Differential
Equations
& Applications
Volume 8, Number 3 (2016), 335–348 doi:10.7153/dea-08-17

SOLOW MULTI–CAPITAL GROWTH MODEL DESCRIBED BY A SYSTEM OF DIFFERENTIAL EQUATIONS ON TIME SCALES

E. BRESTOVANSKÁ AND M. MEDVEĎ

(*Communicated by Ondřej Došlý*)

Abstract. In this paper we derive a system of differential equations on time scales of the Solow type corresponding to a production function depending on several capitals. A sufficient condition for the exponential stability of the steady-state solution with positive coordinates is proved. The obtained results are applied to the case of the Cobb-Douglas type production function.

1. Introduction

The neoclassical Solow growth model, suggested by R. M. Solow [23], [24] and T. W. Swan [25], has many modifications and a great number of papers and books concerning this model have been published also recently (see e. g. [2], [8], [9], [10], [11], [12], [13], [30]). The classical Solow models use to be described by differential or difference equations. So far all dynamic processes were regarded either as solely continuous or solely discrete. These two dynamic theories were unified by S. Hilger in 1988 in his Ph.D thesis [15], published later in [16], [17]. Here so called calculus on time scales is developed. Some elements of time scale calculus and the theory of differential equations formulated in the framework of this calculus is given in Section 2. For more details see [1], [3], [4].

The basic notion of the calculus on time scales is the derivative of a function on a time scale $\mathbb{T}$, which is an arbitrary closed subset of the set $\mathbb{R}$ of real numbers. The special cases of this set are the set of real numbers, the set $\mathbb{Z}$ of integers, the set $\mathbb{N}$ of natural numbers or the Cantor set $\mathbb{C}$, very well-known also from the chaos theory. In the papers [5], [7] and in the recently published paper [6] (see also [14]) Solow models on time scales with one capital are studied. In this paper we study a Solow model with several capitals. The approach used in the papers [5] and [6] is convenient for the multi-capital model and we are using it in this paper.

In this paper we derive a system of differential equations on time scales of the Solow type corresponding to a production function depending on several capitals. A sufficient condition for the exponential stability of the steady-state solution with positive coordinates is proved. The obtained results are applied to the case of the Cobb-Douglas type production function depending on several capitals. We were motivated by the papers [28], [29], where Solow growth models with several capitals are discussed and by the paper [20], where a Solow growth model with a human capital is studied.

Mathematics subject classification (2010): 34N05, 26E70, 97E40, 97M10.
Keywords and phrases: Δ-derivative, time scales, multi-capital, stability, Cobb-Douglas function.

Paper DEA-08-17

2. Solow multi-capital model on time scales

In Section 2 of the paper [7] some basic definitions and results concerning the time scales analysis and the theory of differential equations on time scales are given. We refer the readers to this paper and also to the books [3], [4] [21] for a deeper introduction to these theories. Let us recall some notions and results necessary for the formulation of the obtained results and their proofs. We use the notation $\mathbb{T}$ for the time scale which is defined as an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. It is assumed that the topology on $\mathbb{T}$ is induced by the standard topology on $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\} \tag{1}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$).

A point $t \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, left-dense, if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, $\rho(t) = t$ holds, respectively. The points that are right-scattered and left-scattered at the same time are called isolated. The function $\mu : \mathbb{T} \to R$, $\mu(t) = \sigma(t) - t$ is called the forward graininess function of the time scale $\mathbb{T}$.

DEFINITION 1. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}^k$ provided

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \qquad \text{where } s \to t,\ s \in \mathbb{T} \setminus \{\sigma(t)\}$$

exists. The value $f^{\Delta}(t)$ is called the Δ-derivative of the function f at the point t. The function f is called Δ-differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

LEMMA 1. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable functions and $t \in \mathbb{T}$. Then the following assertions hold:*

(1) $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$;

(2) *If $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}$ with*

$$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t);$$

(3) *$(f.g) : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}$ with*

$$(f.g)^{\Delta}(t) = f^{\Delta}(t) g(t) + d(\sigma(t)) g^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t));$$

(4) *If $g(t) g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is Δ-differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))}.$$

Throughout this work we use the notation f^{σ} for the function $f \circ \sigma$.

3. Exponential stability of differential equations on time scales

In this section we recall a definition of the exponential function, exponential stability of steady-states of differential equations on time scales (see also [3], [4], [19], [21], [26], [27]) and an exponential stability result, proved in the paper [18]. We will apply this result in the proof of a stability result for steady-states of our multi-capital Solow model on time scales.

DEFINITION 2. The function $\exp_p(t,t_0)$, $t \in \mathbb{T}$ is the solution of the initial value problem

$$y^{\Delta} = py, \qquad y(t_0) = 1,$$

where p is a constant.

DEFINITION 3. We say that a steady-state $\bar{x}$ of the differential equation

$$x^{\Delta} = f(x), \qquad x \in D \subset \mathbb{R}^n, \tag{2}$$

on a time scale $\mathbb{T}$ is exponentially stable if there exist $\delta > 0$, $\lambda > 0$, $\beta > 0$ such that if $x(t)$ is a solution a solution of the equation (2) with $\|x(0) - \bar{x}\| < \delta$, then

$$\|x(t) - \bar{x}\| \leqslant \beta \exp_{-\lambda}(t,0)\delta.$$

We also need the following definition.

DEFINITION 4. An $n \times n$ matrix-valued function $A(t)$, $t \in \mathbb{T}$, is called regressive provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$, where $\mu(t)$ is the forward graininess function on $\mathbb{T}$.

Now we can formulate **the Hoffacker-Jackson's stability theorem** (see [18, Theorem 1.1]).

THEOREM 1. *Let $\mathbb{T} \subset \mathbb{R}$ be a time scale with $\mathbb{T}^+ := \mathbb{T} \cap [0,\infty)$ unbounded and $\mu^* := \limsup_{t\to\infty} \mu(t) < \infty$. Let $f \in C^1(D,R^n)$ and $\bar{x} \in D$ be a steady-state of the differential equation (2). Assume that the Jacobi matrix $A = \frac{\partial f(\bar{x})}{\partial x}$ is regressive and having eigenvalues all within the Hilger imaginary circle*

$$I_{\mu^*} := \Big\{ z \in C_{\mu^*} : \big|z + \frac{1}{\mu^*}\big| = \frac{1}{\mu^*} \Big\},$$

if $\mu* \neq 0$, where $C_{\mu^*} = \{z \in \mathbb{C} : z \neq -\frac{1}{\mu^*}\}$ is the Hilger complex plane and $I_0 = \{z_1 + iz_2 \in \mathbb{C} : z_1 < 0\}$, if $\mu^* = 0$. Then the steady-state $\bar{x}$ is exponentially stable.

4. Multi-capital model of Solow type

Now we introduce a general Solow multi-capital growth model defined by the following production function, dynamic equations for capitals, dynamic equations for a human capital, dynamic equation for a technological progress and dynamic equation for a labor:

1. **Production function**

$$Y = f(K_1, K_2, \dots, K_m, H, AL), \tag{3}$$

with L being employment, K_i, $i \in \{1, 2, \dots, m\}$ capital of type i, H is a human capital A and is a variable which can be interpreted as an indicator of the state of the technology, management or government efficiency etc. The production function f is assumed to satisfy the condition:

$$f(\lambda u_1, \lambda u_2, \dots, \lambda u_m, \lambda v, \lambda w) = \lambda f(u_1, u_2, \dots, u_m, v, w) \tag{4}$$

for all $\lambda \in \mathbb{R}_+$.

2. **Dynamic equations for capitals:**

$$K_i^\Delta(t) = s_i(t)Y(t) - \delta_i(t)K_i(t), \qquad i = 1, 2, \dots, m, \;\; 0 < \delta_i(t) \leqslant 1, \;\; t \in \mathbb{T}, \tag{5}$$

where $s_i(t)Y(t)$ with $s_i(t) \in (0, 1)$, is a part of the production $Y(t)$ invested in the i-th capital component and the function $\delta_i(t)$ represents the rate of the depreciation of the i-th capital.

3. **Dynamic Equation for human capital:**

$$H^\Delta(t) = s_H(t)Y(t) - \delta(t)H(t), \qquad 0 < \delta(t) \leqslant 1, \;\; t \in \mathbb{T}, \tag{6}$$

where $s_H(t)Y(t)$ with $s_H(t) \in (0, 1)$, is a part of $Y(t)$ invested in the human capital and $\delta(t)$ is the rate of the depreciation of the human capital.

4. **Dynamic equation for technological progress:**

$$A^\Delta(t) = g(t)A(t), \qquad 0 < g(t) \leqslant 1, \;\; t \in \mathbb{T}; \tag{7}$$

5. **Dynamic equation for labor:**

$$L^\Delta(t) = n(t)L(t), \qquad 0 < n(t) \leqslant 1, \;\; t \in \mathbb{T}. \tag{8}$$

We will use the notations:

$$k_i(t) = \frac{K_i(t)}{A(t)L(t)}, \quad i = 1, 2, \dots, m, \qquad h(t) = \frac{H(t)}{A(t)L(t)}. \tag{9}$$

THEOREM 2. *Let the functions* $g,n\colon \mathbb{T}\to\mathbb{R}$ *be regressive and assume* (3))–((8). *If* $k_i(t)$, $i=1,2,\dots,m$ *and* h *are defined as in* (9)*, then*

$$k_i^{\Delta}(t) = \frac{s_i}{(1+\mu(t)g(t))(1+\mu(t)n(t))}\Phi(k_1(t),k_2(t),\dots,k_m(t),h(t)) - \frac{\delta(t)+n(t)+g(t)(1+\mu(t)n(t))}{(1+\mu(t)g(t))(1+\mu(t)n(t))}k_i(t), \qquad i=1,2,\dots,m \tag{10}$$

$$h^{\Delta}(t) = \frac{s_H}{(1+\mu(t)g(t))(1+\mu(t)n(t))}\Phi(k_1(t),k_2(t),\dots,k_m(t),h(t)) - \frac{\delta(t)+n(t)+g(t)(1+\mu(t)n(t))}{(1+\mu(t)g(t))(1+\mu(t)n(t))}h(t), \tag{11}$$

where $\Phi(k_1,k_2,\dots,k_m,h)) = f(k_1,k_2,\dots,k_m,h,1)$, $t\in\mathbb{T}$.

Proof. For the simplicity we omit the argument t. Using Lemma 1 we obtain:

$$\begin{aligned}
k_i^{\Delta} &= \left(\frac{K_i}{AL}\right)^{\Delta} = \frac{K_i^{\Delta}AL - K_i[A^{\Delta}L^{\sigma}+AL^{\Delta}]}{ALA^{\sigma}L^{\sigma}}\\
&= \frac{K_i}{A^{\sigma}L^{\sigma}} - \frac{K_iA^{\Delta}L^{\sigma}}{ALA^{\sigma}L^{\sigma}} - \frac{K_iAL^{\Delta}}{ALA^{\sigma}L^{\sigma}}\\
&= \frac{K_i^{\Delta}}{A^{\sigma}L^{\sigma}} - \frac{K_ig(1+\mu L)AL}{AL(1+\mu g)(1+\mu n)} - \frac{K_iAnL}{AL(1+\mu g)(1+\mu n)AL}\\
&= \frac{K_i^{\Delta}}{A^{\sigma}L^{\sigma}} - \frac{g}{1+\mu g}k_i - \frac{n}{(1+\mu g)(1+\mu n)}k_i\\
&= \frac{s_if(K_1,K_2,\dots,K_m,H,AL)-\delta_iK_i}{(1+\mu g)(1+\mu n)AL} - \frac{g}{1+\mu g}k_i - \frac{n}{(1+\mu g)(1+\mu n)}k_i\\
&= \frac{s_i}{(1+\mu g)(1+\mu n)}f(k_1,k_2,\dots,k_m,h,1) - \left(\frac{\delta_i+n}{(1+\mu g)(1+\mu n)} + \frac{g}{1+\mu g}\right)k_i\\
&= \frac{s_i}{(1+\mu g)(1+\mu n)}\Phi(k_1,k_2,\dots,k_m,h) - \frac{\delta_i+n+g(t)(1+\mu n)}{(1+\mu g)(1+\mu n)}k_i,
\end{aligned}$$

i. e., we have obtained the equation (10). The proof of the formula (11) is analogous.

The right-hand side of the system (10), (11) is continuously differentiable and one can show using the Lagrange mean value theorem that it is locally Lipschitz and by [4, Theorem 8.2] and also by the result from [26] the local existence and uniqueness of solutions of the initial value problem for this system is guaranteed.

5. Stability of steady-state solutions

In this section we assume that the system (10), (11) is autonomous, i. e., $g(t)\equiv g$, $n(t)\equiv n$, $s_i(t)\equiv s_i$, $\delta_i(t)\equiv\delta_i$, $i=1,2,\dots,m$, $\delta_H(t)\equiv\delta_H$, $\delta(t)\equiv\delta$, $\mu(t)\equiv\mu$ are constant. Let us write this system in the form

$$k_i^{\Delta}(t) = s_i(\mu)\Phi(k_1(t),k_2(t),\dots,k_m(t),h(t)) - \psi_i(\mu)k_i(t), \qquad i=1,2,\dots,m \tag{12}$$

$$h^{\Delta}(t) = s_H(\mu)\Phi(k_1(t), k_2(t), \dots, k_m(t), h(t)) - \psi_H(\mu)h(t), \tag{13}$$

where

$$s_i(\mu) = \frac{s_i}{(1+\mu g)(1+\mu n)}, \qquad s_H(\mu) = \frac{s_H}{(1+\mu g)(1+\mu n)},$$

$$\psi_i(\mu) = \frac{\delta_i + n + g(1+\mu n)}{(1+\mu g)(1+\mu n)},$$
$$\psi_H(\mu) = \frac{\delta + n + g(1+\mu n)}{(1+\mu g)(1+\mu n)}.$$

Let $(k^*, h^*) = (k_1^*, k_2^*, \dots, k_m^*, h^*)$ be a steady-state or equilibrium, respectively, of the system (12), (13) with $k_1^* > 0,\ k_2^* > 0,\ \dots,\ k_m^* > 0$, $h^* > 0$ and $\mu(t) \equiv \mu^*$, $g(t) \equiv g$, $n(t) \equiv n$ be constant. The Jacobi matrix of the right-hand side of the system (12), (13) at the steady-state (k^*, h^*) is the matrix $\mathscr{A} = \mathscr{A}(f, k^*, h^*, \mu^*, g, n) = (\mathbb{A}_{ij})$, where

$$\mathbb{A}_{ij} = s_i(\mu^*)\frac{\partial f(k^*, h^*, 1)}{\partial k_j} - \psi_i(\mu^*)\delta_{ij}, \qquad i, j \in \{1, 2, \dots m\}, \tag{14}$$

where $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$,

$$\mathbb{A}_{im+1} = s_i(\mu^*)\frac{\partial f(k^*, h^*, 1)}{\partial h}, \qquad i = 1, 2, \dots, m, \tag{15}$$

$$\mathbb{A}_{m+1j} = s_H(\mu^*)\frac{\partial f(k^*, h^*, 1)}{\partial k_j}, \qquad j = 1, 2, \dots, m, \tag{16}$$

$$A_{m+1m+1} = s_H(\mu^*)\frac{\partial f(k^*, h^*, 1)}{\partial h} - \psi_H(\mu). \tag{17}$$

Since the graininess function $\mu(t) \equiv \mu^*$, the system (12), (13) is autonomous and we can apply the Hoffacker-Jackson's stability theorem, i.e., Theorem 1. As a consequence of this theorem we obtain the following theorem.

THEOREM 3. *Let $f \in C^1(D, R^n), D \subset \mathbb{R}_+{}^n$ is an open set,where $\mathbb{R}_+ = [0, \infty)$, $(k^*, h^*) = (k_1^*, k_2^*, \dots, k_m^*, h^*) \in D$ be a steady-state or equilibrium, respectively, of the system (12), (13) with $k_1^* > 0,\ k_2^* > 0,\ \dots,\ k_m^* > 0,\ h^* > 0$ and $\mu(t) \equiv \mu^*,\ g(t) \equiv g$, $n(t) \equiv n$ be constant. Let $\mathscr{A} = \mathscr{A}(f, k^*, h^*, \mu, g, n) = (\mathbb{A}_{ij})$ be the matrix defined by (14), (15), (16), (17) and let this matrix is regressive. Assume that the matrix $\mathscr{A}$ has eigenvalues all within the Hilger imaginary circle $I_{\mu^*} := \{z \in C_{\mu^*} : \left|z + \frac{1}{\mu^*}\right| = \frac{1}{\mu^*}\}$, if $\mu* \neq 0$, where $C_{\mu^*} = \{z \in \mathbb{C} : z \neq -\frac{1}{\mu^*}\}$ is the Hilger complex plane and $I_0 = \{z_1 + iz_2 \in \mathbb{C} : z_1 < 0\}$, if $\mu^* = 0$. Then the steady-state (k^*, h^*) is exponentially stable.*

REMARK 1. If $\|\mathscr{A}\| < 1$, then the matrix $\mathscr{A}$ is regressive and $(I + \mathscr{A})^{-1} = \Sigma_{i=0}^{\infty}(-1)^i\mathscr{A}$.

6. Solow multi-capital model on time scales with Cobb-Douglas type production function

In this section we apply Theorem 2 to the case of the Cobb-Douglas type production function with multi-capital, considered in the papers [28] and [29]. This function has the form

$$Y = \big(AL\big)^{1-\sum_{i=1}^{m}\alpha_i}\prod_{i=1}^{m}K_i^{\alpha_i}, \qquad \sum_{i=1}^{m}\alpha_i < 1,\ \alpha_i \in (0,1],\ \ i = 1,2,\dots,m, \tag{18}$$

with L being employment, K_i is capital of type i, as a government, human or private capital, A is a variable representing other, currently unspecified, economic "environment" conditions that may be important to the production process, e. g., a technological progress. The constants α_i reflect the respective shares of the production factor in total output. This production function leads to the system of differential equations

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = s_i y - (n+g+\delta_i)k_i, \qquad i = 1,2,\dots,m, \tag{19}$$

where $k_i = k_i(t) = \frac{K_i(t)}{A(t)L(t)}$, $y = y(t) = \frac{Y(t)}{A(t)L(t)}$. Obviously

$$y(t) = k_1(t)^{\alpha_1}k_2(t)^{\alpha_2}\dots k_m(t)^{\alpha_m}. \tag{20}$$

The formula

$$\tilde{k}_i = \left[\left(\frac{s_i}{n+g+\delta_i}\right)^{1-\sum_{j=1,j\neq i}^{m}\alpha_i}\prod_{r=1,r\neq i}^{m}\left(\frac{s_r}{n+g+\delta_r}\right)^{\alpha_r}\right]^{\frac{1}{1-\sum_{j=1}^{m}\alpha_j}}. \tag{21}$$

for the steady-state values of this system is presented in [29]. Its proof is given in [28]. If this steady-state is asymptotically stable then we can see from this formula that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} k_1(t)^{\alpha_1}k_2(t)^{\alpha_2}\dots k_m(t)^{\alpha_m} = \tilde{k}_1^{\alpha_1}\tilde{k}_2^{\alpha_2}\dots\tilde{k}_m^{\alpha_m}.$$

EXAMPLE 1. Let $Y = K^{\alpha}H^{\beta}(AL)^{1-\alpha-\beta}$. We have $m = 1$, $K_1 = K$, $k_1 = k$, $k_2 = h$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$. We assume $\delta_1 = \delta_2 = \delta$ and denote $s_1 = s_K$, $s_2 = s_K$. Then the system (19) has the form

$$\frac{\mathrm{d}k}{\mathrm{d}t} = s_K k^{\alpha}h^{\beta} - [n+g+\delta]k \tag{22}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = s_H k^{\alpha}h^{\beta} - [n+g+\delta]h. \tag{23}$$

From the formula (21) it follows the formulae for the steady-state (k_*, h_*) of the system (22), (23):

$$k_* = \left(\frac{s_K}{n+g+\delta}\right)^{\frac{1-\beta}{1-\alpha-\beta}}\left(\frac{s_H}{n+g+\delta}\right)^{\frac{\beta}{1-\alpha-\beta}} \tag{24}$$

$$h_* = \left(\frac{s_H}{n+g+\delta}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{s_K}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha-\beta}}. \tag{25}$$

If $F = (F_1, F_2)$ is the right-hand side of the system (22), (23), then the Jacobi matrix $J_{(k_*,h_*)}F$ of F at (k_*, h_*) has the form

$$J_{(k_*,h_*)}F = \begin{pmatrix} s_K \alpha k_*^{\alpha-1} h_*^{\beta} - [n+g+\delta] & s_K \beta k_*^{\alpha} h_*^{\beta-1} \\ s_H \alpha k_{\alpha}^{\alpha-1} h_*^{\beta} & s_H \beta k_*^{\alpha} h_*^{\beta-1} - [n+g+\delta] \end{pmatrix}.$$

The formulae (24), (25) yield

$$\alpha k_*^{\alpha-1} h_*^{\beta} = \frac{n+g+\delta}{s_K}, \qquad \beta k_*^{\alpha} h_*^{\beta-1} = \frac{n+g+\delta}{s_H}$$

and thus we have

$$J_{(k_*,h_*)}F = [n+g+\delta] \begin{pmatrix} \alpha - 1 & -\frac{s_K}{s_H}\beta \\ -\frac{s_H}{s_K}\alpha & \beta - 1 \end{pmatrix}.$$

The eigenvalues of the matrix $J_{(k_*,h_*)}F$ are $\gamma_1 = [n+g+\delta]\lambda_1$, $\gamma_2 = [n+g+\delta]\lambda_2$, where λ_1, λ_2 are roots of the polynomial

$$P(\lambda) = \lambda^2 - [\alpha + \beta - 2]\lambda + [1 - \alpha - \beta].$$

Since

$$D = [\alpha+\beta-2]^2 - 4[1-\alpha-\beta] = (\alpha+\beta)^2,$$
$$\lambda_1 = \alpha+\beta-1 < 0, \quad \lambda_2 = -1$$

and thus

$$\gamma_1 = [n+g+\delta](\alpha+\beta-1) < 0, \qquad \gamma_2 = -[n+g+\delta] < 0. \tag{26}$$

This means that the steady-state (k_*, h_*) is the stable node (see, e. g. [22]).

Now let us study this model in the framework of mathematical analysis on time scales.

$$Y = K_1^{\alpha_1} K_2^{\alpha_2} \dots K_m^{\alpha_m} H^{\beta} (AL)^{1-\beta-\sum_{i=1}^m \alpha_i}, \tag{27}$$

where K_i is capital of type i, H is a human capital and A is as in the above mentioned model on $\mathbb{R}$. We consider the human capital separately, because authors of many papers are studying the production function with one capital K and a human capital H. Then the Solow system (10), (11) has the form

$$k_i^{\Delta}(t) = \frac{s_i}{(1+\mu(t)g(t))(1+\mu(t)n(t))} k^{\alpha}(t) h(t)^{\beta} - \frac{\delta_i(t) + n(t) + g(t)(1+\mu(t)n(t))}{(1+\mu(t)g(t))(1+\mu(t)n(t))} k_i(t), \tag{28}$$

$i = 1, 2, \dots, m,$

$$h^{\Delta}(t) = \frac{s_H}{(1+\mu(t)g(t))(1+\mu(t)n(t))} k^{\alpha}(t)h(t)^{\beta} - \frac{\delta(t)+n(t)+g(t)(1+\mu(t)n(t))}{(1+\mu(t)g(t))(1+\mu(t)n(t))} h(t), \quad (29)$$

where $k^{\alpha}(t) = k_1(t)^{\alpha_1} k_2(t)^{\alpha_2} \dots k_m(t)^{\alpha_m}$, $k = (k_1, k_2, \dots, k_m)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a multi-index with the norm $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_m$, $\alpha_i \geqslant 0$, $i = 1, 2, \dots, m$.

If $\mu(t) \equiv \mu^*$, we can obtain formulas for steady-state values by using their form (21) for the Solow system system (19) in the following way. In the formula (21) we put instead of the coefficients of the system (19) the corresponding coefficients of the system (28), (29):

$$\begin{aligned}
m &\simeq m+1; \\
s_i \simeq s_i(\mu^*) &= \frac{s_i}{(1+\mu^* g)(1+\mu^* n)}, \qquad i = 1, 2, \dots, m, \\
s_{m+1} \simeq s_H(\mu^*) &= \frac{s_H}{(1+\mu^* g)(1+\mu^* n)}; \\
k_i \simeq k_i, \quad i = 1, 2, \dots, m, &\qquad k_{m+1} \simeq h = \frac{H}{AL};
\end{aligned}$$

$$\begin{aligned}
n+g+\delta_i \simeq \psi_i(\mu^*) &= \frac{n+\delta_i+g(1+\mu^* n)}{(1+\mu^* g)(1+\mu^* n)}, \qquad i = 1, 2, \dots, m, \\
n+g+\delta_i \simeq \psi_H(\mu^*) &= \frac{n+\delta+g(1+\mu^* n)}{(1+\mu^* g)(1+\mu^* n)}, \qquad i = m+1; \\
\frac{s_i}{n+g+\delta_i} \simeq \frac{s_i(\mu^*)}{\psi_i(\mu^*)} &= \frac{s_i}{n+\delta_i+g(1+\mu^* n)}, \qquad i = 1, 2, \dots, m, \\
\frac{s_i}{n+g+\delta_i} &\simeq \frac{s_H}{n+\delta+g(1+\mu^* n)}, \qquad i = m+1.
\end{aligned}$$

Applying the formula (21) we obtain the following formula for the steady-state $(k^*, h^*) = (k_1^*, k_2^*, \dots, k_m^*, h^*)$ of the system (28), (29):

$$k_i^* = \left[\left(\frac{s_i}{n+\delta_i+g(1+\mu^* n)} \right)^{1-\sum_{j=1, j\neq i}^{m+1} \alpha_i} \times \prod_{r=1, r\neq i}^{m+1} \left(\frac{s_r(1+\mu^* g)(1+\mu^* n)}{n+\delta_r+g(1+\mu^* n)} \right)^{\alpha_r} \right]^{\frac{1}{1-\sum_{j=1}^{m+1} \alpha_j}}, \qquad i = 1, 2, \cdots, m, \quad (30)$$

$$h_* = \left[\left(\frac{s_i}{n+\delta_i+g(1+\mu^* n)} \right)^{\left(1-\sum_{j=1, j\neq i}^{m+1} \alpha_i\right)} \times \prod_{r=1, r\neq i}^{m+1} \left(\frac{s_r}{n+\delta_r+g(1+\mu^* n)} \right)^{\alpha_r} \right]^{\frac{1}{1-\sum_{j=1}^{m+1} \alpha_j}}, \quad (31)$$

$$s_{m+1} = s_H, \quad \delta_{m+1} = \delta, \quad \alpha_{m+1} = \beta, \quad k_{m+1} = h.$$

The Jacobi-matrix $\mathscr{A}_{COB} = (\mathbb{A}_{ij})$ of the derivative of the right-hand side the system (30), (31) has the coefficients of the form:

$$\mathbb{A}_{ij} = s_i(\mu^*)\alpha_j \hat{k}^{*\alpha_j} h^* - \psi_i \delta_{ij}, \qquad i,j \in \{1,2,\dots m\}, \tag{32}$$

where $\hat{k}^{*\alpha_j} = k_1^{*\alpha_1} k_2^{*\alpha_2} \cdots k_{j-1}^{*\alpha_{j-1}} k_j^{*\alpha_j - 1} k_{j+1}^{*\alpha_{j+1}} \cdots k_m^{*\alpha_m}, \delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$,

$$\mathbb{A}_{m+1j} = s_H(\mu^*)\alpha_j \hat{k}^{*\alpha_j} h^*, \qquad j = 1,2,\dots,m, \tag{33}$$

$$\mathbb{A}_{jm+1} = s_j \beta k^{*\alpha} h^{\beta-1}, \qquad j = 1,2,\dots,m, \tag{34}$$

$$\mathbb{A}_{m+1m+1} = s_H \beta k^{*\alpha} h^{\beta-1} - \psi_H(\mu^*). \tag{35}$$

Now we can formulate the following stability theorem as a consequence of Theorem 3.

THEOREM 4. *Let $(k^*, h^*) = (k_1^*, k_2^*, \dots, k_m^*, h^*)$ be a steady-state or equilibrium, respectively, of the system (28), (29) with $k_1^* > 0, k_2^* > 0, \dots, k_m^* > 0$, $h^* > 0$ and $\mu(t) \equiv \mu^*$, $g(t) \equiv g$, $n(t) \equiv n$ be constant. Let $\mathscr{A}_{COB} = (\mathbb{A}_{ij})$ be the matrix defined by (32)–(35) and let the matrix $\mathscr{A}_{COB}$ be regressive. Assume that the matrix $\mathscr{A}_{COB}$ has eigenvalues all within the Hilger imaginary circle $I_{\mu^*} := \{z \in C_{\mu^*} : \left|z + \frac{1}{\mu^*}\right| = \frac{1}{\mu^*}\}$, if $\mu* \neq 0$, where $C_{\mu^*} = \{z \in \mathbb{C} : z \neq -\frac{1}{\mu^*}\}$ is the Hilger complex plane and $I_0 = \{z_1 + iz_2 \in \mathbb{C} : z_1 < 0\}$, if $\mu^* = 0$. Then the steady-state solution (k^*, h^*) is exponentially stable.*

For the steady-state (k^*, h^*) satisfying the conditions of this theorem we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} k_1(t)^{\alpha_1} k_2(t)^{\alpha_2} \cdots k_m(t)^{\alpha_m} = k_1^{*\alpha_1} k_2^{*\alpha_2} \cdots k_m^{*\alpha_m},$$

where

$$y(t) = \frac{Y(t)}{A(t)L(t)} = k_1(t)^{\alpha_1} k_2(t)^{\alpha_2} \cdots k_m(t)^{\alpha_m}.$$

From this value and the formulas (30), (31) one can see how it depends on the rates s_i, $i \in \{1,2,\dots,m\}, s_H$ of the production invested in the physical capitals and the human capital, respectively. Of course, this value depends also on the coefficients n, g of the dynamic equation for the labor L and the technological progress A, respectively, and on depreciation factors δ_i, $i \in \{1,2,\dots,m\}$ (see the dynamic equations (5) for the capitals K_i) and on the forward graininess μ of the time scale $\mathbb{T}$, which is equal 0, if $\mathbb{T} = \mathbb{R}$ – the set of all real numbers, $\mu = 1$, for $\mathbb{T} = \mathbb{Z}$ – the set of all integers and $\mu = h$ for $\mathbb{T} = h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$, where $h > 0$. In the first case the equations (5)–(8) are differential and in the case $\mathbb{Z}$ and $h\mathbb{Z}$ these are difference.

7. Solow model with one physical capital and a human capital

In this section we apply Theorem 4 to the Solow model with the production function $Y = K^\alpha H^\beta (AL)^{1-\alpha-\beta}$. In this case the system (28), (29) has the form

$$k^\Delta = s_K(\mu) k^\alpha h^\beta - \psi_K(\mu) k \tag{36}$$

$$h^{\Delta} = s_H(\mu)k^{\alpha}h^{\beta} - \psi_H(\mu)h, \tag{37}$$

where

$$s_K(\mu) = \frac{s_K}{(1+\mu g)(1+\mu n)}, \qquad \psi_K(\mu) = \frac{\delta_1 + n + g(1+\mu n)}{(1+\mu g)(1+\mu n)}, \tag{38}$$

$$s_H(\mu) = \frac{s_H}{(1+\mu g)(1+\mu n)}, \qquad \psi_H(\mu) = \frac{\delta_2 + n + g(1+\mu n)}{(1+\mu g)(1+\mu n)}. \tag{39}$$

Here we have $m = 1$, $K_1 = K, \alpha_1 = \alpha$, $s_1 = s_K$. The steady-state of the system (36), (37) has the coordinates:

$$k^* = \left(\frac{s_K}{n+\delta_1+g(1+\mu n)}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{s_H}{n+\delta_2+g(1+\mu n)}\right)^{\frac{\beta}{1-\alpha-\beta}}, \tag{40}$$

$$h^* = \left(\frac{s_H}{n+\delta_2+g(1+\mu n)}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{s_K}{n+\delta_1+g(1+\mu n)}\right)^{\frac{\alpha}{1-\alpha-\beta}}. \tag{41}$$

From (40), (41) it follows that

$$k^{*\alpha-1}h^{*\beta} = \frac{\delta_1+n+g(\mu+n)}{s_K}, \qquad k^{*\alpha}h^{*\beta-1} = \frac{\delta_2+n+g(\mu+n)}{s_H}$$

and (38), (39) yield

$$\frac{s_K(\mu)}{s_K} = \frac{s_H(\mu)}{s_H} = \frac{1}{(1+\mu g)(1+\mu n)}. \tag{42}$$

Applying the equalities (38), (39) and (42) we obtain that the Jacobi matrix $\mathscr{A}_{COB}$ of the right-hand side of (36), (37) at the steady-state $(k*, h^*)$ has the form

$$\mathscr{A}_{COB}(\mu) = \begin{pmatrix} s_K(\mu)\alpha k^{*\alpha-1}h^{*\beta} - \psi_K(\mu) & s_K(\mu)\beta k^{*\alpha}h^{*\beta-1} \\ s_H(\mu)\alpha k^{*\alpha-1}h^{*\beta} & s_H(\mu)\beta k^{*\alpha}h^{*\beta-1} - \psi_H(\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\alpha(\delta_1+n+g(\mu+n))}{(1+\mu g)(1+\mu n)} - \psi_K(\mu) & \frac{\beta(\delta_1+n+g(\mu+n))}{(1+\mu g)(1+\mu n)} \\ \frac{\alpha(\delta_2+n+g(\mu+n))}{(1+\mu g)(1+\mu n)} & \frac{\beta(\delta_2+n+g(\mu+n))}{(1+\mu g)(1+\mu n)} - \psi_H(\mu) \end{pmatrix}.$$

Now let us consider the case $\delta_1 = \delta_2 = \delta$. Then $\psi_K(\mu) = \psi_H(\mu) = \psi(\mu)$ and the matrix $\mathscr{A}_{COB}(\mu)$ has the form

$$\mathscr{A}_{COB}(\mu) = \psi(\mu)\begin{pmatrix} \alpha-1 & -\beta \\ -\alpha & \beta-1 \end{pmatrix}. \tag{43}$$

The eigenvalues of this matrix are

$$\lambda_1(\mu) = \psi(\mu)[\alpha+\beta-1] = \frac{\delta+n+g(\mu+n)}{(1+\mu g)(1+\mu n)}[\alpha+\beta-1],$$

$$\lambda_2(\mu) = -\psi(\mu) = -\frac{\delta+n+g(\mu+n)}{(1+\mu g)(1+\mu n)}.$$

If the forward graininess $\mu(t) = \mu \neq 0$ is constant, then $\lambda_1(\mu)$ ($\lambda_2(\mu)$) is within the Hilger circle I_{μ^*} if and only if

$$\left|1 + \frac{\delta + n + g(\mu + n)}{(1+\mu g)(1+\mu n)}[\alpha + \beta - 1]\right| < 1 \tag{44}$$

$$\left(\left|1 - \frac{\delta + n + g(\mu + n)}{(1+\mu g)(1+\mu n)}\right| < 1\right). \tag{45}$$

Since the eigenvalues $\lambda_1(\mu)$, $\lambda_2(\mu)$ depend continuously on δ we can formulate the following theorem for the system with different rates of depreciation which is a consequence of Theorem 4.

THEOREM 5. *Let the conditions* (44), (45) *be satisfied. Then there exists an* $\varepsilon > 0$ *such that if* $\delta_1 > 0,\ \delta_2 > 0$ *with* $|\delta_i - \delta| < \varepsilon,\ i = 1,2$, *then the equilibrium* (k_*, h_*),

$$k^* = \left(\frac{s_K}{n + \delta_1 + g(1+\mu n)}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{s_H}{n + \delta_1 + g(1+\mu n)}\right)^{\frac{\beta}{1-\alpha-\beta}}, \tag{46}$$

$$h^* = \left(\frac{s_H}{n + \delta_2 + g(1+\mu n)}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{s_K}{n + \delta_2 + g(1+\mu n)}\right)^{\frac{\alpha}{1-\alpha-\beta}} \tag{47}$$

of the system (36), (37) *is exponentially stable.*

If the conditions of this theorem are satisfied then

$$y^* = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{Y(t)}{A(t)L(t)} = \lim_{t\to\infty} k(t)^\alpha h(t)^\beta = k{*}^\alpha h{*}^\beta,$$

where k^*, h^* are given by the formulas (40), (41). In the special case $\delta_1 = \delta_2 = \delta$, we have

$$y^* = \left(\frac{s_K}{n + \delta + g(1+\mu n)}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_H}{n + \delta + g(1+\mu n)}\right)^{\frac{\beta}{1-\alpha-\beta}}.$$

This formula shows how effective is to invest parts $s_K Y(t), s_H Y(t)$ of the production $Y(t)$ in the component of the physical capital and in the component of the human capital, respectively.

Acknowledgements. The authors would like to express sincerely thanks to the anonymous referee for his/her carefully reading the manuscript and valuable comments. This research was supported by the Slovak Grant Agency VEGA-MŠ, project No. 1/0071/14.

REFERENCES

[1] R. AGARWAL, M. BOHNER, D. O'REGAN AND A. PETERSON, *Dynamic Equations on time scales: A survey*, J. Math. Comp. and Appl. Math., **141** (2002), 1–26.

[2] E. ACCINELLI AND J. G. BRIDA, *Population growth and the Solow-Swan model*, International Journal of Ecological Economics and Statistics, **8**, 7 (2007), 54–63.

[3] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhaäuser, Boston, MA 2003.

[4] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhaäuser, Boston, Basel, Berlin 2001.

[5] M. BOHNER, J. HEIM AND A. LIU, *Solow models on time scales*, CUBO, A Mathematica Journal **15**, 1 (2013), 13–32.

[6] M. BOHNER, J. HEIM AND A. LIU, *Qualitative analysis of a Solow model on time scales*, J. Concr. Appl. Math. **13**, 3–4 (2015), 183–197.

[7] E. BRESTOVANSKÁ AND M. MEDVEĎ, *Solow differential equations on time scales: A unified approach to continuous and disrete Solow growth model*, Differ. Equ. Appl., **5**, 4 (2013), 473–488.

[8] S. BRIANZONI, C. MAMMANA AND E. MICHETTI, *Local and global dynamics in discrete time growth model with nonconcave production function*, Discrete Dyn. Nat. Soc., (2012), Article ID 536570, 22 pages, doc:10.1155/2012/53657.

[9] S. BRIANZONI, C. MAMMANA AND E. MICHETTI, *Complex dynamics in the neoclassical growth model with differential saving and non-constant labour force growth*, Stud. Nonlinear Dyn. Econom., **11**, 3 (2007).

[10] S. BRIANZONI, C. MAMMANA AND E. MICHETTI, *Global attractor in Solow growth model with differential savings and endogenic labour force growth*, AMS Periodicals, Modelling Measurement and Control D **29**, 2 (2008), 19–37.

[11] J. G. BRIDA AND E. L. MALDONADO, *Closed form solutions to a generalization of the Solow growth model*, Applied Mathematical Sciences, **1**, 40 (2007), 1991–2000.

[12] J. G. BRIDA AND J. S. PEREYRA, *The Solow model in discrete time and decreasing population growth rate*, Economic Bulletin, **3**, 41 (2008), 1–14.

[13] D. CHEBAN, C. MAMMANA AND E. MICHETTI, *Global attractors of quasi-linear non-autonomous difference equations: A growth model with endogenous population growth*, Nonlinear Anal.: Serie B, **14** (2013), 1716–1731.

[14] J. HEIM, *Economics and Finance on time scale*s, Dissertation, Faculty of the Graduate School, Missouri University of Science and Technology, 2012 (Advisor: M. Bohner).

[15] S. HILGER, *Ein Maßkettenkalkül Anwendung auf Zentrumsmannigfaltigkeiten*, PhD thesis, Universität Würzburg, 1988.

[16] S. HILGER, *Analysis on measure chains – A unified approach to continuous and discrete calculus*, Results Math., **18** (1990), 18–56.

[17] S. HILGER, *Differential and difference calculus – unified!*, Nonlinear Analysis, TMA **30**, 5 (1997), 2683–2694.

[18] J. HOFFACKER AND B. JAKSON, *Stability results for higher dimensional equations on time scales*, Int. J. Dynamical Systems and Differential equations, **3**, No. 1–2 (2011), 48–58.

[19] T. KULIK AND C. C. TISDELL, *Volterra integral equations on time scales, Basic qualitative and quantitative results with applications to initial value problems on unbounded domains*, International Journal of Difference Equations, **3**, 1 (2008), 103–133.

[20] M. G. MANKIW, D. ROMER AND D. N. WEIL, *A contribution to the empirics of economic growth*, The Quarterly J. of Economics, **107**, 2 (1992), 407–437.

[21] V. LAKSHMIKANTAN, S. SIVASUNDRAM, B. KAYAMKCALAN, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Netherlands 1996.

[22] M. MEDVEĎ, *Fundamentals of Dynamical Systems and Bifurcation Theory*, Adam Hilger, Bristol, Philadelphia and New York 1992.

[23] R. M. SOLOW, *Contribution to the theory of economic growth*, Quarterly Journal of Economics, **70**, 1 (1956), 65–94.

[24] R. M. SOLOW, *Technical change and the aggregate production function*, Review of Economics and statistics, **39**, 3 (1957), 312–320.

[25] T. W. SWAN, *Economic growth and capital accumulation*, Economic Record, **32**, 2 (1956), 334–361.

[26] C. C. TISDELL AND A. ZAIDI, *Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales*, Nonlinear Anal., **68**, 11 (2008), 3504–3524.

[27] C. C. TISDELL AND A. H. ZAIDI, *Successive approximations to solutions of dynamic dynamic equations on time scales*, Comm. Appl. Nonlinear Anal., **16**, 1 (2009), 61–87.

[28] P. VANHOUDT, *Are public and private outlays for physical and knowledge capital equally productive?* Applied Economics, **31** (1999), 1401–1410.

[29] J. VANHOUDT, *How productive are capital investments in Europe?* European Investment Bank (EIB). ISSN 0257-7755, **5**, 2 81–106 (http://hdl.handle.net/10419/44797).

[30] H. WIESE, *Intermediate Growth Theory*, Springer-Verlag, Berlin, Heidelberg, New York 2006.

(Received March 13, 2015)
(Revised December 9, 2015)

Eva Brestovanská
Department of Economics and Finance
Faculty of Management, Comenius University
Odbojárov str. 10, 831 04 Bratislava, Slovakia
e-mail: Eva.Brestovanska@fm.uniba.sk

Milan Medved'
Department of Mathematical Analysis and Numerical Mathematics
Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia
e-mail: Milan.Medved@fmph.uniba.sk