (Q) with critical exponential nonlinearity that behaves like u^θ with $\theta > 3$ near the origin and e^{u^2} at ∞. Here, using the critical level obtained in the section 3, we study the critical level for the Kirchhoff problems and we use the Moser functions concentrating on the boundary along with Lion's Lemma on higher integrability to show the strong convergence of Palais-Smale sequences below the critical level.

We now give the organization of the paper. In section 2, we present a version of Moser-Trudinger inequality which is the central idea of the proof of existence result in section 3. In section 3, we consider the critical exponent problem with positive nonlinearity and prove Adimurthi's type existence result. In section 4, we consider the problem (Q) and study the existence result.

2. A Moser-Trudinger inequality

In this section we prove the Moser-Trudinger inequality on $\mathbb{R}^2_+$. We first recall the following result due to LAM-LU [17] (see Theorem 1.7 page 308).

THEOREM 2.1. *Let $0 < \gamma < n$ be an arbitrary real positive number, $p = \frac{n}{\gamma}$ and $\tau > 0$. There holds*

$$\sup_{u \in W^{\gamma,p}(\mathbb{R}^n), \|(\tau I - \Delta)^{\frac{\gamma}{2}} u\|_p \leqslant 1} \int_{\mathbb{R}^n} \phi(\beta_0(n\gamma)|u|^{p'}) \mathrm{d}x < \infty,$$

where

$$p' = \frac{p}{p-1}, \quad \beta_0(n,\gamma) = \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^{\gamma} \Gamma(\gamma/2)}{\Gamma(\frac{n-\gamma}{2})} \right]^{p'}$$

and

$$\phi(t) = e^t - \sum_{j=0}^{j_p - 2} \frac{t^j}{j!}, \quad j_p = \min\{ j \in \mathbb{N} : j \geqslant p \} \geqslant p.$$

Furthermore this inequality is sharp, i.e., if $\beta_0(n,\gamma)$ is replaced by any $\beta > \beta_0(n,\gamma)$, then the supremum is infinite.

Now we will prove the following theorem:

THEOREM 2.2. *There exists a constant $C > 0$ such that*

$$\sup_{w \in E_V,\, \|w\| \leqslant 1} \int_{\mathbb{R}} (e^{\alpha |w(x,0)|^2} - 1) dx \leqslant C, \quad \textit{for all } 0 \leqslant \alpha \leqslant \pi. \tag{2.1}$$

PROOF OF THEOREM 2.2 :. We use Theorem 2.1 in the case $n = 1$, $\gamma = \frac{1}{2}$ and $\tau > 0$ small enough. Precisely, we have

$$\sup_{u \in W^{1/2,2}(\mathbb{R}), \|(\tau I - \Delta)^{\frac{1}{4}} u\|_2 \leqslant 1} \int_{\mathbb{R}} (e^{\pi |u|^2} - 1) \mathrm{d}x < \infty. \tag{2.2}$$

Now, observing that

$$\begin{aligned} \|(\tau I - \Delta)^{\frac{1}{4}} u\|^2_{L^2(\mathbb{R})} &= \|\widehat{(\tau I - \Delta)^{\frac{1}{4}} u}\|^2_{L^2(\mathbb{R})} \\ &= \|(\tau + |\xi|^2)^{\frac{1}{4}} \hat{u}\|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} (\tau + |\xi|^2)^{\frac{1}{2}} |\hat{u}|^2 \mathrm{d}x \\ &\leqslant \int_{\mathbb{R}} (\tau^{\frac{1}{2}} + |\xi|) |\hat{u}|^2 \mathrm{d}x. \end{aligned} \tag{2.3}$$

In the other hand, let $w \in E_V$, we have from the harmonic extension property that

$$\int_{\mathbb{R}^2_+} |\nabla w|^2 \mathrm{d}x = \|(-\Delta)^{\frac{1}{4}} (w(x,0))\|^2_{L^2(\mathbb{R})}.$$

Therefore, from **(V)** and (2.3), we infer that for $\tau^{\frac{1}{2}} \leqslant V_0$

$$\|w\|^2 \geqslant \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} w(x,0)|^2 + V_0 \cdot w(x,0)^2 \mathrm{d}x \geqslant \|(\tau I - \Delta)^{\frac{1}{4}} w(x,0)\|^2_{L^2(\mathbb{R})}.$$

Hence, from (2.2), (2.1) follows. This completes the proof of Theorem 2.2.

In addition, we have the following lemma.

LEMMA 2.1. *For any* $u \in H^{\frac{1}{2}}(\mathbb{R})$ *and any* $\alpha > 0$*, we have* $\int_{\mathbb{R}}(e^{\alpha u^2} - 1)dx < \infty$*.*

Proof. We use some ideas from [27] (see also [15] for the bounded domain case). Let $u \in H^{\frac{1}{2}}(\mathbb{R})$ and $\alpha > 0$. From Proposition 1 in [22] page 261, there exists $M > 0$ such that for any $q \geqslant 2$ and any $f \in H^{\frac{1}{2}}(\mathbb{R})$,

$$\|f\|_{L^q(\mathbb{R})} \leqslant Mq^{1/2}\|(-\Delta)^{1/4}f\|_{L^2(\mathbb{R})}^{1-2/q}\|f\|_{L^2(\mathbb{R})}^{2/q}.$$

Therefore, for $k \geqslant 1$

$$\|u\|_{L^{2k}(\mathbb{R})}^{2k} \leqslant M^{2k}(2k)^k\|(-\Delta)^{1/4}u\|_2^{2k-2}\|u\|_{L^2(\mathbb{R})}^2.$$

Hence,

$$\int_{\mathbb{R}}(e^{\alpha u^2} - 1)dx = \sum_{k=1}^{\infty}\frac{\alpha^k\|u\|_{L^{2k}(\mathbb{R})}^{2k}}{(k)!} \leqslant \sum_{k=1}^{\infty}\frac{\alpha^k}{(k)!}M^{2k}(2k)^k\|(-\Delta)^{1/4}u\|_2^{2k-2}\|u\|_{L^2(\mathbb{R})}^2,$$

which is a convergent sequence for $0 < \alpha \leqslant \alpha_0$ small enough. Furthermore, we infer that there exists $C_0 > 0$ such that

$$\int_{\mathbb{R}}\left(e^{\alpha_0 u^2} - 1\right)dx \leqslant C_0\|u\|_{L^2(\mathbb{R})}^2, \tag{2.4}$$

for every $u \in H^{\frac{1}{2}}(\mathbb{R})$ with $[u] \leqslant 1$. Let Φ the function defined as $\Phi(t) = \frac{e^{\alpha_0 t^2}-1}{C_0}$ and consider the corresponding Orlicz class and Orlicz space $K_\Phi(\mathbb{R})$, $L_\Phi(\mathbb{R})$ respectively (see [1] page 232 and 233 for definitions). $L_\Phi(\mathbb{R})$, equipped with the Luxemburg norm $\|\cdot\|_\Phi$ is defined as

$$\|u\|_\Phi \stackrel{\text{def}}{=} \inf\left\{k > 0 : \int_{\mathbb{R}}\Phi\left(\frac{|u(x)|}{k}\right)dx \leqslant 1\right\}.$$

We first prove that $H^{\frac{1}{2}}(\mathbb{R})$ is continuously imbedded in $L_\Phi(\mathbb{R})$. For that, let $u \in H^{\frac{1}{2}}(\mathbb{R})$. Then we define $w \stackrel{\text{def}}{=} \frac{u}{\|u\|_{1/2}}$ which satisfies $[w] \leqslant 1$. Thus, in virtue of (2.4),

$$\int_{\mathbb{R}}\Phi(w) \leqslant \|w\|_{L^2(\mathbb{R})} \leqslant 1.$$

Hence, we obtain that .

$$\|w\|_\Phi \leqslant \|w\|_{1/2}.$$

This proves that $(H^{\frac{1}{2}}(\mathbb{R}), \|\cdot\|_{1/2})$ is continuously imbedded in $(L_\Phi(\mathbb{R}), \|\cdot\|_\Phi)$. Consider $E_\Phi(\mathbb{R})$, the closure of the space functions u which are bounded and have a compact support in $\mathbb{R}$. It is easy to prove that $E_\Phi(\mathbb{R}) \subset K_\Phi(\mathbb{R})$ (see [1] page 236). Furthermore from the fact that $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L_\Phi(\mathbb{R})$ continuously and the density of $C_c^\infty(\mathbb{R})$

in $H^{\frac{1}{2}}(\mathbb{R})$ (see [1]), we deduce that $H^{\frac{1}{2}}(\mathbb{R}) \subset E_\Phi(\mathbb{R}) \subset K_\Phi(\mathbb{R})$. Therefore, for any $u \in H^{\frac{1}{2}}(\mathbb{R})$,

$$\int_{\mathbb{R}} \Phi\left(\frac{\alpha^{\frac{1}{2}} u}{\alpha_0^{\frac{1}{2}}}\right) dx < \infty.$$

This ends the proof of the lemma.

3. Critical growth problem

In this section we study the existence of positive solutions for the problem

$$(P) \quad (-\Delta)^{1/2} u + V(x)u = h(u) \text{ in } \mathbb{R},$$

where $V(x)$ satisfies the assumption **(V)** and $h(u)$ satisfies the following critical growth conditions:

(h1) $h \in C^1(\mathbb{R})$, $h(t) = 0$ for $t \leqslant 0$, $h(t) > 0$ for $t > 0$ and h satisfies for any $\varepsilon > 0$, $\lim_{t\to\infty} h(t)e^{-(1+\varepsilon)t^2} = 0$.

(h2) There exists $\mu > 2$ such that for all $u > 0$,

$$0 \leqslant \mu H(u) \leqslant uh(u), \text{ where } H(u) = \int_0^u h(s)ds.$$

(h3) There exist positive constants t_0, M such that

$$H(t) \leqslant Mh(t) \text{ for all } t \in [t_0, +\infty).$$

(h4) $\lim_{t\to\infty} th(t)e^{-t^2} = \infty$.

(h5) $\limsup_{u\to 0} \frac{2H(u)}{u^2} < \lambda_1$.

REMARK 3.1. Prototype examples of h satisfying (h1)-(h5) are $t^p e^{t^2}$ with $p > 1$ and $t^p(e^{t^\beta} - 1)e^{t^2}$ with $p > 1$ and $\beta \in (0,2)$. Nonlinearities of the form $t^p e^{\beta t^2}$ ($\beta > 0$, $p > 1$) can be also dealt with according modifications in the assumptions and minor changes in the proofs. Note in this case that the first critical level of the energy functional I is $\frac{\pi}{2\beta}$.

The variational functional associated to the problem (P) is given as

$$\tilde{I}(u) = \frac{1}{2}\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}} V(x)|u|^2 dx - \int_{\mathbb{R}} H(u)dx.$$

The harmonic extension problem corresponding to (P) is

$$(P_E) \begin{cases} \Delta w = 0, \quad w > 0 \quad \text{in } \mathbb{R}^2_+, \\ -\frac{\partial w}{\partial y} = -w(x,0)V(x) + h(w(x,0)) \text{ on } \mathbb{R}. \end{cases}$$

The variational functional $I : E_V \to \mathbb{R}$ related to the problem (P_E) is given as

$$I(w) = \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla w|^2 dxdy + \frac{1}{2}\int_{\mathbb{R}} V(x)|w(x,0)|^2 dx - \int_{\mathbb{R}} H(w(x,0))dx.$$

Any positive function $w \in E$ is called the weak solution of the problem (P_E) if for any $\phi \in E_V$

$$\int_{\mathbb{R}^2_+} \nabla w.\nabla\phi dxdy + \int_{\mathbb{R}} V(x)w(x,0)\phi(x,0)dx - \int_{\mathbb{R}} h(w(x,0))\phi(x,0)dx = 0. \tag{3.1}$$

It is clear that critical points of I in E_V correspond to the critical points of $\tilde{I}$ in $H^{\frac{1}{2}}(\mathbb{R})$. Thus if w solves (P_E) then $u = \text{trace}(w) = w(x,0)$ is the solution of problem (P) and vice versa.

With this introduction we state the main result of this section.

THEOREM 3.1. *Suppose (h1)-(h5) are satisfied. Then the problem (P) has a weak solution, $u \neq 0$. If $V \in C^{1,\gamma}_{\text{loc}}(\mathbb{R})$, with $\gamma \in (0,1)$, then $u \in C^2(\mathbb{R})$.*

3.1. Mountain-pass solution

We will use the mountain pass lemma to show the existence of a solution in the critical case. The assumption $(\mathbf{V})$ implies that $u \mapsto \int_{\mathbb{R}} |u|^q dx$ is weakly continuous for $q \in [2,\infty)$. Next we have the following:

LEMMA 3.1. *Assume that the conditions (h1)-(h5) hold. Then I satisfies the mountain pass geometry around* 0.

Proof. Using assumption $(h2)$, we get

$$H(s) \geqslant C_1|s|^\mu - C_2$$

for some $C_1, C_2 > 0$ and $\mu > 2$. Hence for function $w \in E_V$ with compact support, we get

$$I(tw) \leqslant \frac{t^2}{2}\|w\|^2 - C_1 t^\mu \int_{\mathbb{R}} |w(x,0)|^\mu dx + C_2|\operatorname{supp} w(x,0)|.$$

Hence $I(tw) \to -\infty$ as $t \to \infty$. Next we will show that there exists $\alpha, \rho > 0$ such that $I(w) > \alpha$ for all $\|w\| = \rho$. From (h1) and (h5), for $\varepsilon > 0, r > 2$ there exists $C_1 > 0$ such that

$$|H(s)| \leqslant \frac{\lambda_1 - \varepsilon}{2}s^2 + C_1|s|^r(e^{(1+\varepsilon)s^2} - 1).$$

Hence, using Hölder's inequality, we get for $t > 1$ and $t' = \frac{t}{t-1}$,

$$\int_{\mathbb{R}} |H(w(x,0))| \leqslant \frac{\lambda_1 - \varepsilon}{2}\int_{\mathbb{R}} |w(x,0)|^2 dx + C_1\int_{\mathbb{R}} |w(x,0)|^r\left(e^{(1+\varepsilon)w(x,0)^2} - 1\right)dx$$

$$\leqslant \frac{\lambda_1-\varepsilon}{2}\|w(\cdot,0)\|^2_{L^2(\mathbb{R})}+C_2\|w(\cdot,0)\|^r_{L^{t'}(\mathbb{R})}\left(\int_{\mathbb{R}}\left(e^{(1+\varepsilon)tw(x,0)^2}-1\right)dx\right)^{\frac{1}{t}}$$
$$\leqslant \frac{\lambda_1-\varepsilon}{2}\|w(\cdot,0)\|^2_{L^2(\mathbb{R})}+C_2\|w(\cdot,0)\|^r_{L^{t'}(\mathbb{R})}\left(\int_{\mathbb{R}}\left(e^{(1+\varepsilon)t\|w\|^2\left(\frac{w(x,0)}{\|w(x,0)\|}\right)^2}-1\right)dx\right)^{\frac{1}{t}}.$$

Now let w such that $\|w\|=\rho$ for sufficiently small ρ and t close to 1 such that $(1+\varepsilon)t\|w\|^2\leqslant\pi$. Then using Moser-Trudinger inequality in (2.1), we get

$$I(w)\geqslant \frac{1}{2}\|w\|^2-\frac{\lambda_1-\varepsilon}{2}\|w(\cdot,0)\|^2_{L^2(\mathbb{R})}-C_3\|w(\cdot,0)\|^r_{L^{t'}(\mathbb{R})}$$
$$\geqslant \frac{1}{2}\left(1-\frac{\lambda_1-\varepsilon}{\lambda_1}\right)\|w\|^2-C_3\|w(\cdot,0)\|^r_{L^{t'}(\mathbb{R})}.$$

Hence there exists $\alpha>0$ such that $I(w)>\alpha$ for all $\|w\|=\rho$ for sufficiently small ρ. Next we show the boundedness of Palais-Smale sequences.

LEMMA 3.2. *Every Palais-Smale sequence of I is bounded in E_V.*

Proof. Let $\{w_k\}$ be a $(PS)_c$ sequence, that is

$$I(w_k)=c+o(1) \text{ and } I'(w_k)=o(1). \tag{3.2}$$

Then,

$$\frac{1}{2}\|w_k\|^2-\int_{\mathbb{R}}H(w_k(x,0))dx=c+o(1),$$
$$\|w_k\|^2-\int_{\mathbb{R}}h(w_k(x,0))w_k(x,0)dx=o(\|w_k\|).$$

Therefore,

$$\left(\frac{1}{2}-\frac{1}{\mu}\right)\|w_k\|^2-\frac{1}{\mu}\int_{\mathbb{R}}\Big(\mu H(w_k(x,0))-h(w_k(x,0))w_k(x,0)\Big)dx=c+o(\|w_k\|).$$

Using assumption (h2), we get $\|w_k\|\leqslant C$ for some $C>0$.

We have the following version of Compactness Lemma that is derived from the Vitali's convergence theorem:

LEMMA 3.3. *For any $(PS)_c$ sequence $\{w_k\}$ of I, there exists $w_0\in E_V$ such that, up to subsequence, $h(w_k(\cdot,0))\to h(w_0(\cdot,0))$ in $L^1_{loc}(\mathbb{R})$ and $H(w_k(\cdot,0))\to H(w_0(\cdot,0))$ in $L^1(\mathbb{R})$.*

Proof. From Lemma 3.2, we get that the sequence $\{w_k\}$ is bounded in E_V. Therefore, up to subsequence, $w_k\rightharpoonup w_0$ weakly in E_V, for some $w_0\in E_V$. Also from equation (3.2), we get $C>0$ such that

$$\int_{\mathbb{R}}h(w_k(x,0))w_k(x,0)dx\leqslant C \text{ and } \int_{\mathbb{R}}H(w_k(x,0))dx\leqslant C.$$

Now using Vitali's convergence theorem, we get

$$h(w_k(\cdot,0)) \to h(w_0(\cdot,0)) \text{ in } L^1_{\text{loc}}(\mathbb{R}).$$

Now to show second part of the above Lemma, we use (h3) and generalized Lebesgue dominated convergence theorem to get

$$H(w_k(\cdot,0)) \to H(w_0(\cdot,0)) \text{ in } L^1_{\text{loc}}(\mathbb{R}). \tag{3.3}$$

Now by assumption (h3) for $R,\ A > 0$ large enough,

$$\int_{\substack{|x|>R\\ |w_k(\cdot,0)|>A}} H(w_k(x,0))dx \leqslant \frac{C}{A}\int_{\substack{|x|>R\\ |w_k|>A}} h(w_k(x,0))w_k(x,0)dx.$$

Since sequence w_k is bounded, for any $\delta > 0$, we can choose A sufficiently large such that

$$\int_{\substack{|x|>R\\ |w_k|>A}} H(w_k(x,0))dx < \frac{\delta}{2}. \tag{3.4}$$

Next, note that from (h5) there exists $C_A > 0$ only depending on A such that

$$\begin{aligned}
\int_{\substack{|x|>R\\ |w_k(\cdot,0)|\leqslant A}} H(w_k(x,0))dx &\leqslant C_A \int_{\substack{|x|>R\\ |w_k(\cdot,0)|\leqslant A}} |w_k(x,0)|^2 dx\\
&\leqslant 2C_A \int_{\substack{|x|>R\\ |w_k(\cdot,0)|\leqslant A}} |w_k(x,0)-w_0(x,0)|^2 dx\\
&\quad + 2C_A \int_{\substack{|x|>R\\ |w_k(\cdot,0)|\leqslant A}} |w_0(x,0)|^2 dx.
\end{aligned}$$

Using the compact embedding of E_V into $L^r(\mathbb{R}), r \geqslant 2$, we can choose R such that

$$\int_{\substack{|x|>R\\ |w_k|\leqslant A}} H(w_k(x,0))dx < \frac{\delta}{2}. \tag{3.5}$$

Hence combining equations (3.3), (3.4), (3.5), the proof follows.

We use the following version of "Moser functions concentrated on the boundary" in the spirit of [4]:

LEMMA 3.4. *There exists a sequence $\{\phi_k\} \subset E_V$ satisfying*

(1) $\phi_k \geqslant 0$, *supp*$(\phi_k) \subset B(0,1)\cap\overline{\mathbb{R}^2_+}$,

(2) $\|\phi_k\| = 1$,

(3) ϕ_k *is constant in* $B(0,\frac{1}{k})\cap\mathbb{R}^2_+$, *and* $\phi_k^2 = \frac{1}{\pi}\log k + O(1)$.

Proof. Let

$$\psi_k(x,y) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log k} & 0 \leqslant \sqrt{x^2+y^2} \leqslant \frac{1}{k} \\ \frac{\log \frac{1}{\sqrt{x^2+y^2}}}{\sqrt{\log k}} & \frac{1}{k} \leqslant \sqrt{x^2+y^2} \leqslant 1 \\ 0 & \sqrt{x^2+y^2} \geqslant 1. \end{cases}$$

Then

$$\int_{\mathbb{R}^2} |\nabla \psi_k|^2 dxdy = 1 \text{ and } \int_{\mathbb{R}^2} |\psi_k|^2 dxdy = O\left(\frac{1}{\log k}\right).$$

Let $\bar{\psi}_k = \psi_k|_{\overline{\mathbb{R}^2_+}}$ and $\phi_k = \dfrac{\bar{\psi}_k}{\|\bar{\psi}_k\|}$. Then $\phi_k \geqslant 0$ and $\|\phi_k\| = 1$. Also

$$\int_{\mathbb{R}^2_+} |\nabla \bar{\psi}_k|^2 dxdy = \frac{1}{2} \text{ and } \int_{\mathbb{R}} |\bar{\psi}_k|^2 dxdy = O\left(\frac{1}{\log k}\right).$$

Therefore $\phi_k^2 = \dfrac{1}{\pi} \log k + O(1)$.

Define

$$\Gamma = \{\gamma \in C([0,1]; E_V) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$$

and mountain pass level as

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)). \tag{3.6}$$

LEMMA 3.5. *Let c be defined as in (3.6). Then $c < \frac{\pi}{2}$.*

Proof. We prove by contradiction. Suppose $c \geqslant \frac{\pi}{2}$. Then we have

$$\sup_{t \geqslant 0} I(t\phi_k) = I(t_k \phi_k) \geqslant \frac{\pi}{2}, \tag{3.7}$$

where functions ϕ_k are given by Lemma 3.4. From equation (3.7), we get

$$t_k^2 \geqslant \pi. \tag{3.8}$$

Now as t_k is point of maximum, we get $\frac{d}{dt} I(t\phi_k)|_{t=t_k} = 0$. Therefore

$$t_k^2 \|\phi_k\|^2 = \int_{\mathbb{R}} h(t_k \phi_k) t_k \phi_k dx. \tag{3.9}$$

Now we estimate the right hand side of equation (3.9) using assumption (h4) as

$$\begin{aligned} t_k^2 = \int_{\mathbb{R}} h(t_k\phi_k) t_k \phi_k dx &\geqslant \int_{B(0,\frac{1}{k})} h(t_k \phi_k) t_k \phi_k dx \\ &\geqslant \frac{2}{k} h(t_k \phi_k(0,0)) t_k \phi_k(0,0) \end{aligned}$$

$$\geqslant Ce^{\frac{1}{\pi}(t_k^2-\pi)(\log k+O(1))}\ \frac{h(t_k\phi_k(0,0))t_k\phi_k(0,0)}{e^{t_k^2\phi_k^2(0,0)}}. \tag{3.10}$$

Now since t_k is bounded we have that $t_k^2 \to \pi$. Thus, (3.10) together with assumption (h4) contradict equation (3.8).

Next we prove Theorem 3.1 using the above Lemmas.

PROOF OF THEOREM 3.1: Using Lemma 3.2, we get a bounded $(PS)_c$ sequence. So there exists $w_0 \in E_V$ such that, up to subsequence, $w_k \rightharpoonup w_0$ in E_V and $w_k(x,0) \to w_0(x,0)$ a.e. in $\mathbb{R}$. We first prove that w_0 solves the problem, then we show that w_0 is non zero. From Lemma 3.2 and equation (3.2), there exists $C>0$ such that

$$\int_{\mathbb{R}} h(w_k(x,0))w_k(x,0)dx \leqslant C \ \text{ and } \ \int_{\mathbb{R}} H(w_k(x,0))dx \leqslant C.$$

Now from Lemma 3.3, we get $h(w_k(\cdot,0)) \to h(w_0(\cdot,0))$ in $L^1(\mathbb{R})$. So for $\phi \in C_c^\infty$ the equation (3.1) holds. Hence from the density of $C_c^\infty(\mathbb{R}^2_+)$ in E_V, w_0 is weak solution of (P_E).

Next we claim that $w_0 \not\equiv 0$. Suppose not. Then from Lemma 3.3, we get

$$H(w_k(x,0)) \to 0 \text{ in } L^1(\mathbb{R}).$$

Hence from equation (3.2), we get $\frac{1}{2}\|w_k\|^2 \to c$ as $k\to\infty$ which implies $\|w_k\|^2 \leqslant \pi-\varepsilon$ for some $\varepsilon>0$. Let

$$0<\delta<\frac{\varepsilon}{\pi} \ \text{ and } \ q=\frac{\pi}{(1+\delta)(\pi-\varepsilon)}>1.$$

Using $\mathrm{sh}(s) \leqslant C(e^{(1+\delta)s^2}-1)$ for some $C>0$ large enough and $(e^s-1)^q \leqslant (e^{sq}-1)$ for $q\geqslant 1$ and Moser-Trudinger inequality (2.1) we get

$$\begin{aligned}\int_{\mathbb{R}} |h(w_k(x,0))w_k(x,0)|^q dx &\leqslant C\int_{\mathbb{R}} (e^{q(1+\delta)w_k(x,0)^2}-1)dx\\ &\leqslant C\int_{\mathbb{R}} (e^{q(1+\delta)\|w_k\|^2\frac{w_k^2(x,0)}{\|w_k\|^2}}-1)dx<\infty.\end{aligned}$$

Therefore by Vitali's convergence theorem, $\int_{\mathbb{R}} h(w_k(x,0))w_k(x,0)dx \to 0$ and from equation (3.2), we get $\lim_k \|w_k\|^2=0$, which is a contradiction. Hence w_0 is a nontrivial solution of the problem (P_E). Now by Theorem 5.2 of [7], we get $w_0 \in L^\infty_{loc}(\mathbb{R}^2_+)$.

To show the positivity and regularity of the solution (in case $V\in C^{1,\gamma}_{\text{loc}}$), w e take the cylinder $\mathscr{C}_a=(-a,a)\times(0,\infty)$ for $a>0$. Then w_0 satisfies the elliptic problem

$$\begin{cases} -\Delta v=0,\ v\geqslant 0 \ \text{in } \mathscr{C}_a\\ \quad v = w_0 \text{ on } \{-a,a\}\times(0,\infty),\\ \frac{\partial v}{\partial \nu} = h(w_0)-V(x)w_0 \text{ on } (-a,a)\times\{0\}\end{cases}$$

Now by taking odd extension to the whole cylinder $(-a,a)\times\mathbb{R}$ as in [7] and noting that $w_0, h(w_0)\in L^\infty(\mathscr{C}_a)$ we get $w_0\in C^{2,\alpha}(\overline{\mathscr{C}_a})$ for some $0<\alpha<1$. Therefore, $u(x)=w_0(x,0)\in C^2(\mathbb{R})$. The positivity of the solution follows from Lemma 4.2 of [7].

REMARK 3.2. We remark that Theorem 3.1 holds for the weighted problem

$$(-\Delta)^{1/2}u+u=K(x)h(u) \text{ in } \mathbb{R}$$

with $h(u)\sim|u|^p, p>1$ is super-quadratic near 0 and $K(x)$ satisfies the assumption introduced in [19]: **(K)** $K\in L^\infty(\mathbb{R})\cap C(\mathbb{R})$. Further, for any sequence $\{A_n\}$ of Borel sets of $\mathbb{R}$ with $|A_n|\leqslant R$ for some $R>0$,

$$\lim_{r\to\infty}\int_{A_n\bigcap\mathbb{R}\setminus[-r,r]} K(x)dx=0 \text{ uniformly for all } n.$$

In this case the embedding E_1 into $L^1(\mathbb{R};K)$ is compact for $r\in(2,\infty)$. If $K\in L^1(\mathbb{R})$ then the embedding is compact for $r\in[2,\infty)$.

REMARK 3.3. We remark that the methods and ideas applied in the present section can be used to show the existence of two solutions (for small λ) for the following problem with convex-concave nonlinearity:

$$(-\Delta)^{\frac{1}{2}}u+V(x)u=\lambda u^q+h(u), u>0 \text{ in } \mathbb{R},$$

where $0<q<1$. The harmonic extension $w(x,y)$ of $u(x)$ satisfies the problem:

$$\Delta w=0 \text{ in } R^2_+,$$
$$-\tfrac{\partial w}{\partial y}(\cdot,0)=(\lambda u^q+h(u)-Vu)(\cdot) \text{ in } \mathbb{R}.$$

The variational functional $J_\lambda : E_V\to\mathbb{R}$ is defined as

$$J_\lambda(w)=\frac{1}{2}\|w\|^2-\frac{\lambda}{q+1}\int_{\mathbb{R}}|w(x,0)|^{q+1}dx-\int_{\mathbb{R}}H(w(x,0))dx.$$

Then it is not difficult to show that J_λ satisfies

$$J_\lambda(w)\geqslant\left(\frac{1}{2}-\frac{\lambda_1-\varepsilon}{\lambda_1}\right)\|w\|^2-\frac{\lambda}{q+1}\|w\|^{q+1}-C\|w\|^p,$$

for some $p>2$. So by taking $\|w\|$ small enough we can show that there exists ρ,R_0,λ_0 such that for all $\lambda<\lambda_0$

$$J_\lambda(w)\geqslant\rho>0 \text{ on } \|w\|=R_0.$$

Also, for a fixed ϕ with compact support in $\mathbb{R}^2_+$, for all small t $J_\lambda(t\phi)<0$. So, we can consider the minimization problem:

$$\min_{\|w\|\leqslant R_0} J_\lambda(w).$$

This minimum is clearly negative and so one can follow Lemma 3.5 and Theorem 3.1 to show the existence of a solution w_λ. Also, w_λ is strict local minimum of J_λ for $\lambda\in(0,\lambda_0)$.

To show the existence of second solution, we can translate the problem to the origin. In other words, a second solution can be obtained as $v_\lambda = w_\lambda + v$ where v satisfies

$$\begin{cases} -\Delta v = 0\,,\ v \geqslant 0 & \text{in } \mathbb{R}^2_+,\\ \frac{\partial v}{\partial \nu} = \lambda g(v+w_\lambda)+h(v+w_\lambda)-\lambda g(w_\lambda)-h(w_\lambda)-V(v+w_\lambda)+Vw_\lambda & \text{on } \mathbb{R}, \end{cases}$$

where $g(s) = s^q$. The corresponding functional $\tilde{J}_\lambda : E_V \to \mathbb{R}$ is defined by

$$\tilde{J}^t_\lambda(v) = \frac{1}{2}\|v\|^2 - \lambda \int_{\mathbb{R}} \tilde{G}(v(x,0)) - \int_{\mathbb{R}} \tilde{H}(v(x,0)),$$

where

$$\tilde{G}(v) = \int_0^v (g(s+w_\lambda)-g(w_\lambda))ds \ \text{ and } \ \tilde{H}(v) = \int_0^v (h(s+w_\lambda)-h(w_\lambda))ds.$$

Since w_λ is a strict local minimum and $\lim\limits_{t\to\infty} \tilde{J}_\lambda(tv) = -\infty$ for fixed v, we can fix $e \in E_V$ such that $\tilde{J}_\lambda(e) < 0$. Let

$$\Gamma = \{\gamma : [0,1] \to E_V : \gamma \text{ continuous on } [0,1], \gamma(0) = 0, \gamma(1) = e\}.$$

Define the mountain pass level

$$\rho_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \tilde{J}_\lambda(\gamma(t)).$$

Now using the growth assumptions on g and h, one can show as in Lemma 3.5 that there exists a Palais-Smale sequence below the mini-max level ρ_0 and the existence of a positive solution v follows similar arguments as in the proof of Theorem 3.1.

4. Kirchhoff type equation with critical nonlinearity

In this section we study the problem

$$(Q) \quad m\left(\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}}u|^2dx + \int_{\mathbb{R}} V(x)u^2dx\right)\left((-\Delta)^{\frac{1}{2}}u + V(x)u\right) = f(u) \text{ in } \mathbb{R},$$

where $V \in C^{0,\gamma}_{\mathrm{loc}}(\mathbb{R})$ with $0<\gamma<1$ verifies **(V)** and $m : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy the following assumptions:

(m1) There exists $m_0 > 0$ such that $m(t) \geqslant m_0$ for all $t \geqslant 0$ and

$$M(t+s) \geqslant M(t) + M(s) \text{ for all } s,t \geqslant 0,$$

where $M(t) = \int_0^t m(s)ds$, the primitive of m.

(m2) There exists constants $a_1, a_2 > 0$ and $t_0 > 0$ such that for some $\sigma \in \mathbb{R}$

$$m(t) \leqslant a_1 + a_2 t^\sigma, \text{ for all } t \geqslant t_0.$$

(m3) $\frac{m(t)}{t}$ is decreasing for $t > 0$.

A typical example of a function satisfying **(m1)-(m3)** is $m(t) = m_0 + at$, where $m_0 > 0$ and $a \geqslant 0$. Also, from **(m3)** one can deduce that

$$\frac{1}{2}M(t) - \frac{1}{4}m(t)t \geqslant 0 \text{ for all } t \geqslant 0.$$

The nonlinearity f satisfies

(f1) $f \in C^1(\mathbb{R}), f(t) = 0$ for $t \leqslant 0$, $f(t) > 0$ for $t > 0$ and f satisfies for any $\varepsilon > 0$, $\lim_{t\to\infty} f(t)e^{-(1+\varepsilon)t^2} = 0$. Moreover, $\lim_{t\to 0} \frac{f(t)}{t^3} = 0$, $\frac{f(t)}{t^3}$ is increasing in $(0,\infty)$ and $\frac{F(t)}{t^\theta}$ is increasing in $(0,\infty)$ for some $\theta > 4$ and where $F(t) = \int_0^t f(s)ds$.

(f2) There exist positive constants t_0, K_0 such that

$$F(t) \leqslant K_0 f(t), \text{ for all } t \geqslant t_0.$$

(f3) $\lim_{t\to\infty} tf(t)e^{-t^2} = \infty$.

REMARK 4.1. Note that the assumption $f(t)/t^{\theta-1}$ is increasing in $(0,\infty)$ with $\theta > 4$ implies $f(t)/t^3$ is increasing in $(0,\infty)$ and $F(t)/t^\theta$ is increasing in $(0,\infty)$.

REMARK 4.2. Prototype examples of f satisfying **(f1)-(f3)** are $t^p e^{t^2}$ with $p \geqslant \theta - 1$ and $t^p(e^{t^\beta} - 1)e^{t^2}$ with $p \geqslant \theta - 1$ and $\beta \in (0,2)$.

Let $w(x,y)$ be the harmonic extension of $u(x)$. Then $w(x,y)$ satisfies the problem

$$\begin{aligned} \Delta w &= 0 \text{ in } \mathbb{R}^2_+ \\ -\tfrac{\partial w}{\partial y}(\cdot,0) &= -V(\cdot)w(\cdot,0) + \frac{f(w(\cdot,0))}{m(\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx + \int_{\mathbb{R}} V u^2 dx)} \quad \text{on } \mathbb{R}. \end{aligned} \tag{4.1}$$

From the definition of $\mathscr{E}_{\frac{1}{2}}$, we have

$$\mathscr{E}_{\frac{1}{2}}(u) = \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx = \int_{\mathbb{R}^2_+} |\nabla w(x,y)|^2 dxdy \text{ and } w(x,0) = u(x).$$

Therefore, the problem in (4.1) is equivalent to

$$\begin{aligned} -m\Big(\|w\|^2\Big)\Delta w &= 0 \text{ in } \mathbb{R}^2_+, \\ m\Big(\|w\|^2\Big)\Big(\tfrac{\partial w}{\partial y}(\cdot,0) + V(\cdot)w(\cdot,0)\Big) &= f(w(\cdot,0)) \quad \text{on } \mathbb{R}, \end{aligned} \tag{4.2}$$

where $\|w\| := \Big(\int_{\mathbb{R}^2_+} |\nabla w|^2 dx + \int_{\mathbb{R}} V(x)w(x,0)^2 dx\Big)^{\frac{1}{2}}$.

DEFINITION 4.1. *We say that $w \in E_V$ is a weak solution of (4.2) if*

$$m\left(\|w\|^2\right)\left(\int_{\mathbb{R}^2_+} \nabla w \nabla \phi dxdy + \int_{\mathbb{R}} V(x)w(x,0)\phi(x,0)dx\right) - \int_{\mathbb{R}} f(w(x,0))\phi(x,0)dx = 0$$

holds for all $\phi \in E_V$.

The variational functional corresponding to (4.2) is defined as $J : E_V \to \mathbb{R}$ as

$$J(w) = \frac{1}{2}M\left(\|w\|^2\right) - \int_{\mathbb{R}} F(w(x,0))dx.$$

Then the trace of critical points of the functional J are weak solutions to (Q). We prove the following theorem in this section.

THEOREM 4.1. *Suppose* **(m1)-(m3)** *and* **(f1)-(f3)** *are satisfied. Then the problem (Q) has a positive weak solution.*

We prove this theorem by using the mountain pass lemma. In the next few lemmas we study the mountain pass structure and properties of Palais-Smale sequences to the functional J. Our proofs closely follow [12].

LEMMA 4.1. *Assume that the conditions* **(m1)**, **(f1)-(f3)** *hold. Then J satisfies the mountain-pass geometry around 0.*

Proof. From the assumptions **(f1)-(f3)**, for $\varepsilon > 0$, $r > 2$, there exists $C > 0$ such that

$$|F(t)| \leqslant \varepsilon|t|^2 + C|t|^r(e^{(1+\varepsilon)|t|^2} - 1), \text{ for all } t \in \mathbb{R}.$$

Therefore, using Sobolev and Hölder inequalities, for $w \in E_V$, we get

$$\begin{aligned}\int_{\mathbb{R}} F(w(x,0))\, dx &\leqslant \varepsilon \int_{\mathbb{R}} |w(x,0)|^2 + C\int_{\mathbb{R}} |w(x,0)|^r(e^{(1+\varepsilon)|w(x,0)|^2} - 1)dx \\ &\leqslant \varepsilon C_1\|w\|^2 + C\|w(\cdot,0)\|^r_{L^{2r}(\mathbb{R})}\left(\int_{\mathbb{R}}\left(e^{2(1+\varepsilon)\|w(\cdot,0)\|^2(\frac{w}{\|w\|})^2} - 1\right)dx\right)^{1/2} \\ &\leqslant \varepsilon C_1\|w\|^2 + C_2\|w\|^r\end{aligned}$$

for $\|w\| \leqslant R_1$, where $(1+\varepsilon)^{1/2}R_1 \leqslant \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$, thanks to Moser-Trudinger inequality in (2.1). Hence

$$J(w) \geqslant \|w\|^2\left(\frac{m_0}{2} - \varepsilon C_1 - C_2\|w\|^{r-2}\right).$$

Since $r > 2$, we can choose ε, $0 < R \leqslant R_1$ small such that $J(w) \geqslant \tau$ for some τ on $\|w\| = R$.

Now by **(f1)** and **(f3)**, for some $\tilde{\theta} > \max\{2, 2(\sigma+1)\}$, there exist C_1, $C_2 > 0$ such that

$$F(t) \geqslant C_1 t^{\tilde{\theta}} - C_2 \text{ for all } t \geqslant 0 \tag{4.3}$$

and condition **(m2)** implies that for all $t \geqslant t_0$

$$M(t) \leqslant \begin{cases} a_0 + a_1 t + \frac{a_2}{\sigma+1} t^{\sigma+1}, & \text{if } \sigma \neq -1, \\ b_0 + a_1 t + a_2 \ln t, & \text{if } \sigma = -1, \end{cases} \tag{4.4}$$

where $a_0 = M(t_0) - a_1 t_0 - \frac{a_2}{\sigma+1} t_0^{\sigma+1}$ and $b_0 = M(t_0) - a_1 t_0 - a_2 \ln t_0$. Now, choose a function $\phi_0 \in E_V$ with compact support, $\phi_0 \geqslant 0$ and $\|\phi_0\| = 1$. Then from (4.3) and (4.4), for all $t \geqslant t_0$, we obtain

$$J(t\phi_0) \leqslant \begin{cases} \frac{a_0}{2} + \frac{a_1}{2} t^2 + \frac{a_2}{2\sigma+2} t^{2\sigma+2} - C_1 t^{\tilde{\theta}} \|\phi_0\|_{\tilde{\theta}}^{\tilde{\theta}} + C_2, & \text{if } \sigma \neq -1, \\ \frac{b_0}{2} + \frac{a_1}{2} t^2 + \frac{a_2}{2} \ln t - C_1 t^{\tilde{\theta}} \|\phi_0\|_{\tilde{\theta}}^{\tilde{\theta}} + C_2, & \text{if } \sigma = -1, \end{cases}$$

from which we conclude that $J(t\phi_0) \to -\infty$ as $t \to +\infty$ since $\tilde{\theta} > \max\{2, 2\sigma+2\}$. Therefore, J satisfies the mountain-pass geometry near 0.

LEMMA 4.2. *Every Palais-Smale sequence of J is bounded in E_V.*

Proof. Let $\{w_k\} \subset E_V$ be a Palais-Smale sequence for J at level c, that is

$$\frac{1}{2} M(\|w_k\|^2) - \int_{\mathbb{R}} F(w_k(x,0))dx \to c \tag{4.5}$$

and for all $\phi \in E_V$

$$\begin{aligned} \Bigg| - m\Big(\|w_k\|^2\Big) \Big(\int_{\mathbb{R}^2_+} \nabla w_k \nabla \phi \; dxdy + \int_{\mathbb{R}} V(x) w_k(x,0) \phi(x,0) dx \Big) \\ - \int_{\mathbb{R}} f(w_k(x,0)) \phi(x,0) dx \Bigg| \\ \leqslant \varepsilon_k \|\phi\|, \end{aligned} \tag{4.6}$$

where $\varepsilon_k \to 0$ as $k \to \infty$. From **(m3)**, **(f1)**, (4.5) and (4.6), we obtain that there exists $C > 0$ independent of k such that

$$\begin{aligned} C + \varepsilon_k \|w_k\| &\geqslant \frac{1}{2} M(\|w_k\|^2) - \frac{1}{\theta} m(\|w_k\|^2) \|w_k\|^2 \\ &\quad - \int_{\mathbb{R}} \Big(F(w_k(x,0)) - \frac{1}{\theta} f(w_k(x,0)) w_k(x,0) \Big) dx \\ &\geqslant \Big(\frac{1}{4} - \frac{1}{\theta} \Big) m(\|w_k\|^2) \|w_k\|^2. \end{aligned}$$

From this and since $\theta > 4$, we obtain the boundedness of the sequence.

Let $\Gamma = \Big\{ \gamma \in C\Big([0,1], E_V\Big) : \gamma(0) = 0,\ J(\gamma(1)) < 0 \Big\}$ and define the mountain-pass level

$$c_* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)). \tag{4.7}$$

Then we have,

LEMMA 4.3. *Let c_* be defined as in (4.7). Then* $c_* < \frac{1}{2}M(\pi)$.

Proof. Let ϕ_k be the sequence of Moser functions as in Lemma 3.4. Assume by contradiction that $c_* \geqslant \frac{1}{2}M(\pi)$. Then for each k, there exists t_k such that

$$\sup_{t>0} J(t\phi_k) = J(t_k\phi_k) = \frac{1}{2}M(\|t_k\phi_k\|^2) - \int_{\mathbb{R}} F(t_k\phi_k(x,0))dx \geqslant \frac{1}{2}M(\pi). \tag{4.8}$$

From (4.8), we see that t_k is a bounded sequence since $J(t_k\phi_k) \to -\infty$ as $t_k \to \infty$. Also using the fact that M is monotone increasing and $F(t_k\phi_k(x,0)) \geqslant 0$ in (4.8), we obtain

$$t_k^2 \geqslant \pi. \tag{4.9}$$

Now since t_k is a point of maximum for the one dimensional map $t \mapsto J(t\phi_k)$, we have $\frac{d}{dt}J(t\phi_k)|_{t=t_k} = 0$. From this it follows that

$$\begin{aligned} m(t_k^2)t_k^2 =& m(t_k^2\|\phi_k\|^2)t_k^2\|\phi_k\|^2 = \int_{\mathbb{R}} f(t_k\phi_k(x,0))t_k\phi_k(x,0)dx \qquad (4.10)\\ &\geqslant \int_{-\frac{1}{k}}^{\frac{1}{k}} f(t_k\phi_k(x,0))t_k\phi_k(x,0)dx \\ &= \frac{2}{k}t_k\phi_k(0,0)f(t_k\phi_k(0,0)). \qquad (4.11) \end{aligned}$$

Then from the above inequality and (4.9), it follows that $t_k^2 \to \pi$. Now as in Lemma 3.4, **(f3)** and (4.10) give the required contradiction. Hence $c_* < \frac{1}{2}M(\pi)$.

Now we have the following version of Lions' Lemma. The proof here is an adaptation of Lemma 2.3 of [29].

LEMMA 4.4. *Let $\{w_k\}$ be a sequence in E_V with $\|w_k\| = 1$ and $w_k \rightharpoonup w_0$ weakly in E_V. Then for any p such that $1 < p < \frac{1}{1-\|w_0\|^2}$, we have*

$$\sup_k \int_{\mathbb{R}} \left(e^{\alpha p(w_k(x,0))^2} - 1\right)dx < \infty, \quad \textit{for all } 0 < \alpha < \pi.$$

Proof. First we note that from the Young inequality, for $\frac{1}{\mu} + \frac{1}{\nu} = 1$,

$$e^{s+t} - 1 \leqslant \frac{1}{\mu}(e^{\mu s} - 1) + \frac{1}{\nu}(e^{\nu t} - 1). \tag{4.12}$$

Now using again the Young inequality $w_k^2 \leqslant (1+\varepsilon)(w_k - w_0)^2 + C(\varepsilon)w_0^2$ and (4.12), we get

$$e^{\alpha p w_k^2} - 1 \leqslant e^{\alpha p\left((1+\varepsilon)(w_k-w_0)^2 + C(\varepsilon)w_0^2\right)} - 1$$

$$\leqslant \frac{1}{\mu}\left(e^{\alpha p\mu\left((1+\varepsilon)(w_k-w_0)^2\right)}-1\right)+\frac{1}{\nu}\left(e^{\alpha p\nu\left(C(\varepsilon)w_0^2\right)}-1\right).$$

Now using the fact that $\|w_k-w_0\|^2=1-\|w_0\|^2+o_k(1)$, we get

$$\alpha p\mu\Big((1+\varepsilon)(w_k-w_0)^2\Big)=\Big(\alpha p\mu(1+\varepsilon)(1-\|w_0\|^2+o_k(1))\Big)\left(\frac{(w_k-w_0)}{\|w_k-w_0\|}\right)^2.$$

Hence for any $1<p<\frac{1}{1-\|w_0\|^2}$, and $\varepsilon>0$ small enough and $\mu>1$ close to 1, we have that

$$\alpha p\mu(1+\varepsilon)(1-\|w_0\|^2)<\pi$$

and the proof follows from (2.1) and Lemma 2.1.
From the fact that $\frac{F(t)}{t^4}$ is increasing (since $\theta\geqslant 4$), we deduce easily the following lemma.

LEMMA 4.5. *Let condition* **(f1)** *holds. Then,* $sf(s)-4F(s)$ *is increasing for* $s\geqslant 0$. *In particular* $sf(s)-4F(s)\geqslant 0$ *for all* $s\geqslant 0$.

Now we define the Nehari manifold associated to the functional J, as

$$\mathcal{N}:=\{0\not\equiv w\in E_V:\langle J'(w),w\rangle=0\}$$

and let $b:=\inf_{w\in\mathcal{N}}J(w)$. Then we need the following Lemma that compare c_* and b.

LEMMA 4.6. $c_*\leqslant b$.

Proof. Let $w\in\mathcal{N}$, define $h:(0,+\infty)\to\mathbb{R}$ by $h(t)=J(tw)$. Then

$$h'(t)=\langle J'(tw),w\rangle=m(t^2\|w\|^2)t\|w\|^2-\int_{\mathbb{R}}f(tw)w\,dx \text{ for all } t>0.$$

Since $\langle J'(w),w\rangle=0$, we have

$$h'(t)=\|w\|^4t^3\left(\frac{m(t^2\|w\|^2)}{t^2\|w\|^2}-\frac{m(\|w\|^2)}{\|w\|^2}\right)+t^3\int_{\mathbb{R}}\left(\frac{f(w)}{w^3}-\frac{f(tw)}{(tw)^3}\right)w^4dx.$$

From (**m3**) and (**f1**), we get $h'(1)=0$, $h'(t)\geqslant 0$ for $0<t<1$ and $h'(t)<0$ for $t>1$. Hence $J(w)=\max_{t\geqslant 0}J(tw)$. Now define $g:[0,1]\to E_V$ as $g(t)=(t_0w)t$, where t_0 is such that $J(t_0w)<0$. We have $g\in\Gamma$ and therefore

$$c_*\leqslant\max_{t\in[0,1]}J(g(t))\leqslant\max_{t\geqslant 0}J(tw)=J(w).$$

Since $w\in\mathcal{N}$ is arbitrary, $c_*\leqslant b$ and the proof is complete. □

PROOF OF THEOREM 4.1: Let $\{w_k\}\in E_V$ be a Palais-Smale sequence of J at level $c_*>0$. That is $J(w_k)\to c_*$ and $J'(w_k)\to 0$. Then by Lemma 4.2, there exists $w_0\in E_V$ such that

$w_k \rightharpoonup w_0$ weakly in E_V. Note that under assumptions **(m1)-(m3)** and **(f1)-(f3)**, a compactness result similar to Lemma 3.3 holds, that is

$$f(w_k(\cdot,0)) \to f(w_0(\cdot,0)) \text{ in } L^1_{loc}(\mathbb{R}) \quad \text{and} \quad F(w_k(\cdot,0)) \to F(w_0(\cdot,0)) \text{ in } L^1(\mathbb{R}).$$

Now we claim that w_0 is the required positive solution. It is not difficult to see that $w_0 \not\equiv 0$. Indeed, if $w_0 \equiv 0$. Then $\int_{\mathbb{R}} F(w_k(x,0))dx \to 0$ and hence

$$\frac{1}{2}M(\|w_k\|^2) \to c_* < \frac{1}{2}M(\pi).$$

Therefore, there exists β such that $\|w_k\|^2 < \beta < \pi$ for all k large. So we can find $q > 1$ and close to 1 so that $q\|w_k\|^2 < \pi$. Now it is easy to show that $\int_{\mathbb{R}} f(w_k(x,0))w_k(x,0)dx \to 0$. Hence

$$o_k(1) = \langle J'(w_k), w_k\rangle = m(\|w_k\|^2)\|w_k\|^2 - \int_{\mathbb{R}} f(w_k(x,0))w_k(x,0)dx = m(\|w_k\|^2)\|w_k\|^2 + o_k(1).$$

That is, $\|w_k\| \to 0$ and $J(w_k) \to 0$ which provides a contradiction.

To show the positivity of the solution, we see that $\|w_k\| \to \rho_0 > 0$ (up to a subsequence). So $J'(w_k) \to 0$ implies that for all $\phi \in E_V$, we have

$$m(\rho_0^2)\left(\int_{\mathbb{R}^2_+} \nabla w_0 \nabla \phi \, dxdy + \int_{\mathbb{R}} V(x)w_0(x,0)\phi(x,0)dx\right) - \int_{\mathbb{R}} f(w_0(x,0))\phi(x,0)dx = 0.$$

That is $u_0(x) = w_0(x,0)$ satisfies the equation

$$(-\Delta)^{\frac{1}{2}}u_0 + Vu_0 = \frac{1}{m(\rho_0^2)}f(u_0) \text{ in } \mathbb{R}.$$

Using the growth condition in **(f1)** of f and Trudinger-Moser inequality, we get $f(u_0) \in L^p_{loc}(\mathbb{R})$ for all $1 < p < \infty$. Therefore by regularity result in proposition 3.1, page 21 in [7], $u_0 \in C^{1,\gamma}_{loc}(\mathbb{R})$ and hence by strong maximum principle (see Lemma 4.2 of [7]), we get $u_0 > 0$ in $\mathbb{R}$.

Claim 1: $m(\|w_0\|^2)\|w_0\|^2 \geqslant \int_{\mathbb{R}} f(w_0(x,0))w_0(x,0)dx$.

Proof. The proof follows ideas in Lemma 5.1 of [12]. For completeness, we give the details here. Suppose by contradiction that $m(\|w_0\|^2)\|w_0\|^2 < \int_{\mathbb{R}} f(w_0(x,0))w_0(x,0)dx$. That is $\langle J'(w_0), w_0\rangle < 0$. Using **(f1)** and the Sobolev imbedding, we can see that $\langle J'(tw_0), w_0\rangle > 0$ for $t > 0$ sufficiently small. Thus there exists $\sigma \in (0,1)$ such that $\langle J'(\sigma w_0), w_0\rangle = 0$. That is $\sigma w_0 \in \mathcal{N}$. Thus according to Lemma 4.5, Lemma 4.6 and **(m3)**,

$$\begin{aligned} c_* \leqslant b \leqslant J(\sigma w_0) &= J(\sigma w_0) - \frac{1}{4}\langle J'(\sigma w_0), \sigma w_0\rangle \\ &= \frac{M(\|\sigma w_0\|^2)}{2} - \frac{m(\|\sigma u_0\|^2)\|\sigma u_0\|^2}{4} + \int_{\mathbb{R}} \frac{(f(\sigma w_0)\sigma w_0 - 4F(\sigma w_0))}{4} \\ &< \frac{1}{2}M(\|w_0\|^2) - \frac{1}{4}m(\|w_0\|^2)\|w_0\|^2 + \frac{1}{4}\int_{\mathbb{R}} (f(u_0)u_0 - 4F(u_0)). \end{aligned}$$

By the lower semicontinuity of the norm and the Fatou's Lemma, we get

$$c_* < \liminf_{k\to\infty} \frac{1}{2}\left(M(\|w_k\|^2) - \frac{1}{2}m(\|w_k\|^2)\|w_k\|^2\right) + \liminf_{k\to\infty} \frac{1}{4}\int_{\mathbb{R}} \left(f(w_k)u_k - 4F(w_k)\right)dx$$

$$\leqslant \lim_{k\to\infty}\Big(J(w_k)-\frac{1}{4}\langle J'(w_k),w_k\rangle\Big)=c_*$$

which is a contradiction and the Claim 1 is proved.

Claim 2: $J(w_0)=c_*$.

Proof. Using $\int_{\mathbb{R}}F(w_k(x,0))\to\int_{\mathbb{R}}F(w_0(x,0))$ as $k\to\infty$ and the lower semicontinuity of the norm we have $J(w_0)\leqslant c_*$. We are going to show that the case $J(w_0)<c_*$ can not occur. Indeed, if $J(w_0)<c_*$ then $\|w_0\|^2<\rho_0^2$. Moreover,

$$\frac{1}{2}M(\rho_0^2)=\lim_{k\to\infty}\frac{1}{2}M(\|w_k\|^2)=c_*+\int_{\mathbb{R}}F(w_0)dx \tag{4.13}$$

which implies

$$\rho_0^2=M^{-1}(2c_*+2\int_{\mathbb{R}}F(w_0)dx).$$

Next defining $v_k=\frac{w_k}{\|w_k\|}$ and $v_0=\frac{w_0}{\rho_0}$, we have $v_k\rightharpoonup v_0$ in E_V and $\|v_0\|<1$.

On the other hand, by Claim 1 and Lemma 4.5, we have

$$J(w_0)\geqslant\frac{M(\|w_0\|^2)}{2}-\frac{m(\|w_0\|^2)\|w_0\|^2}{4}+\int_{\mathbb{R}}\frac{(f(w_0)w_0-4F(w_0))}{4}\geqslant 0.$$

Using this together with Lemma 4.3 and the equality: $2(c_*-J(w_0))=M(\rho_0^2)-M(\|w_0\|^2)$, we get $M(\rho_0^2)\leqslant 2c_*+M(\|w_0\|^2)<M(\pi)+M(\|w_0\|^2)$. Therefore by **(m1)**

$$\rho_0^2<M^{-1}\Big(M(\pi)+M(\|w_0\|^2)\Big)\leqslant\pi+\|w_0\|^2. \tag{4.14}$$

Since $\rho_0^2(1-\|v_0\|^2)=\rho_0^2-\|w_0\|^2$, from (4.14) it follows that $\rho_0^2(1-\|v_0\|^2)<\pi$. Thus, there exists $\beta>0$ such that

$$\|w_k\|^2<\beta<\frac{\pi}{1-\|v_0\|^2}\quad\text{for } k \text{ large.}$$

We can choose $q>1$ close to 1 such that $q\|w_k\|^2\leqslant\beta<\frac{\pi}{1-\|v_0\|^2}$ and using Lemma 4.4, we conclude that for k large

$$\int_{\mathbb{R}}\Big(e^{q|w_k(x,0)|^2}-1\Big)dx\leqslant\int_{\mathbb{R}}\Big(e^{\beta|v_k(x,0)|^2}-1\Big)\leqslant C.$$

Now by standard calculations, using Hölder's inequality and weak convergence of $\{w_k\}$ to w_0, we get $\int_{\mathbb{R}}f(w_k(x,0))(w_k(x,0)-w_0(x,0))dx\to 0$ as $k\to\infty$. Since $\langle J'(w_k),w_k-w_0\rangle\to 0$, it follows that

$$m(\|w_k\|^2)\int_{\mathbb{R}^2_+}\nabla w_k(\nabla w_k-\nabla w_0)\to 0.$$

Now by weak convergence of w_k and $m(t)>0$, we get $w_k\to w_0$ strongly in E_V and $J(w_0)=c_*$. This ends the proof of Claim 2.

Now by Claim 2 and (4.13) we can see that $M(\rho_0^2)=M(\|w_0\|^2)$ which implies that $\rho_0^2=\|w_0\|^2$. Hence, w_0 is a weak solution of (4.2). □

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REFERENCES

[1] D. ADAMS, *Sobolev Spaces*, Academic Press, New york, 1975.

[2] ADIMURTHI, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **17**, (1990), 393–413.

[3] ADIMURTHI AND S. K. PRASHANTH, *Critical exponent problem in* $\mathbb{R}^2$ *-borderline between existence and nonexistence of positive solutions for Dirichlet problem*, Adv. Differential Equations, **5**, (2000), 67–95.

[4] ADIMURTHI AND S. L. YADAVA, *Critical exponent problem in* $\mathbb{R}^2$ *with Neumann boundary conditions*, Commun. in Partial Differential Equations, **15**, 4 (1990), 461–501.

[5] D. APPLEBAUM, *Levy processes - From probability to finance and quantum groups*, Notices Amer. Math. Soc., **51**, (2004), 1336–1347.

[6] X. CABRÉ AND Y. SIRE, *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **31**, 1 (2014), 23–53.

[7] X. CABRÉ AND J.G. TAN, *Positive solutions of nonlinear problem involving the square root of the Laplacian*, Adv. Math., **224**, 2 (2010), 2052–2093.

[8] L. CAFFARELLI AND L. SILVESTRE, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, **32**, (2007), 1245–1260.

[9] A. CAPELLA, *Solutions of a pure critical exponent problem involving the half-Laplacian in annular-shaped domains*, Commun. Pure Appl. Anal., **10**, 6 (2011), 1645–1662.

[10] W. CHOI, S. KIM AND K.A. LEE, *Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian*, J. Funct. Anal., **266**, 11 (2014), 6531–6598.

[11] D. G. DE FIGUEIREDO, O. H. MIYAGAKI AND B. RUF, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range,* Calc. Var. Partial Differential Equations, **3**, 2 (1995), 139–153.

[12] G. M. FIGUEIREDO AND U. B. SEVERO, *Ground state solution for a Kirchhoff problem with exponential critical growth*, (2015), to appear in Milan Journal of Mathematics.

[13] J. GIACOMONI, P. K. MISHRA AND K. SREENADH, *Fractional elliptic equations with critical exponential nonlinearity*, to appear in Advances in Nonlinear Analysis, DOI: 10.1515/anona-2015-0081.

[14] J. GIACOMONI, S. PRASHANTH AND K. SREENADH, *A global multiplicity result for N-Laplacian with critical nonlinearity of concave-convex type*, J. Differential Equations, **232**, (2007), 544–572.

[15] A. IANNIZZOTTO AND M. SQUASSINA, *1/2-Laplacian problems with exponential nonlinearity*, J. Math. Anal. Appl., **414**, (2014), 372–385.

[16] H. KOZONO, T. SATO AND H. WADADE, *Upper bound of the best constant of a Trudinger-Moser inequality and its application to Gagliardo-Nirenberg inequality*, Indiana Univ. Math. J., **55**, (2006), 1951–1974.

[17] N. LAM AND G. LU, *A new approach to sharp Moser-Trudinger and Adams type inequalities: A rearrangement-free argument*, J. Differential Equations, **255**, (2013), 298–325.

[18] J. MARCUS DO Ó, *N-Laplacian equations in* R^N *with critical growth*, Abstract and Applied Analysis, **2**, (1997), 301–315.

[19] J. MARCUS DO Ó, O.H. MIYAGAKI AND M. SQUASSINA, *Nonautonomous fractional problems with exponential growth*, Nonlinear Differential Equations and Applications NoDEA, **22**, 5 (2015), 1395–1410.

[20] L. MARTINAZZI, *Fractional Adams Moser-Trudinger inequalities*, Nonlinear Analysis, **127**, (2015), 263–278.

[21] J. MOSER, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J., **20**, 1 (1970), 1077–1092.

[22] T. OZAWA, *On critical cases of Sobolev's inequalities*, J. Funct. Anal., **127**, 1 (1995), 259–269.

[23] R. PANDA, *Nontrivial solution of a quasilinear elliptic equation with critical growth in* R^n, Proc. Indian Acad. Sci. (Math. Sci.), **105**, (1995), 425–444.

[24] F. PUNZO AND G. TERRONE, *On a fractional sublinear elliptic equation with a variable coefficient, Applicable Analysis*, **94**, 4 (2015), 800–818.

[25] S. SECCHI, *Perturbation results for some nonlinear equations involving fractional operators*, Differ. Equ. Appl., **5**, 2 (2013), 221–236.

[26] J. TAN, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, **36**, 1-2(2011), 21–41.

[27] N.S. TRUDINGER, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech, **17**, (1967), 473–483.

[28] S. YAN, J. YANG AND X. YU, *Equations involving fractional Laplacian operator: Compactness and application*, J. Funct. Anal., **269**, 1 (2015), 47–79.

[29] Y. YANG, *Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space*, J. Funct. Anal., **262**, (2012), 1679–1704.

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