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DIFFUSIVE SOLUTIONS OF THE COMPETITIVE LOTKA–VOLTERRA SYSTEM

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Abstract. In the present work we show by means of explicit construction that three new types of solutions exist for the one dimensional competitive Lotka-Volterra reaction-diffusion system. The new solutions constructed are (i) space-time separated solutions, (ii) unbounded solutions, and (iii) solutions of Gaussian type, with the constructions being based largely on the standard methods for constructing solutions to the one-dimensional heat equation. From these exact solutions a new and interesting phenomena is found, namely diffusion-induced long-term coexistence of three species. In addition, the approach to constructing explicit solutions presented here can readily be applied to other reaction-diffusion systems.

1. Introduction

In the present paper we study solutions to the competitive three species Lotka-Volterra system, which is a frequently-used model to describe the competition among three distinct biological species. The system of equations is

$$\begin{cases} u_t = \delta_1 u_{xx} + u(r_1 - l_{11}u - l_{12}v - l_{13}w), \\ v_t = \delta_2 v_{xx} + v(r_2 - l_{21}u - l_{22}v - l_{23}w), \quad x \in \mathbb{R}, \quad t > 0, \\ w_t = \delta_3 w_{xx} + w(r_3 - l_{31}u - c_{32}v - l_{33}w), \end{cases} \tag{1.1}$$

where $u(x,t)$, $v(x,t)$ and $w(x,t)$ stand for the density of the three distinct species; δ_i, r_i, l_{ii} $(i = 1,2,3)$, and l_{ij} $(i,j = 1,2,3)$ are the diffusion rates, the intrinsic growth rates, the intra-specific competition rates, and the inter-specific competition rates, which are all assumed to be positive constants. Under suitable scalings of the dependent and independent variables, (1.1) can be rewritten as

$$(LV)\begin{cases} u_t = u_{xx} + u(1 - u - a_1 v - b_1 w), \\ v_t = d_1 v_{xx} + \lambda_1 v(1 - a_2 u - v - b_2 w), \quad x \in \mathbb{R}, \quad t > 0, \\ w_t = d_2 w_{xx} + \lambda_2 w(1 - a_3 u - b_3 v - w), \end{cases}$$

where a_i, b_i $(i = 1,2,3)$, and d_i, λ_i $(i = 1,2)$ are positive constants. Throughout the remainder of the paper, it is assumed that $d_1 = d_2 = 1$ unless otherwise specified.

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In ecology, determining which species will survive in a competitive system is of fundamental importance. In order to study this problem, we can use traveling wave solutions, which are solutions of the form

$$(u(x,t),v(x,t),w(x,t)) = (u(z),v(z),w(z)), \quad z = px - \theta t. \tag{1.2}$$

Here $p > 0$ is a constant and θ/p represents the wave velocity of the traveling wave. The sign of the propagation speed θ/p determines which species survive(s) in the competition among the three species.

When v and w are absent in (LV), we obtain the well-known Fisher-KPP (Kolmogoroff, Petrovsky, and Piscounoff) equation, i.e.

$$u_t = u_{xx} + u(1-u), \quad x \in \mathbb{R}, \quad t > 0. \tag{1.3}$$

Fisher was the first to use (1.3) as a model to describe the propagation of an advantageous gene in a population ([11]).

Following the pioneering work of KPP ([19]) on traveling wave solutions of (1.3), there have been numerous studies of this special form of solution. We note in particular that for a specific wave velocity, (1.3) admits an exact traveling wave solution ([1]). This solution was found by applying Painlevé analysis of ordinary differential equations. Related results on the existence of traveling wave solutions can be found, for example, in [2, 9, 10, 23, 28] and references cited therein. In [9, 10], the existence, uniqueness (up to a translation), minimum wave speed and global stability of traveling front solutions for (1.3) were established by employing comparison theorems and a priori estimates. Excepting the pure initial value problem in the space $(x,t) \in (-\infty,\infty) \times (0,\infty)$, the initial-boundary value problem in the space $(x,t) \in (0,\infty) \times (0,\infty)$ was also considered in [2].

For the case where w is absent, (LV) reduces to the two-species system:

$$\begin{cases} u_t = u_{xx} + u(1-u-a_1 v), \\ v_t = d_1 v_{xx} + \lambda_1 v(1-a_2 u - v), \quad x \in \mathbb{R}, \quad t > 0. \end{cases} \tag{1.4}$$

Much effort has been devoted to studying the existence of traveling wave solutions for (1.4). See for instance, [8, 12, 14, 15, 16, 17, 18, 20, 21, 26]. In particular, Mimura and Rodrigo ([24, 25]) constructed exact traveling wave solutions of (1.4) by applying a judicious ansatz for solutions. However, very little is known about the existence of nontrivial solutions (i.e. $u,v,w > 0$) for (LV). Under certain assumptions on the parameters, existence of nontrivial solutions for (LV) is shown by giving exact traveling wave solutions as well as numerical simulations ([3, 4]). A question naturally arises: In addition to traveling wave solutions, does (LV) admit *other* classes of solutions? Motivated by the work of Cherniha and Davydovych (Theorem 3. in [6]), we partially answer this question affirmatively, i.e. it turns out that, under certain conditions on the parameters, (LV) admits *space-time separated solutions*, *unbounded solutions*, and *solutions of Gaussian type* (Theorem 2.5). By assuming hypotheses $(H1) \sim (H2)$ in Theorem 2.1, we are able to construct the three classes of solutions from the solutions of the heat equation. As a result, in this work we present a completely new approach to explore solutions of (LV), which have not yet been found in the literature.

In [6], Q-conditional symmetries are employed to find exact solutions, which are not positive for all $x \in \mathbb{R}$ and $t > 0$, of (1.4), while our method is elementary and can be applied to find positive solutions for (LV). All [6, 24, 25] are devoted to investigating exact solutions of (1.4). The essential difference of their work lies in that two distinct types of solutions are found, i.e., exact traveling wave solutions are constructed in [24, 25] and space-time separated solutions are found in [6], respectively. For multi-component Lotka-Volterra systems, refer to [22] and [27] for example.

We also propose a question which serves as one motivation for studying (LV). In the absence of diffusion effects, (LV) becomes the following system of ODEs:

$$\begin{cases} u_t = u(1-u-a_1 v - b_1 w), \\ v_t = v(1-a_2 u - v - b_2 w)\,\lambda_1, \quad t>0, \\ w_t = w(1-a_3 u - b_3 v - w)\,\lambda_2, \end{cases} \tag{1.5}$$

which in general has eight spatially homogeneous equilibria. Among these equilibria, we denote by $(u^\sharp, v^\sharp, w^\sharp)$, with $u^\sharp, v^\sharp, w^\sharp > 0$, the unique coexistence equilibrium of the three species if it exists. Except for the coexistence equilibrium $(u^\sharp, v^\sharp, w^\sharp)$, any of the other seven equilibria, if they exist, represent a state in which at least one species is extinct. Assuming the parameters $a_i, b_i (i = 1,2,3)$, and $\lambda_i (i = 1,2)$ are such that $(u^\sharp, v^\sharp, w^\sharp)$ is an unstable equilibrium, the solution of (1.5) with any initial condition $u(0), v(0), w(0) > 0$ will not eventually tend to $(u^\sharp, v^\sharp, w^\sharp)$ when t approaches infinity. Now, if diffusion is incorporated into (1.5) in a manner similar to (LV), we are led to the following question: do there exist solutions $(u(x,t), v(x,t), w(x,t))$ of (LV) with the asymptotic behaviour $(u(x,t), v(x,t), w(x,t)) \to (u^\sharp, v^\sharp, w^\sharp)$ uniformly in x as $t \to \infty$? If such a situation occurs, we refer to this phenomenon as *diffusion-induced long-term coexistence*. Indeed, we show in this paper that the answer to this question is affirmative by giving exact solutions of (LV). We remark that the coexistence here is proved only under special perturbations of the initial data. To the author's knowledge, the phenomenon related to diffusion-induced coexistence (see also [29]) have not yet been extensively studied in the literature.

The remainder of this paper is organized as follows. Section 2 is devoted to the main results (Theorem 2.1 and Theorem 2.5) as well as the proof. For some parameter regimes, it is shown in Section 3 that diffusion-induced long-term coexistence occurs for the solutions of (LV). We provide in Section 4 an alternative approach, which is based on a judicious ansatz for solutions, to space-time separated solutions. In Section 5, examples are given to illustrate the main theorems. Finally, we conclude the present paper with some remarks in Section 6.

2. Main results

The goal of this section is to prove our main results i.e. Theorem 2.1 and Theorem 2.5. In Theorem 2.1, we give a sufficient condition under which solutions of (LV) are constructed from solutions of the initial value problem for the heat equation. Depending on various initial conditions, three new types of solutions, including *space-time* *separated solutions*, *unbounded solutions*, and *solutions of Gaussian type* are found in Theorem 2.5.

THEOREM 2.1. *Assume that* $d_1 = d_2 = 1$ *and that the following hypotheses hold:*
$(H1)$: $1-u-a_1 v-b_1 w$, $1-a_2 u-v-b_2 w$, *and* $1-a_3 u-b_3 v-w$ *are linearly dependent, in the sense that there exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $c_1 c_2 \neq 0$ *such that* $c_1(1-u-a_1 v-b_1 w)+c_2(1-a_2 u-v-b_2 w)+(1-a_3 u-b_3 v-w)=0$ *for* u, v, $w \in \mathbb{R}$. *In other words, the solution of the system*

$$\begin{cases} 1-u-a_1 v-b_1 w=0, \\ 1-a_2 u-v-b_2 w=0, \\ 1-a_3 u-b_3 v-w=0 \end{cases} \tag{2.1}$$

can be expressed by

$$(u_0, v_0, w_0) = \left(\frac{-1+a_1}{-1+a_1 a_2} + \frac{w_0\,(b_1-a_1 b_2)}{-1+a_1 a_2}, \frac{-1+a_2}{-1+a_1 a_2} + \frac{w_0\,(-a_2 b_1+b_2)}{-1+a_1 a_2}, w_0 \right)$$

with $u_0, v_0, w_0 > 0$;
$(H2)$: $f(x)$ *is an arbitrary function which is bounded and continuous, and satisfies for all* $x \in \mathbb{R}$ *the following inequalities:*

$$u_0 + \frac{b_1-a_1 b_2}{-1+a_1 a_2} f(x) > 0, \quad v_0 + \frac{-a_2 b_1+b_2}{-1+a_1 a_2} f(x) > 0, \quad w_0 + f(x) > 0. \tag{2.2}$$

Then (LV) *admits a positive solution of the form*

$$u(x,t) = u_0 + \frac{b_1-a_1 b_2}{-1+a_1 a_2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi)\, d\xi, \tag{2.3a}$$

$$v(x,t) = v_0 + \frac{-a_2 b_1+b_2}{-1+a_1 a_2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi)\, d\xi, \tag{2.3b}$$

$$w(x,t) = w_0 + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi)\, d\xi. \tag{2.3c}$$

Proof. The key step in the proof consists in constructing a solution of (LV) by means of the representation formula for the initial value problem of the heat equation. Under $(H1)$, this can be achieved by making a suitable translation of the solution for (LV) and applying an appropriate ansatz for the translated solution. $(H2)$ then guarantees the positivity of the constructed solution.

The proof is elementary. We first translate the solution of (LV) by introducing the new variables $(\tilde{u}, \tilde{v}, \tilde{w}) = (\tilde{u}(x,t), \tilde{v}(x,t), \tilde{w}(x,t))$ as

$$u(x,t) = \tilde{u}(x,t) + u_0, \quad v(x,t) = \tilde{v}(x,t) + v_0, \quad w(x,t) = \tilde{w}(x,t) + w_0. \tag{2.4}$$

A direct computation yields that $\tilde{u}(x,t)$ satisfies

$$\begin{aligned}\tilde{u}_t &= \tilde{u}_{xx} + (\tilde{u}+u_0)\,(-\tilde{u} - a_1\,\tilde{v} - b_1\,\tilde{w} + 1 - u_0 - a_1\,v_0 - b_1\,w_0)\\ &= \tilde{u}_{xx} + (\tilde{u}+u_0)\,(-\tilde{u} - a_1\,\tilde{v} - b_1\,\tilde{w}).\end{aligned} \tag{2.5}$$

The last equality is true due to $(H1)$. Similarly, $\tilde{v}(x,t)$ and $\tilde{w}(x,t)$ satisfy respectively,

$$\tilde{v}_t = \tilde{v}_{xx} + (\tilde{v}+v_0)\,(-a_2\,\tilde{u} - \tilde{v} - b_2\,\tilde{w}) \tag{2.6}$$

and

$$\tilde{w}_t = \tilde{w}_{xx} + (\tilde{w}+w_0)\,(-a_3\,\tilde{u} - b_3\,\tilde{v} - \tilde{w}). \tag{2.7}$$

Since we assume $(H1)$, $-\tilde{u} - a_1\,\tilde{v} - b_1\,\tilde{w}$, $-a_2\,\tilde{u} - \tilde{v} - b_2\,\tilde{w}$, and $-a_3\,\tilde{u} - b_3\,\tilde{v} - \tilde{w}$ are linearly dependent and the solution of

$$-\tilde{u} - a_1\,\tilde{v} - b_1\,\tilde{w} = 0, \quad -a_2\,\tilde{u} - \tilde{v} - b_2\,\tilde{w} = 0, \quad -a_3\,\tilde{u} - b_3\,\tilde{v} - \tilde{w} = 0 \tag{2.8}$$

can be expressed in terms of $\tilde{w}$ by

$$\tilde{u} = \frac{b_1 - a_1\,b_2}{-1 + a_1\,a_2}\,\tilde{w}, \quad \tilde{v} = \frac{-a_2\,b_1 + b_2}{-1 + a_1\,a_2}\,\tilde{w}. \tag{2.9}$$

We use (2.9) as the ansatz for solutions of (2.5), (2.6), and (2.7). The problem is then reduced to solving the heat equation

$$\tilde{w}_t = \tilde{w}_{xx}, \quad x \in \mathbb{R},\ t > 0. \tag{2.10}$$

It is known that the bounded solution of the initial value problem for the heat equation can be uniquely represented by the convolution formula

$$\tilde{w}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\, f(\xi)\, d\xi. \tag{2.11}$$

Here $f(x)$ is the initial condition $\tilde{w}(x,0) = f(x)$, which is assumed to be bounded and continuous. We note that the representation formula (2.11) remains valid for unbounded $f(x)$ which satisfy the growth condition $|f(x)| \leqslant K\,e^{\varepsilon x^2}$, for some positive constants K and ε. Some examples of such f can be seen in Proposition 2.3. The positivity of the constructed solutions follows immediately from $(H2)$ and the elementary result

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\, d\xi = 1, \tag{2.12}$$

for any $x \in \mathbb{R}$ and $t > 0$.

To explain the biological meaning of $(H1)$, for convenience let $g_1 = 1 - u - a_1\,v - b_1\,w$, $g_2 = 1 - a_2\,u - v - b_2\,w$ and $g_3 = 1 - a_3\,u - b_3\,v - w$. Then g_1, g_2 and g_3 are the "net" birth rates of the three species. Therefore, the relationship among the net birth rates of the three species u, v, and w are related by $c_1\,g_1 + c_2\,g_2 + g_3 = 0$. This is a balance among the three species in some sense.

We remark that, due to $(H1)$, the solutions obtained in Theorem 2.1 are linearly dependent. In [5], such idea was also used for the two-component Lotka-Volterra system. Also, $(H1)$ means that (LV) under consideration contains infinite number of steady-state points, which form a straight line. Note that Theorem 2.1 can also be proved with the aid of Rouché-Capelli Theorem, which allows us to compute the number of solutions to a system of linear algebraic equations.

By means of Theorem 2.1, our attention turns to the following initial value problem for the heat equation:

$$\begin{cases} \omega_t = \omega_{xx}, & x \in \mathbb{R},\ t > 0, \\ \omega(x,0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{2.13}$$

As simple consequences of the Poisson formula (2.11) and the fact that for $\mu \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(2\mu x)\, dx = \sqrt{\pi}\, e^{-\mu^2}, \tag{2.14}$$

we find for various initial conditions $f(x)$ solutions of (2.13), including *space-time separated solutions*, *unbounded solutions*, and *solutions of Gaussian type*.

PROPOSITION 2.2. (**Space-Time Separated Solutions**) *Suppose that A_s, p_s, A_c, p_c are non-zero constants. Then*

(i) When $f(x) = A_s \sin(p_s x)$, a solution of (2.13) is given by

$$\omega(x,t) = A_s e^{-p_s^2 t} \sin(p_s x). \tag{2.15}$$

(ii) When $f(x) = A_c \cos(p_c x)$, a solution of (2.13) is given by

$$\omega(x,t) = A_c e^{-p_c^2 t} \cos(p_c x). \tag{2.16}$$

PROPOSITION 2.3. (**Unbounded Solutions**) *Suppose that B_s, q_s, B_c, q_c are non-zero constants. Then*

(i) When $f(x) = B_s e^{-\lambda_s x} \sin(q_s x)$, a solution of (2.13) is given by

$$\omega(x,t) = B_s e^{-\lambda_s x + (\lambda_s^2 - q_s^2) t} \sin(q_s x - 2 q_s \lambda_s t). \tag{2.17}$$

(ii) When $f(x) = B_c e^{-\lambda_c x} \cos(q_c x)$, a solution of (2.13) is given by

$$\omega(x,t) = B_c e^{-\lambda_c x + (\lambda_c^2 - q_c^2) t} \cos(q_c x - 2 q_c \lambda_c t). \tag{2.18}$$

PROPOSITION 2.4. (**Solutions of Gaussian Type**) *Suppose that $f(x) = C e^{-\nu x^2}$, where C and ν are positive constants. Then a solution of (2.13) is given by*

$$\omega(x,t) = \frac{C}{\sqrt{1 + 4t\nu}} e^{-\frac{\nu}{1+4t\nu} x^2}. \tag{2.19}$$

REMARK 2.1. Proposition 2.3 can be viewed as an extension of Proposition 2.2 in the sense that when $\lambda_s = \lambda_c = 0$, Proposition 2.3 becomes Proposition 2.2. Elementary techniques are used to prove Proposition 2.2, Proposition 2.3, and Proposition 2.4. However, we note that the three propositions can also be shown by employing the properties of the Fourier transform. By taking real and imaginary parts, Proposition 2.3 follows from the Fourier transform (with a imaginary shift) of the Gaussian with a translation, while the property that the Fourier transform of $e^{-\xi^2}$ is itself yields Proposition 2.4.

As an immediate consequence of combining the results of Theorem 2.1, Proposition 2.2, Proposition 2.3, and Proposition 2.4, we obtain the following theorem.

THEOREM 2.5. *Assume that $d_1 = d_2 = 1$ and hypotheses $(H1) \sim (H2)$ in Theorem 2.1 hold. Then*

(i) (***space-time separated solutions***) *if $f(x) = A_s \sin(p_s x)$, where A_s and p_s are non-zero constants, then (LV) admits the solution*

$$\begin{cases} u(x,t) = u_0 + \dfrac{b_1 - a_1 b_2}{-1 + a_1 a_2} A_s e^{-p_s^2 t} \sin(p_s x), \\ v(x,t) = v_0 + \dfrac{-a_2 b_1 + b_2}{-1 + a_1 a_2} A_s e^{-p_s^2 t} \sin(p_s x), \\ w(x,t) = w_0 + A_s e^{-p_s^2 t} \sin(p_s x), \end{cases}$$

(ii) (***unbounded solutions***) *if $f(x) = B_s e^{-\lambda_s x} \sin(q_s x)$, where B_s and q_s are non-zero constants, then (LV) admits the solution*

$$\begin{cases} u(x,t) = u_0 + \dfrac{b_1 - a_1 b_2}{-1 + a_1 a_2} B_s e^{-\lambda_s x + (\lambda_s^2 - q_s^2)t} \sin(q_s x - 2 q_s \lambda_s t), \\ v(x,t) = v_0 + \dfrac{-a_2 b_1 + b_2}{-1 + a_1 a_2} B_s e^{-\lambda_s x + (\lambda_s^2 - q_s^2)t} \sin(q_s x - 2 q_s \lambda_s t), \\ w(x,t) = w_0 + B_s e^{-\lambda_s x + (\lambda_s^2 - q_s^2)t} \sin(q_s x - 2 q_s \lambda_s t), \end{cases}$$

(iii) (***solutions of Gaussian type***) *if $f(x) = C e^{-\nu x^2}$, where C and ν are positive constants, then (LV) admits the solution*

$$\begin{cases} u(x,t) = u_0 + \dfrac{b_1 - a_1 b_2}{-1 + a_1 a_2} \dfrac{C}{\sqrt{1 + 4t\nu}} e^{-\frac{\nu}{1+4t\nu} x^2}, \\ v(x,t) = v_0 + \dfrac{-a_2 b_1 + b_2}{-1 + a_1 a_2} \dfrac{C}{\sqrt{1 + 4t\nu}} e^{-\frac{\nu}{1+4t\nu} x^2}, \\ w(x,t) = w_0 + \dfrac{C}{\sqrt{1 + 4t\nu}} e^{-\frac{\nu}{1+4t\nu} x^2}, \end{cases}$$

Proof. The theorem follows immediately from Theorem 2.1, Proposition 2.2, Proposition 2.3, and Proposition 2.4. We note that it is clear that for case (ii), $f(x) =$

$B_s e^{-\lambda_s x}\sin(q_s x)$ can not satisfy $(H2)$ in Theorem 2.1. However, the result of (ii) remains valid with the solution being negative for some $x\in\mathbb{R}$, $t>0$.

We note that both (i) and (ii) in Theorem 2.5 remain valid if sin is replaced by cos. In particular, the solution in case (ii) takes negative values for some $x\in\mathbb{R}, t>0$. This fact will be made clear in Section 5.

3. Diffusion-induced long-term coexistence

In this section, diffusion-induced long-term coexistence is shown to occur. We first consider (1.5). In order to analyze the stability of the equilibria of (1.5), we linearize the system around (u_0,v_0,w_0), obtaining

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = \mathscr{L}(u_0,v_0,w_0)\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{3.1}$$

where u_0, v_0, $w_0\in\mathbb{R}$ and the linearized operator $\mathscr{L}(u_0,v_0,w_0)$ is defined by

$$\mathscr{L}(u_0,v_0,w_0)=\begin{pmatrix} f_u(u_0,v_0,w_0) & f_v(u_0,v_0,w_0) & f_w(u_0,v_0,w_0) \\ g_u(u_0,v_0,w_0) & g_v(u_0,v_0,w_0) & g_w(u_0,v_0,w_0) \\ h_u(u_0,v_0,w_0) & h_v(u_0,v_0,w_0) & h_w(u_0,v_0,w_0) \end{pmatrix}, \tag{3.2}$$

where $f=u(1-u-a_1 v-b_1 w)$, $g=\lambda_1 v(1-a_2 u-v-b_2 w)$, and $h=\lambda_2 w(1-a_3 u-b_3 v-w)$. We prove the following lemma, which is related to the sign of the eigenvalues of $\mathscr{L}(u_0,v_0,w_0)$.

LEMMA 3.1. *Assume that $(H1)$ and $(H2)$ in Theorem 2.1 hold and let*

$$\kappa_0 = u_0 w_0\,(a_2 b_1-b_2)\,c_2\,\lambda_2+v_0\lambda_1\,(u_0-u_0 a_1 a_2-w_0\,(b_1-a_1 b_2)\,c_1\,\lambda_2); \tag{3.3}$$

$$\kappa_1 = u_0+v_0\lambda_1-w_0\,(b_1 c_1+b_2 c_2)\,\lambda_2. \tag{3.4}$$

Then

(i) one eigenvalue of $\mathscr{L}(u_0,v_0,w_0)$ is zero;

(ii) the other two eigenvalues of $\mathscr{L}(u_0,v_0,w_0)$ satisfy the quadratic equation

$$\mu^2+\kappa_1\mu+\kappa_0=0; \tag{3.5}$$

(iii) at least one eigenvalue of $\mathscr{L}(u_0,v_0,w_0)$ has positive real part if and only if

$$\{\kappa_0<0\}\cup\{\kappa_0=0,\kappa_1<0\}\cup\{\kappa_0>0,\kappa_1\leqslant -2\sqrt{k_0}\}. \tag{3.6}$$

Proof. By definition, the three eigenvalues of $\mathscr{L}(u_0,v_0,w_0)$ satisfy the cubic equation $\det\big(\mathscr{L}(u_0,v_0,w_0)-\mu I\big)=0$, where I is the identity matrix in $\mathbb{R}^3$. To prove the desired result, it suffices to show $\det\big(\mathscr{L}(u_0,v_0,w_0)\big)=0$. Under $(H1)$, we have for some $c_1,\ c_2 \in \mathbb{R},\ c_1 c_2 \neq 0,\ c_1(1-u-a_1 v-b_1 w)+c_2(1-a_2 u-v-b_2 w)+(1-a_3 u-b_3 v-w)=0$, for $u,v,w \in \mathbb{R}$. This gives

$$h=-w\left(c_1(1-u-a_1 v-b_1 w)+c_2(1-a_2 u-v-b_2 w)\right)\lambda_2, \tag{3.7}$$

and therefore

$$h_u = w(c_1+c_2 a_2)\lambda_2,\ h_v = w(c_1 a_1+c_2)\lambda_2, \tag{3.8}$$

$$h_w=-\left(c_1(1-u-a_1 v-b_1 w)+c_2(1-a_2 u-v-b_2 w)\right)\lambda_2+w(c_1 b_1+c_2 b_2)\lambda_2. \tag{3.9}$$

A straightforward calculation also yields

$$f_u = 1-2u-a_1 v-b_1 w,\quad f_v=-a_1 u,\quad f_w=-b_1 u, \tag{3.10}$$

$$g_u=-\lambda_1 a_2 v,\quad g_u=(1-a_1 u-2v-b_2 w)\lambda_1,\quad g_w=-\lambda_1 b_2 v. \tag{3.11}$$

Because of $(H1)$, we have

$$\begin{aligned}
&\det\left(\mathscr{L}(u_0,v_0,w_0)\right) \qquad (3.12)\\
&= \det\begin{pmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{pmatrix}\Bigg|_{(u,v,w)=(u_0,v_0,w_0)}\\
&= \det\begin{pmatrix} -u_0 & -a_1 u_0 & -b_1 u_0 \\ -\lambda_1 a_2 v_0 & -\lambda_1 v_0 & -\lambda_1 b_2 v_0 \\ w_0(c_1+c_2 a_2)\lambda_2 & w_0(c_1 a_1+c_2)\lambda_2 & w_0(c_1 b_1+c_2 b_2)\lambda_2 \end{pmatrix}\\
&= \det\begin{pmatrix} 1 & a_1 & b_1 \\ a_2 & 1 & b_2 \\ c_1+c_2 a_2 & c_1 a_1+c_2 & c_1 b_1+c_2 b_2 \end{pmatrix}\lambda_1\lambda_2 u_0 v_0 w_0\\
&= \det\begin{pmatrix} c_1 & c_1 a_1 & c_1 b_1 \\ c_2 a_2 & c_2 & c_2 b_2 \\ c_1+c_2 a_2 & c_1 a_1+c_2 & c_1 b_1+c_2 b_2 \end{pmatrix}\lambda_1\lambda_2 u_0 v_0 w_0 (c_1 c_2)^{-1}\\
&= 0.
\end{aligned}$$

This proves the assertion of (i). Using `Mathematica` software, we find the cubic equation that the eigenvalues of $\mathscr{L}(u_0,v_0,w_0)$ satisfy, i.e.

$$\begin{aligned}
0 &= \det\left(\mathscr{L}(u_0,v_0,w_0)-\mu I\right)\\
&= \det\begin{pmatrix} -u_0-\mu & -a_1 u_0 & -b_1 u_0 \\ -\lambda_1 a_2 v_0 & -\lambda_1 v_0-\mu & -\lambda_1 b_2 v_0 \\ w_0(c_1+c_2 a_2)\lambda_2 & w_0(c_1 a_1+c_2)\lambda_2 & w_0(c_1 b_1+c_2 b_2)\lambda_2-\mu \end{pmatrix}\\
&= -\mu\Big(\mu^2+\left(u_0+v_0\lambda_1-w_0(b_1 c_1+b_2 c_2)\lambda_2\right)\mu\\
&\qquad +u_0 w_0(a_2 b_1-b_2)c_2\lambda_2+v_0\lambda_1\left(u_0-u_0 a_1 a_2-w_0(b_1-a_1 b_2)c_1\lambda_2\right)\Big).
\end{aligned} \tag{3.13}$$

The assertions of (ii) and (iii) follow immediately.

We are now in the position to assert the occurrence of diffusion-induced long-term coexistence for the solutions of (1.5).

THEOREM 3.2. (**Diffusion-induced long-term coexistence**) *Suppose that* $(H1) \sim (H2)$ *are fulfilled with* f *satisfying one of the following additional assumptions:*

(i) $f \in L^p(\mathbb{R})$, *where* $p \in [1,\infty]$;

(ii) $f \in C^2(\mathbb{R})$, *is* 2π*-periodic, and satisfies*

$$\int_0^{2\pi} f(\xi)\,d\xi = 0. \tag{3.14}$$

Let κ_0 *and* κ_1 *be as defined in Lemma 3.1. Then diffusion-induced coexistence occurs for the solution of* (1.5) *provided that* κ_0 *and* κ_1 *belong to*

$$\{\kappa_0 < 0\} \cup \{\kappa_0 = 0, \kappa_1 < 0\} \cup \{\kappa_0 > 0, \kappa_1 \leqslant -2\sqrt{k_0}\}. \tag{3.15}$$

Proof. In view of Lemma 3.1, $\mathscr{L}(u_0,v_0,w_0)$ has at least one positive eigenvalue. It follows that (u_0,v_0,w_0) is an unstable equilibrium of (1.5). However, the solution satisfying (1.5) in the presence of equal diffusion effects, i.e. satisfying (LV) with $d_1 = d_2 = 1$, in Theorem 2.1 has the asymptotic behavior $(u,v,w)(x,t) \to (u_0,v_0,w_0)$ as t tends to infinity. This asymptotic behavior is due to Lemma 3.3 and Lemma 3.4.

LEMMA 3.3. *Suppose that* $p \in [1,\infty]$ *and* $q := p/(p-1) \in [1,\infty]$. *If* $f \in L^p(\mathbb{R})$, *then*

$$\frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi)\,d\xi \leqslant (2\sqrt{\pi t})^{-\frac{1}{p}} (\sqrt{q})^{-\frac{1}{q}} \|f\|_p \tag{3.16}$$

for all $x \in \mathbb{R}, t > 0$.

Proof. The desired result is an immediate consequence of Hölder inequality.

The following Lemma is found in [13].

LEMMA 3.4. *Suppose that* $f(x) \in C^2(\mathbb{R})$, *is* 2π*-periodic, i.e.* $f(x) = f(x+2\pi)$ *for all* $x \in \mathbb{R}$, *and satisfies*

$$\int_0^{2\pi} f(\xi)\,d\xi = 0. \tag{3.17}$$

The solution $\psi(x,t)$ *of the Cauchy problem for the heat equation*

$$\begin{cases} \psi_t = \psi_{xx}, & x \in \mathbb{R},\ t > 0, \\ \psi(x,0) = f(x), & x \in \mathbb{R} \end{cases} \tag{3.18}$$

has the property that $\psi(x,t) \to 0$ *as* $t \to \infty$.

REMARK 3.1. Examining Lemma 3.4, it is natural to consider the consequences of replacing the condition $\int_0^{2\pi} f(\xi)d\xi = 0$ with $\int_0^{2\pi} f(\xi)d\xi = 2\pi\bar{f}$ for some $\bar{f} \in \mathbb{R}\setminus\{0\}$. If $\int_0^{2\pi} f(\xi)d\xi = 2\pi\bar{f}$, then $\int_0^{2\pi}(f(\xi)-\bar{f})\,d\xi = 0$. Let $f(x)-\bar{f} = f^*(x)$ and $\psi(x,t)-\bar{f} = \psi^*(x,t)$, then

$$\begin{cases} \psi_t^* = \psi_{xx}^*, & x\in\mathbb{R},\ t>0, \\ \psi^*(x,0) = f^*(x), & x\in\mathbb{R}, \end{cases} \tag{3.19}$$

where $f^* \in C^2(\mathbb{R})$, is 2π-periodic, and satisfies

$$\int_0^{2\pi} f^*(\xi)d\xi = 0. \tag{3.20}$$

It follows from Lemma 3.4 that $\psi^*(x,t)\to 0$ as $t\to\infty$, i.e., $\psi(x,t)\to\bar{f}$ as $t\to\infty$.

Next we show that initial conditions can be found for the Cauchy problem of the heat equation to have solutions which converge to *nonzero* constants. Such initial conditions cannot belong to the space $L^p(\mathbb{R})$, as shown by Lemma 3.3.

PROPOSITION 3.5. ([30]) *Suppose that* $f\in C(\mathbb{R})$ *with* $\lim_{x\to-\infty} f(x) = f_-$ *and* $\lim_{x\to+\infty} f(x) = f_+$. *Then the bounded solution* $\psi(x,t)$ *to the Cauchy problem for the heat equation*

$$\begin{cases} \psi_t = \psi_{xx}, & x\in\mathbb{R},\ t>0, \\ \psi(x,0) = f(x), & x\in\mathbb{R}, \end{cases} \tag{3.21}$$

has the property $\lim_{t\to\infty}\psi(x,t) = \frac{1}{2}(f_- + f_+)$ *for all* $x\in\mathbb{R}$.

Proposition 3.5 allows us to construct solutions of (LV) when initial condition $f(x)$ does not belong to $L^p(\mathbb{R})$. For instance, let $f(x)$ be

$$f(x) = \begin{cases} 2, & \text{if} \quad x\in(-\infty,-1), \\ x+3, & \text{if} \quad x\in[-1,0], \\ -2x+3, & \text{if} \quad x\in(0,1), \\ 1, & \text{if} \quad x\in[1,\infty). \end{cases} \tag{3.22}$$

In this case, $\lim_{x\to-\infty} f(x) = 2$ and $\lim_{x\to+\infty} f(x) = 1$. Then the bounded solution to problem (3.21) is given by

$$\begin{aligned} \psi(x,t) = \frac{1}{2}\Bigg[3 + 2\sqrt{\frac{t}{\pi}}\left(2e^{-\frac{(-1+x)^2}{4t}} - 3e^{-\frac{x^2}{4t}} + e^{\frac{-(1+x)^2}{4t}}\right) + 2\,(-1+x)\,\mathrm{erf}\left(\frac{-1+x}{2\sqrt{t}}\right) \\ - 3x\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) + (1+x)\,\mathrm{erf}\left(\frac{1+x}{2\sqrt{t}}\right)\Bigg]. \end{aligned} \tag{3.23}$$

Here the error function $\mathrm{erf}(z)$ is the integral of the Gaussian distribution defined by $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}\,d\xi$ and $\mathrm{erf}_c(z) = 1-\mathrm{erf}(z)$. It is easily verified that $\lim_{t\to\infty}\psi(x,t) = \frac{3}{2}$ for all $x \in \mathbb{R}$. In Section 6, $\psi(x,t)$ will be used to construct a solution of (LV).

4. Alternative approach to space-time separated solutions: the method of exp-sin functions

In Section 2, we obtained a sufficient condition under which solutions of (LV) can be constructed from solutions of the heat equation. We demonstrate in this section that for space-time separated solutions it is also a necessary condition. This is done by employing an approach which is in the spirit of the approach developed in establishing exact traveling wave solutions of (LV) ([3, 4]). For unbounded solutions and solutions of Gaussian type, it is conjectured that the sufficient condition can also be proved to be necessary by a similar approach to the one presented here. In particular, the method of exp-sin functions employed also provides an alternative approach to establishing space-time separated solutions.

To begin with, we make the following ansatz for space-time separated solutions of (LV):

$$\begin{cases} u(x,t) = k_1 + m_1 e^{\beta t}\sin(\alpha x), \\ v(x,t) = k_2 + m_2 e^{\beta t}\sin(\alpha x), \\ w(x,t) = k_3 + m_3 e^{\beta t}\sin(\alpha x), \end{cases} \tag{4.1}$$

where $m_i\,(i=1,2,3.)$ are constants; $k_i\,(i=1,2,3.)$, α, and $-\beta$ are positive constants, which are to be determined. Substituting this ansatz into (LV) leads to

$$\begin{aligned} u_t - u_{xx} - u(1-u-a_1 v - b_1 w) = {} & k_1\,(-1+k_1+a_1k_2+b_1k_3) \\ & + \Big[\,(-1+\alpha^2+\beta+2k_1+a_1k_2+b_1k_3)\,m_1 \\ & + k_1\,(a_1m_2+b_1m_3)\Big]\,e^{\beta t}\sin(\alpha x) \\ & + \big[m_1\,(m_1+a_1m_2+b_1m_3)\big]\,e^{2\beta t}\sin^2(\alpha x), \end{aligned} \tag{4.2}$$

$$v_t - d_1 v_{xx} - \lambda_1 v(1-a_2u-v-b_2w) \tag{4.3}$$

$$\begin{aligned} = {} & k_2\,(-1+a_2k_1+k_2+b_2k_3)\,\lambda_1 \\ & + \Big[m_2\,\big(\beta+\alpha^2 d_1+(-1+a_2k_1+2k_2+b_2k_3)\,\lambda_1\big) \\ & + k_2\,(a_2m_1+b_2m_3)\,\lambda_1\Big]\,e^{\beta t}\sin(\alpha x) \\ & + \big[m_2\,(a_2m_1+m_2+b_2m_3)\,\lambda_1\big]\,e^{2\beta t}\sin^2(\alpha x), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} w_t - d_2 w_{xx} - \lambda_2 w (1 - a_3 u - b_3 v - w) \quad (4.5) \\ = k_3 \left(-1 + a_3 k_1 + b_3 k_2 + k_3\right) \lambda_2 \\ + \Big[m_3 \left(\beta + \alpha^2 d_2 + (-1 + a_3 k_1 + b_3 k_2 + 2 k_3) \lambda_2\right) \\ + k_3 \left(a_3 m_1 + b_3 m_2\right) \lambda_2 \Big] e^{\beta t} \sin(\alpha x) \\ + \left[m_3 \left(a_3 m_1 + b_3 m_2 + m_3\right) \lambda_2 \right] e^{2\beta t} \sin^2(\alpha x). \quad (4.6) \end{aligned}$$

By equating the coefficients of powers of $e^{n\beta t} \sin^n(\alpha x)\,(n = 0,1,2)$ in the last three equations to zero, we obtain a system of nine algebraic equations. In terms of certain parameters, this system can be solved with the aid of `Mathematica` software. It turns out that there are four families of nontrivial solutions, in which $d_1 = d_2 = 1$ and $\beta = -\alpha^2$. More precisely, the four families $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$, and $\mathscr{F}_4$ can be expressed as follows:

$$\begin{aligned} \mathscr{F}_1 = \Bigg\{ (a_1,a_2,a_3,b_1,b_2,b_3) \Bigg| \; & a_1 = \frac{k_3 m_1 + m_3 - k_1 m_3}{-k_3 m_2 + k_2 m_3}, a_2 = \frac{k_3 m_2 + m_3 - k_2 m_3}{-k_3 m_1 + k_1 m_3}, \\ & a_3 = \frac{m_2 - k_3 m_2 + k_2 m_3}{-k_2 m_1 + k_1 m_2}, b_1 = \frac{k_2 m_1 + m_2 - k_1 m_2}{k_3 m_2 - k_2 m_3}, \\ & b_2 = \frac{m_1 - k_2 m_1 + k_1 m_2}{k_3 m_1 - k_1 m_3}, b_3 = \frac{m_1 - k_3 m_1 + k_1 m_3}{k_2 m_1 - k_1 m_2} \Bigg\}, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \mathscr{F}_2 = \Bigg\{ (a_1,a_2,a_3,b_1,b_3,k_1,k_3) \Bigg| \; & a_1 = 1, a_2 = -\frac{m_2 + b_2 m_3}{m_1}, a_3 = -\frac{m_2 + m_3}{m_1}, \\ & b_1 = -\frac{m_1 + m_2}{m_3}, b_3 = 1, k_1 = \frac{(-1 + k_2)\, m_1}{m_2}, \\ & k_3 = \frac{(-1 + k_2)\, m_3}{m_2} \Bigg\}, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \mathscr{F}_3 = \Bigg\{ (a_1,a_2,a_3,b_1,b_2,k_1,k_3) \Bigg| \; & a_1 = -\frac{m_1 + m_3}{m_2}, a_2 = -\frac{m_2 + m_3}{m_1}, \\ & a_3 = -\frac{b_3 m_2 + m_3}{m_1}, \\ & b_1 = 1, b_2 = 1, k_1 = \frac{k_2 m_1}{m_2}, k_3 = 1 + \frac{k_2 m_3}{m_2} \Bigg\}, \end{aligned} \quad (4.9)$$

$$\mathscr{F}_4 = \Bigg\{ (a_1,a_2,a_3,b_2,b_3,k_1,k_3) \Bigg| \; a_1 = -\frac{m_1 + b_1 m_3}{m_2}, a_2 = 1, a_3 = 1,$$

$$\left.\begin{aligned} b_2 &= -\frac{m_1+m_2}{m_3}, \\ b_3 &= -\frac{m_1+m_3}{m_2}, \\ k_1 &= 1+\frac{k_2\,m_1}{m_2}, k_3 = \frac{k_2\,m_3}{m_2} \end{aligned}\right\}, \tag{4.10}$$

where family $\mathscr{F}_3$ and family $\mathscr{F}_4$ are equivalent. This is readily seen from the fact that the roles of v and w can be interchanged. We note that λ_1 and λ_2 are not restricted in all the four families, namely, λ_1 and λ_2 can be chosen arbitrarily, provided that λ_1 and λ_2 are positive.

LEMMA 4.1. *For each family $\mathscr{F}_i\,(i=1,2,3,4.)$, condition $(H1)$ is fulfilled.*

Proof. To show $(H1)$ holds for each family $\mathscr{F}_i$, it suffices to find nonzero ε_1, $\varepsilon_2 \in \mathbb{R}$ such that $\varepsilon_1\,(1-u-a_1\,v-b_1\,w)+\varepsilon_2\,(1-a_2\,u-v-b_2\,w)+(1-a_3\,u-b_3\,v-w)=0$ for $a_i, b_i\,(i=1,2,3.) \in \mathscr{F}_i$, and $u,v,w \in \mathbb{R}$. Indeed, it is readily verified that

- for family $\mathscr{F}_1$, $\varepsilon_1 = -1+\dfrac{1+\frac{m_2-k_3\,m_2+k_2\,m_3}{k_2\,m_1-k_1\,m_2}}{1+\frac{k_3\,m_2+m_3-\,k_2\,m_3}{k_3\,m_1-k_1\,m_3}}$, $\varepsilon_2 = -\dfrac{1+\frac{m_2-k_3\,m_2+k_2\,m_3}{k_2\,m_1-k_1\,m_2}}{1+\frac{k_3\,m_2+m_3-\,k_2\,m_3}{k_3\,m_1-k_1\,m_3}}$;
- for family $\mathscr{F}_2$, $\varepsilon_1 = -\frac{(-1+b_2)\,m_3}{m_1+\,m_2+b_2\,m_3}$, $\varepsilon_2 = -\frac{m_1+m_2+m_3}{m_1+m_2+\,b_2\,m_3}$;
- for family $\mathscr{F}_3$, $\varepsilon_1 = \frac{(-1+b_3)\,m_2}{m_1+m_2+m_3}$, $\varepsilon_2 = -\frac{m_1+b_3\,m_2+m_3}{m_1+m_2+m_3}$;
- for family $\mathscr{F}_4$, $\varepsilon_1 = -\frac{m_1+m_2+m_3}{m_1+m_2+\,b_1\,m_3}$, $\varepsilon_2 = -\frac{(-1+b_1)\,m_3}{m_1+\,m_2+b_1\,m_3}$.

This completes the proof.

Lemma 4.1, together with the result obtained by the method of exp-sin functions yields the following theorem. In the next theorem, we give a necessary and sufficient condition for the existence of space-time separated solutions

THEOREM 4.2. *When $d_1=d_2=1$, (LV) admits space-time separated solutions of the form* (4.1) *if and only if $(H1)$ holds.*

5. Illustrative examples

There *do* exist parameters a_i, b_i $(i=1,2,3.)$, λ_i $(i=1,2.)$ and functions $f(x)$ which satisfy hypotheses $(H1) \sim (H2)$. Indeed, we illustrate in this section cases (i) and (iii) of Theorem 2.5 by an example.

EXAMPLE 1. We first choose

$$a_1=\frac{48}{35}, a_2=\frac{55}{42}, a_3=\frac{235}{126}, b_1=1, b_2=1, b_3=\frac{1}{3}, \lambda_1=1, \lambda_2=3 \tag{5.1}$$

so that the solution (u_0, v_0, w_0) of (2.1)

$$\begin{cases} 1-u-\frac{48}{35}v-w=0, \\ 1-\frac{55}{42}u-v-w=0, \\ 1-\frac{235}{126}u-\frac{1}{3}v-w=0 \end{cases} \tag{5.2}$$

is given in terms of w_0 by

$$u_0 = \frac{7}{15} - \frac{7w_0}{15},\ v_0 = \frac{7}{18} - \frac{7w_0}{18}. \tag{5.3}$$

We then choose $w_0 = \frac{13}{28}$ in (5.3), from which it follows that $u_0 = \frac{1}{4}$ and $v_0 = \frac{5}{24}$. It is easily verified that

$$\frac{70}{39}\left(1-u-\frac{48}{35}v-w\right)-\frac{109}{39}\left(1-\frac{55}{42}u-v-w\right)+\left(1-\frac{235}{126}u-\frac{1}{3}v-w\right)=0. \tag{5.4}$$

Hence in this case, $(H1)$ is satisfied. To satisfy $(H2)$, we take $f(x) = \frac{3}{7}\sin(2x)$ in case (i) such that for $x \in \mathbb{R}$,

$$u_0 + \frac{b_1 - a_1 b_2}{-1 + a_1 a_2} f(x) = \frac{1}{4} - \frac{1}{5}\sin(2x) > 0, \tag{5.5a}$$

$$v_0 + \frac{-a_2 b_1 + b_2}{-1 + a_1 a_2} f(x) = \frac{5}{24} - \frac{1}{6}\sin(2x) > 0, \tag{5.5b}$$

$$w_0 + f(x) = \frac{13}{28} + \frac{3}{7}\sin(2x) > 0. \tag{5.5c}$$

This shows that $(H2)$ is fulfilled. The resulting solution is given by

$$\begin{cases} u(x,t) = \frac{1}{4} - \frac{1}{5}e^{-4t}\sin(2x), \\ v(x,t) = \frac{5}{24} - \frac{1}{6}e^{-4t}\sin(2x), \\ w(x,t) = \frac{13}{28} + \frac{3}{7}e^{-4t}\sin(2x). \end{cases} \tag{5.6}$$

On the other hand, suppose that we take $f(x) = \frac{3}{7}e^{-x}\sin(2x)$ in case (ii). According to Theorem 2.5, the resulting solution is

$$\begin{cases} u(x,t) = \frac{1}{4} - \frac{1}{5}e^{-x-3t}\sin(2x-4t), \\ v(x,t) = \frac{5}{24} - \frac{1}{6}e^{-x-3t}\sin(2x-4t), \\ w(x,t) = \frac{13}{28} + \frac{3}{7}e^{-x-3t}\sin(2x-4t). \end{cases} \tag{5.7}$$

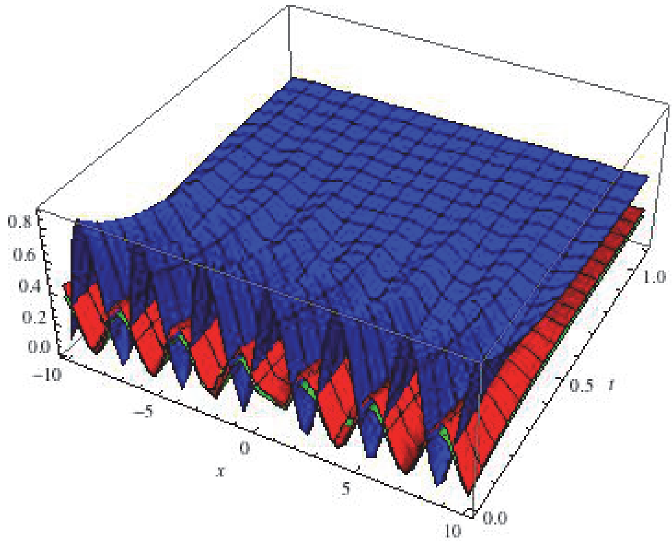

Figure 1: Space-time separated solution (5.7): u (red), v (green) and w (blue)

It should be noted that each component of this solution is not positive for all $x \in \mathbb{R}$, $t > 0$. For case (iii) we take $f(x) = \frac{3}{7}e^{-x^2}$. Similarly, we have for $x \in \mathbb{R}$,

$$u_0 + \frac{b_1 - a_1 b_2}{-1 + a_1 a_2} f(x) = \frac{1}{4} - \frac{1}{5} e^{-x^2} > 0, \tag{5.8a}$$

$$v_0 + \frac{-a_2 b_1 + b_2}{-1 + a_1 a_2} f(x) = \frac{5}{24} - \frac{1}{6} e^{-x^2} > 0, \tag{5.8b}$$

$$w_0 + f(x) = \frac{13}{28} + \frac{3}{7} e^{-x^2} > 0 \tag{5.8c}$$

and the resulting solution is

$$\begin{cases} u(x,t) = \frac{1}{4} - \frac{1}{5}\frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+4t}x^2}, \\ v(x,t) = \frac{5}{24} - \frac{1}{6}\frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+4t}x^2}, \\ w(x,t) = \frac{13}{28} + \frac{3}{7}\frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+4t}x^2}. \end{cases} \tag{5.9}$$

Profiles of the (5.7) and (5.9) with different values of t are shown respectively in Figure 1 and Figure 2. As mentioned at the end of Section 2, we see that the unbounded solution here is not positive for all $x \in \mathbb{R}, t > 0$. From the viewpoint of biology, the unbounded solution is unlikely to be of interest.

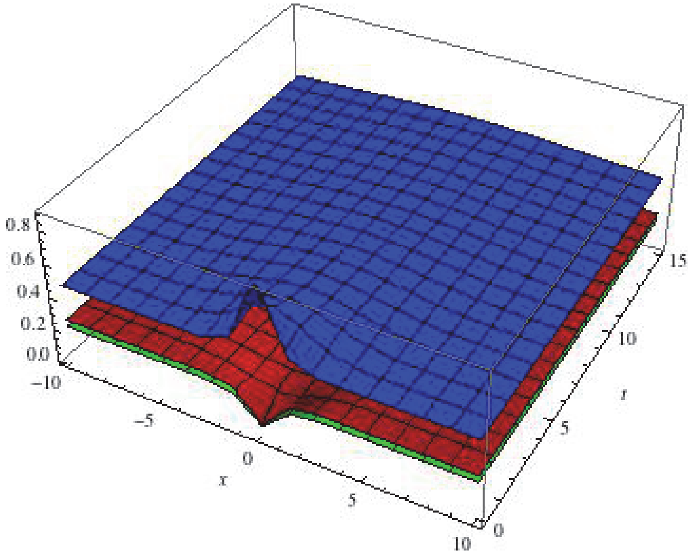

Figure 2: Solution of Gaussian type (5.9): u (red), v (green) and w (blue)

This example also shows that diffusion-induced coexistence for the parameters in (5.1) occurs. Indeed, first it is readily verified that $\frac{3}{7}\sin(2x)$ is a 2π-periodic function with $\int_0^{2\pi}\frac{3}{7}\sin(2\xi)\,d\xi = 0$ and $\frac{3}{7}e^{-x^2}\in L^2(\mathbb{R})$. Thus conditions (i) and (ii) in Theorem 3.2 are satisfied. A direct calculation yields

$$\mathscr{L}(u_0,v_0,w_0)\Big|_{(u_0,v_0,w_0)=(\frac{1}{4},\frac{5}{24},\frac{13}{28})} = \begin{pmatrix} -\frac{1}{4} & -\frac{275}{1008} & -\frac{3055}{1176} \\ -\frac{12}{35} & -\frac{5}{24} & -\frac{13}{28} \\ -\frac{1}{4} & -\frac{5}{24} & -\frac{39}{28} \end{pmatrix}, \tag{5.10}$$

and the eigenvalues of $\mathscr{L}(\frac{1}{4},\frac{5}{24},\frac{13}{28})$ are 0, $-\frac{27}{14}$, and $\frac{13}{168}$.

6. Concluding Remarks

We would like first to remark that, besides the work of Cherniha and Davydovych ([6]), the present investigations were motivated by the following naive observation. Consider the following problem:

$$\begin{cases} u_t = u_{xx}, & x\in\mathbb{R},\ t>0, \\ u(x,0)=\sin(x), & x\in\mathbb{R}. \end{cases} \tag{6.1}$$

The unique bounded solution to this problem can be represented as

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\sin(\xi)\,d\xi. \tag{6.2}$$

On the other hand, it is easy to verify that

$$u(x,t) = e^{-t}\sin(x) \tag{6.3}$$

is also a bounded solution satisfying the given initial condition. By uniqueness, the two solutions (6.2) and (6.3) should be identical. This observation results in the three types of solutions of (6.1), with $\sin(x)$ replaced by various initial conditions.

In this paper we have found that, under certain conditions on the parameters, space-time separated solutions, unbounded solutions, and solutions of Gaussian type of (LV) exist by giving explicit forms of these solutions. Each type of solution is essentially different from that of traveling wave solutions (for non-traveling wave type solutions, refer to [7]), which were previously established in [3, 4]. In addition, the traveling wave solutions found in [3, 4] have the asymptotic behavior $(u(x,t),v(x,t),w(x,t)) \to (1,0,0)$ or $(0,1,0)$ as $t \to \infty$, meaning that two of the three competing species become extinct while the remaining species survive long term. This phenomenon is known as *Gause's competitive exclusion principle*. However, our space-time separated solutions and solutions of Gaussian type both have the asymptotic behavior

$$(u(x,t),v(x,t),w(x,t)) \to (u_0,v_0,w_0)$$

with $u_0,v_0,w_0 > 0$ as $t \to \infty$. Ecologically, this can be interpreted as the long-term coexistence of the three competing species. This remarkable feature of space-time separated solutions and solutions of Gaussian type differs greatly from that of traveling wave solutions.

One of the essential purposes of this paper is to investigate diffusion-induced long-term coexistence of diffusive Lotka-Volterra systems of three competing species, which is new and interesting phenomenon and seems to be explored for the first time for Lotka-Volterra systems. As indicated in the introduction, we show in addition to traveling wave solutions, other classes of solutions exist for diffusive Lotka-Volterra systems of three competing species. In spite of the fact that the parameter region where the assumption $(H1)$ is satisfied is not generic, the solutions, not yet described in the literature, are found in this paper. In order to find more solutions, an elementary approach is employed in this paper. The key idea is to make use of the linear dependence of nonlinearity in (LV). We remark that this approach has been previously used for two-component Lotka-Volterra systems (see [5] for instance). Although this approach is elementary, it will be shown in subsequent studies that, by applying this approach diffusive Lotka-Volterra systems of three species can be reduced to one single reaction-diffusion equation. Moreover, we also extend Cherniha and Davydovych's results [6] from systems of two species to systems of three species.

We believe that the approach developed here, i.e. construction of solutions for a system from the solution of a single, linear equation, can be applied to certain classes of reaction-diffusion systems other than (LV). These problems will be investigated in future work.

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