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EXACT TRAVELING WAVES SOLUTIONS FOR LONG WAVES AND BLOW–UP PHENOMENA

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Abstract. In this work we find exact traveling waves solutions to the fifth-order KDV-BBM type model that appear to describe the propagation of long waves in shallow water. We study the possibility of blow-up phenomenon of the fifth-order KDV-BBM type model under certain restrictions on the coefficients. Moreover, by applying the Ince transformation we also establish exact traveling waves solutions to the nonlinear evolution equation Benney-Lin type.

1. Introduction

In this work we will consider the fifth order BBM-KdV type equation

$$\eta_t+\eta_x-\frac{1}{6}\eta_{xxt}+\delta_1\eta_{xxxxt}+\delta_2\eta_{xxxxx}+\frac{3}{4}(\eta^2)_x+\gamma(\eta^2)_{xxx}-\frac{1}{12}(\eta_x^2)_x-\frac{1}{4}(\eta^3)_x=0, \tag{1.1}$$

where $\eta=\eta(x,t)$ is a real-valued function, and $\delta_1>0$, $\delta_2,\gamma\in\mathbb{R}$. This model was recently introduced by Bona et al [7] to describe the unidirectional propagation of water waves. It was formally obtained as a second order approximation from the higher order generalized Boussinesq system derived by Bona et al [10], which describes the two-way propagation of water waves. The authors in [10] derived a first-order and second-order correct Boussinesq systems from the the original Euler equations using respectively the first and second order approximations.

In particular, to obtain an approximate one-way model, in the Boussinesq regime, one generally uses a relation

$$u_x=-u_t+O(a,b) \qquad \text{as} \quad a,b\to 0, \tag{1.2}$$

where a,b are small parameters related to small amplitude and long wavelength (see references [10, 11]). For instance, if A is a typical amplitude of the wave in the channel with constant depth h and l is a typical wavelengh, the conditions of the models are $a=\frac{A}{h}$, $b=h^2/l^2$, and then the Stokes number S will be $S=a/b\approx 1$.

Mathematics subject classification (2010): 35A01, 35Q53.

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In fact, using this sort of argument Korteweg-de Vries (KdV) and Benjamin-Bona-Mahony (BBM) equations are derived from the Boussinesq system correct to first order (see for example [6, 3] and references therein). However, if one uses the relation (1.2) to obtain one-way model like the one in (1.1) from the Boussinesq system correct to second order, there is some loss of information coming from the interacting terms, because that could be not so small. Consequently the resulting equation does not have a correct dispersion relation. Taking this into consideration, a correction term is introduced in [7] to obtain the fifth order mathematical model (1.1) describing long waves propagating mainly in one direction. For the detailed explanation about derivation and well-posedness theory of this model we refer to [7]. In particular the authors prove the following local well-posedness result to the IVP associated to (1.1).

THEOREM 1. *([7]) Assume* $\delta_1 > 0$. *For any* $s \geqslant 1$ *and for given* $\eta_0 \in H^s(\mathbb{R})$, *there exist a time* $T = T(\|\eta_0\|_{H^s})$ *and a unique solution* $\eta \in C([0,T];H^s)$ *to the IVP for* (1.1) *that depends continuously to the initial data.*

Also, they prove the following global well-posedness result with more regularity assumptions on the data and a further restriction on the coefficients appearing in the equation.

THEOREM 2. *([7]) Assume* $\delta_1 > 0$. *Let* $s \geqslant 3/2$ *and* $\gamma = 1/12$. *Then the solution to the IVP associated to* (1.1) *given by Theorem 1 can be extended to arbitrarily large time intervals* $[0,T]$. *Hence the problem is globally well posed in this case.*

Finally we consider an equation of Benney-Lin type, that is,

$$u_t + \lambda_1 u_{xxxxx} + \lambda_2 u_{xxxx} + u_{xxx} + \lambda_3 u_{xx} + u u_x = 0, \tag{1.3}$$

where $x \in \mathbb{R}$, $t > 0$. $u = u(x,t)$ is a unknown real-valued function. $\lambda_j \in \mathbb{R}$ $j = 1,2,3$ are constant to be defined. When $\lambda_2 = \lambda_3 \neq 0$, the above equation is known as Benney-Lin equation and was derived from fluid mechanics by Benney [4] and Lin [22] (see also [31]). Indeed, we have

$$u_t + \lambda_1 u_{xxxxx} + \lambda_2 (u_{xxxx} + u_{xx}) + u_{xxx} + u u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.4}$$

where $u = u(x,t)$ is a unknown real-valued function, $\lambda_1 \in \mathbb{R}$ and $\lambda_2 > 0$. It describes the propagation of one-dimensional small but finite amplitude long waves in certain problems in fluids dynamics. An important feature of this model is that it includes both conservative dispersive effects and nonconservative dissipative ones represented by the terms $\lambda_1 u_{xxxxx} + u_{xxx}$ and $u_{xxxx} + u_{xx}$, respectively. (1.3) can also be regarded as an hybrid of the well-known Kawahara equation ($\lambda_2 = \lambda_3 = 0$) and the derivative Korteweg-de Vries-Kuramoto-Sivashinsky(KdV-KS) equation ($\lambda_1 = 0$ and $\lambda_2 = \lambda_3 \neq 0$). Integrable systems, both classical and quantum, are a fascinating subject. Decades of research in this area have led to mathematical developments which are quite beautiful. However, not all systems posed in physics are integrable (ref. [20]), for instance, the Korteweg-de Vries-Burgers equation. Therefore the direct methods to solve nonlinear systems appear to be more powerful and important. In this paper we will propose an exact solution to a general equation of Benney-Lin type (1.3).

Other fifth order long wave models that describe the unidirectional propagation are also available in the literature, see for instance [16], [19], [21], [26], [28] and references therein. Most of these models are derived as a second-order approximation from Euler equations in the so-called Boussinesq regime. A well known fifth order such model is the so called Kawahara equation (see [5] and references therein). We note that very few fifth order models, as far as we know, have Hamiltonian structure and global solutions, see for example [28] and [19] and references therein.

2. Exact traveling waves solutions for the fifth order BBM-KdV type equation

In this section we will prove the following theorem

THEOREM 3. *If* $\gamma = -1/30$ *and* δ_1, δ_2 *satisfy the relation*

$$9\big(388\sqrt{1069} - 8269\big)\delta_1 - 25650\,\delta_2 = 190,$$

then a exact traveling wave solution of (1.1) *is*

$$\eta(x,t) = \frac{1}{\left(\frac{\sqrt{3}}{2|k|} + \sinh(kx - \omega t) - \cosh(kx - \omega t)\right)^2} + \alpha, \tag{2.1}$$

where the constants k, α *and* ω *are given by* (2.34), (2.35) *and* (2.36) *respectively.*

Proof. The solution η of the equation (1.1) is redefined by

$$\eta(x,t) = \varphi(\xi) \tag{2.2}$$

where

$$\xi = kx - \omega t \tag{2.3}$$

and k and ω being constants to be determined. Substituting into (1.1) we have

$$(\delta_2 k - \delta_1 \omega)k^4 \varphi^{(5)} + \frac{k^2\omega}{6}\varphi''' + \gamma k^3 (\varphi^2)''' + (k-\omega)\varphi' + \frac{3}{4}k(\varphi^2)' - \frac{k^3}{12}[(\varphi')^2]' - \frac{k}{4}(\varphi^3)' = 0. \tag{2.4}$$

Now we integrate (2.4) to get

$$(\delta_2 k - \delta_1 \omega)k^4 \varphi^{(4)} + \frac{k^2\omega}{6}\varphi'' + \gamma k^3 (\varphi^2)'' + (k-\omega)\varphi + \frac{3}{4}k\varphi^2 - \frac{k^3}{12}(\varphi')^2 - \frac{k}{4}\varphi^3 = \vartheta, \tag{2.5}$$

where ϑ is a constant of integration. Notice that equation (2.5) can not be integrated directly, in this case to be a solution of (2.5) will use the same ideas given in [24, 30], the novelty in our work is in solve a system of ordinary differential equations of high order, see the system (2.9)-(2.14). Indeed, we consider the Ince transformation method, see [18]. Let

$$\varphi(\xi) = r(e^{\xi})e^{2\xi} + \alpha \tag{2.6}$$

where α is a constant. Deriving the equation (2.6) and replace it results in equation (2.5)

$$\begin{cases} (\delta_2 k - \delta_1 \omega)k^4\{ r^{(4)}e^{6\xi} + 14r'''e^{5\xi} + 55r''e^{4\xi} + 65r'e^{3\xi} + 16re^{2\xi} \} \\ \qquad\qquad + \dfrac{k^2\omega}{6}\{ r''e^{4\xi} + 5r'e^{3\xi} + 4re^{2\xi} \} \\ +\gamma k^3\{ 2\big(rr'' + (r')^2\big)e^{6\xi} + 18rr'e^{5\xi} + 2(\alpha r'' + 8r^2)e^{4\xi} + 10\alpha r'e^{3\xi} + 8\alpha re^{2\xi} \} \\ -\dfrac{k^3}{12}\{ (r')^2e^{6\xi} + 4rr'e^{5\xi} + 4r^2e^{4\xi} \} + (k-\omega)\{ re^{2\xi} + \alpha \} \\ +\dfrac{3k}{4}\{ r^2e^{4\xi} + 2\alpha re^{2\xi} + \alpha^2 \} - \dfrac{k}{4}\{ r^3e^{6\xi} + 3r^2\alpha e^{4\xi} + 3r\alpha^2e^{2\xi} + \alpha^3 \} = \vartheta. \end{cases} \tag{2.7}$$

Organizing conveniently equation (2.7), we arrive at the expression

$$\begin{cases} \{ (\delta_2 k - \delta_1 \omega)k^4 r^{(4)} + 2\gamma k^3 rr'' + (2\gamma k^3 - \dfrac{k^3}{12})(r')^2 - \dfrac{k}{4}r^3 \}e^{6\xi} \\ +\{ 14(\delta_2 k - \delta_1 \omega)k^4 r''' + (18\gamma - \dfrac{1}{3})k^3\, rr' \}e^{5\xi} \\ +\{ [55(\delta_2 k - \delta_1 \omega)k^4 + \dfrac{k^2\omega}{6} + 2\alpha\gamma k^3]r'' + [\dfrac{3k}{4} + 16\gamma k^3 - \dfrac{k^3}{3} - \dfrac{3}{4}k\alpha]r^2 \}e^{4\xi} \\ +\{ 65(\delta_2 k - \delta_1 \omega)k^4 + \dfrac{5}{6}k^2\omega + 10\alpha\gamma k^3 \}\, r'\, e^{3\xi} \\ +\{ 16(\delta_2 k - \delta_1 \omega)k^4 + \dfrac{2}{3}k^2 c + 8\alpha\gamma k^3 + (k-\omega) + \dfrac{3}{2}k\alpha - \dfrac{3}{4}k\alpha^2 \}\, r\, e^{2\xi} \\ +\{ \alpha(k-\omega) + \dfrac{3}{4}k\alpha^2 - \dfrac{k}{4}\alpha^3 \} = \vartheta. \end{cases} \tag{2.8}$$

Considering $\vartheta \neq 0$ we conclude that

$$(\delta_2 k - \delta_1 \omega)k^4 r^{(4)} + 2\gamma k^3 (rr')' - \frac{k^3}{12}(r')^2 - \frac{k}{4}r^3 = 0, \tag{2.9}$$

$$14(\delta_2 k - \delta_1 \omega)k^4 r''' + (18\gamma k^3 - \frac{k^3}{3})rr' = 0, \tag{2.10}$$

$$\left[55(\delta_2 k - \delta_1 \omega)k^4 + \frac{k^2\omega}{6} + 2\alpha\gamma k^3\right]r'' + \left[\frac{3k}{4} + 16\gamma k^3 - \frac{k^3}{3} - \frac{3}{4}k\alpha\right]r^2 = 0, \tag{2.11}$$

$$\left[65(\delta_2 k-\delta_1\omega)k^4+\frac{5}{6}k^2\omega+10\alpha\gamma k^3\right]r'=0, \tag{2.12}$$

and

$$\left[16(\delta_2 k-\delta_1\omega)k^4+\frac{2}{3}k^2\omega+8\alpha\gamma k^3+(k-\omega)+\frac{3}{2}k\alpha-\frac{3}{4}k\alpha^2\right]r=0, \tag{2.13}$$

$$\alpha(k-\omega)+\frac{3}{4}k\alpha^2-\alpha^3\frac{k}{4}=\vartheta, \tag{2.14}$$

where in the first equation was used that $rr''+(r')^2=(rr')'$. In the next we will go to find a solution to the six equations above.

In order to find the constant ω,α,k,γ and the function r, firstly we proceeded to compare the equations (2.9) and (2.10), and in this process we will need that r should to satisfy the following equation

$$\frac{k^3}{12}(r')^2+\frac{k}{4}r^3=0. \tag{2.15}$$

Note that (2.15) is equivalent with the following equation

$$\boxed{k^2r'^2+3r^3=0,} \tag{2.16}$$

differentiating we obtain

$$2k^2r'\,r''+9r^2\,r'=0,$$

and to $r'\neq 0$, this equation is equivalent with

$$2k^2r''+9r^2=0. \tag{2.17}$$

In the equation (2.11), we have

$$\mathscr{A}\,r''+\mathscr{B}r^2=0, \tag{2.18}$$

where

$$\mathscr{A}=55\,\mathscr{X}\,k^4+\frac{k^2\omega}{6}+2\alpha\gamma k^3\neq 0 \qquad \text{and} \qquad \mathscr{B}=16\,\gamma k^3-\frac{k^3}{3}+\frac{3}{4}k(1-\alpha)\neq 0, \tag{2.19}$$

and $\mathscr{X}=\delta_2 k-\delta_1\omega$.

In order to obtain equation (2.17) is equal to equation (2.18), we need that

$$\mathscr{A}=\frac{2}{9}\mathscr{B}k^2. \tag{2.20}$$

Observe that (2.16) and

$$\gamma=-\frac{1}{30}, \tag{2.21}$$

implies that r satisfies the differential equations (2.9) and (2.10) (see the exact traveling wave solution of (2.16) in formula (2.42)).

Let C_1 such that

$$C_1\,\mathscr{X}\,k^4 = \frac{k^2\omega}{6} + 2\alpha\gamma k^3, \tag{2.22}$$

thus

$$\mathscr{A} = (55 + C_1)\,\mathscr{X}\,k^4. \tag{2.23}$$

Considering the equation (2.12) and the equation (2.22) we obtain

$$65\,\mathscr{X}\,k^4 + 5\left(\frac{k^2\omega}{6} + 2\alpha\gamma k^3\right) = 5(13 + C_1)\,\mathscr{X}\,k^4 = 0,$$

which implies

$$C_1 = -13. \tag{2.24}$$

And also from (2.23) that

$$\mathscr{A} = 42\,\mathscr{X}\,k^4. \tag{2.25}$$

Differentiating the equation (2.18) we arrive to

$$\mathscr{A}\,r''' + 2\mathscr{B}rr' = 0. \tag{2.26}$$

So, the equations (2.10) and (2.26) are equal if we take $\mathscr{B}$ such that

$$\mathscr{B} = -\frac{7}{5}k^3, \tag{2.27}$$

combining (2.19), (2.21) and (2.27) it is easy to see that

$$\alpha = \frac{32}{45}k^2 + 1, \tag{2.28}$$

and combining (2.20), (2.25) and (2.27) gives

$$\mathscr{B} = -\frac{7}{5}k^3 = \frac{9\mathscr{A}}{2k^2} = 189\,\mathscr{X}\,k^2, \tag{2.29}$$

thus

$$\mathscr{X} = -\frac{k}{135}. \tag{2.30}$$

Now (2.13), (2.22), (2.24) and (2.30) give

$$\omega = \frac{-16(-\frac{k}{135})k^4 + \frac{3}{4}(\alpha-2)k\alpha + \frac{4}{15}\alpha k^3 - k}{\frac{2}{3}k^2 - 1} = \frac{2}{5}k\alpha - 78(-\frac{k}{135})k^2. \tag{2.31}$$

From (2.31) it is not hard to show that k satisfies

$$3(16k^4 - 45\alpha^2 + 66\alpha + 60) = 104k^2, \tag{2.32}$$

and using (2.28) we can show that

$$38k^4+291k^2=\frac{3645}{8}, \tag{2.33}$$

and from this equation we obtain the following solutions

$$k_1=\frac{1}{2}\sqrt{\frac{3}{19}\left(4\sqrt{1069}-97\right)}\approx 1.1548, \qquad k_2=-k_1. \tag{2.34}$$

The function $f(x)=38x^4+291x^2-3645/8$ is shown in the Figure 1.

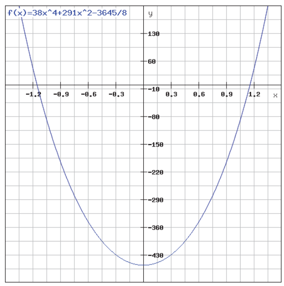

Figure 1: The graph of the function $f(x)=38x^4+291x^2-3645/8$.

Considering the value of $k:=k_1$, the equality (2.28) gives

$$\alpha=\frac{32\sqrt{1069}-491}{285}\approx 1.9483. \tag{2.35}$$

From (2.31) we obtain

$$\omega=k\left(\frac{194}{225}k^2+\frac{2}{5}\right)=\frac{1}{2}\sqrt{\frac{3}{19}\left(4\sqrt{1069}-97\right)}\frac{388\sqrt{1069}-8269}{2850} \approx 1.78965. \tag{2.36}$$

Also from (2.30) and (2.36) we have

$$\mathscr{X}=\delta_2 k-\delta_1 k\left(\frac{194}{225}k^2+\frac{2}{5}\right)=-\frac{k}{135} \tag{2.37}$$

consequently

$$\delta_2-\delta_1\frac{388\sqrt{1069}-8269}{2850}=-\frac{1}{135}, \tag{2.38}$$

or equivalently

$$\delta_1\frac{388\sqrt{1069}-8269}{190}=\frac{1}{9}(135\delta_2+1).$$

We will to find a solution to (2.16). First we make a change of variable $s(x) = -r(x)$, then s is solution of

$$k^2 s'^2 - 3s^3 = 0, \tag{2.39}$$

or equivalently

$$|k|s' = \sqrt{3}s^{3/2}.$$

Now, let $s(x) = v(x)^2$, then v satisfies the following equation

$$\left(-\frac{1}{v}\right)' = \frac{v'}{v^2} = \frac{\sqrt{3}}{2|k|}, \tag{2.40}$$

and integrating this equation we arrive to the solution

$$v(x) = \frac{v(0)}{1 - k_0 v(0)x}, \tag{2.41}$$

where $k_0 = \dfrac{\sqrt{3}}{2|k|} \approx 0,74995$, and as $r(x) = -v(x)^2$, we have

$$r(x) = \frac{r(0)}{(1 - k_0(-r(0))^{1/2}x)^2}, \tag{2.42}$$

we need that $r(0) < 0$. Considering $r(0) = -1$ and by (2.2), (2.6) and (2.42) we obtain the following solution of (1.1)

$$\begin{aligned}\eta(x,t) &= -\frac{e^{2(kx-\omega t)}}{(1 - k_0 e^{kx-\omega t})^2} + \alpha \\ &= -\frac{1}{(e^{-(kx-\omega t)} - k_0)^2} + \alpha \\ &= \frac{\operatorname{sech}^2(kx-\omega t)}{\big(1 - \tanh(kx-\omega t) - k_0 \operatorname{sech}(kx-\omega t)\big)^2} + \alpha.\end{aligned}$$

□

REMARK 1. We have the following observations:

(i) In order to obtain the compatibility of the equations (2.9) and (2.10) using this method, we obtain the unique value $\gamma = -1/30$. We do not know whether other methods given other values of γ to obtain exact traveling wave solution of (1.1).

(ii) If we define

$$C_k^+ = \begin{cases} 1 & \text{if } \; k > 0, \\ 0 & \text{if } \; k < 0, \end{cases} \quad \text{and} \quad C_k^- = \begin{cases} 1 & \text{if } \; k < 0, \\ 0 & \text{if } \; k > 0, \end{cases} \tag{2.43}$$

then

$$\lim_{x \to \pm\infty} \eta(x,t) = \alpha - \frac{4k^2}{3} C_k^{\pm}. \tag{2.44}$$

3. Blow up phenomena

We start by recalling the concept of the blow-up solution. Let T be the maximal time of existence of the solution $\eta(x,t)$. We say that the solution η has the blow-up property in the space X if and only if

$$\sup_{t\in[0,T)} \|\eta(t)\|_X = \infty.$$

We say that the solution η does not have blow-up property in the space X if

$$\sup_{t\in[0,T)} \|\eta(t)\|_X < \infty.$$

The solution in (2.1) have singularity along the line

$$s(t) = \frac{w}{k}t - \frac{\ln k_0}{k}, \qquad t \geqslant 0, \tag{3.1}$$

where

$$\frac{w}{k} = \frac{388\sqrt{1069} - 8269}{2850} \approx 1.54978, \qquad -\frac{\ln k_0}{k} = -\frac{\ln\left(\frac{19}{4\sqrt{1069}-97}\right)}{\sqrt{\frac{3}{19}\left(4\sqrt{1069}-97\right)}} \approx 0.2492.$$

Considering the approximate values of k, ω, k_0 and α we draw the graphics of the approximate solution

$$\eta(x,t) = 1.9483 - \frac{e^{-3.5793t}}{(e^{-1.1548x} - 0.74995\, e^{-1.78965t})^2}.$$

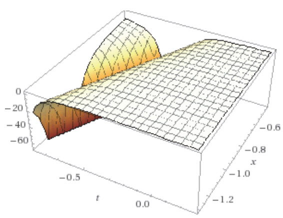

Figure 2: The graph of the approximate solution $\eta(x,t)$.

In Figure 2 we present the graphics of the approximate solution $\eta(x,t)$ and in Figure 3 we have the graph of the blow-up contour of $\eta(x,t)$. In figure 4 we have the graphics of the function $f(x) = \eta(x,1)$, observe the blow-up in

$$s(1) = \frac{w}{k} - \frac{\ln k_0}{k} \approx 1.79898.$$

In [15] was proved the following result. For the purpose of completing our paper we present the proof here

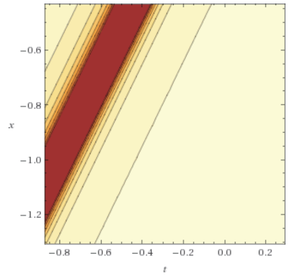

Figure 3: The graph of the blow-up contour of $\eta(x,t)$.

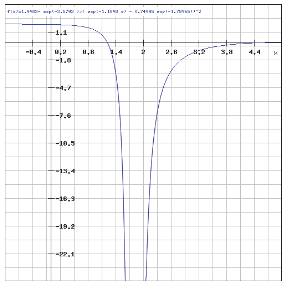

Figure 4: The graph of the approximate solution $f(x) = \eta(x,1)$ with blow-up in $s(1)$.

THEOREM 4. *Let T be the maximal time of existence of the solution $\eta(x,t)$ to the IVP (1.1). If $\delta_1 > 0$ and $\gamma \leqslant 1/42$, then the corresponding solution blows-up in H^4 if and only if*

$$\liminf_{t\to T^-} \inf_{x\in\mathbb{R}} \eta_x(x,t) = -\infty \text{ or } \limsup_{t\to T^-} \sup_{x\in\mathbb{R}} |\eta(x,t)| = \infty. \tag{3.2}$$

Proof. Suppose that

$$\liminf_{t\to T^-} \inf_{x\in\mathbb{R}} \eta_x(x,t) > -\infty \quad \text{and} \quad \limsup_{t\to T^-} \sup_{x\in\mathbb{R}} |\eta(x,t)| < \infty. \tag{3.3}$$

We will prove that, under this assumption, the solution to the IVP (1.1) does not blow-up in H^4. In turn, this would show that, if the solution satisfies condition (3.2), it will blow-up in finite time.

Note that, our assumption (3.3) implies that there are constants $L_1 > 0$ and $L_2 > 0$ such that for any $x \in \mathbb{R}$ and $t \in [0,T)$, one has

$$\eta_x(x,t) \geqslant -L_1 \quad \text{and} \quad |\eta(x,t)| \leqslant L_2. \tag{3.4}$$

Using the relation $(\eta^2)_{xxx} = 2\eta\eta_{xxx} + 6\eta_x\eta_{xx}$, we write the equation in the initial value problem (1.1) in the following form

$$\eta_t + \eta_x - \frac{1}{6}\eta_{xxt} + \delta_1\eta_{xxxxt} + \delta_2\eta_{xxxxx} + \frac{3}{2}\eta\eta_x + 2\gamma\eta\eta_{xxx} + K_\gamma\eta_x\eta_{xx} - \frac{3}{4}\eta^2\eta_x = 0, \tag{3.5}$$

where $K_\gamma = 6\gamma - 1/6$.

Multiplying (3.5) by $\Psi(\eta) := \eta - \frac{1}{6}\eta_{xx} + \delta_1\eta_{xxxx}$ and integrating, we get

$$\begin{aligned}\frac{1}{2}\partial_t \int \Big[\eta - \frac{1}{6}\eta_{xx} + \delta_1\eta_{xxxx}\Big]^2 dx = & -\int (\eta_x + \delta_2\eta_{xxxxx})\Psi(\eta)dx - \frac{3}{2}\int \eta\eta_x\Psi(\eta)dx \\ & -2\gamma\int \eta\eta_{xxx}\Psi(\eta)dx - K_\gamma\int \eta_x\eta_{xx}\Psi(\eta)dx \\ & +\frac{3}{4}\int \eta^2\eta_x\Psi(\eta)dx \\ =: & \sum_{i=1}^{5} J_i.\end{aligned} \tag{3.6}$$

We use integration by parts, to obtain

$$J_1 = 0, \tag{3.7}$$

$$J_2 = -\frac{1}{8}\int \eta_x(\eta_x)^2 dx - \frac{15}{4}\delta_1\int \eta_x(\eta_{xx})^2 dx, \tag{3.8}$$

$$J_3 = -2\gamma\int \eta_x(\eta_x)^2 dx - \frac{\gamma}{6}\int \eta_x(\eta_{xx})^2 dx + \gamma\delta_1\int \eta_x(\eta_{xxx})^2 dx, \tag{3.9}$$

$$J_4 = \frac{K_\gamma}{2}\int \eta_x(\eta_x)^2 dx + \frac{K_\gamma}{6}\int \eta_x(\eta_{xx})^2 dx + K_\gamma\delta_1\int \eta_x(\eta_{xxx})^2 dx. \tag{3.10}$$

Now, combining (3.6)-(3.10), we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t \int \Big[\eta - \frac{1}{6}\eta_{xx} + \delta_1\eta_{xxxx}\Big]^2 dx \\ &= \Big(-\frac{1}{8} + \frac{K_\gamma}{2} - 2\gamma\Big)\int \eta_x(\eta_x)^2 dx + \Big(-\frac{15\delta_1}{4} + \frac{K_\gamma}{6} - \frac{\gamma}{6}\Big)\int \eta_x(\eta_{xx})^2 dx \\ &\qquad + \delta_1(K_\gamma + \gamma)\int \eta_x(\eta)^2 dx + \frac{3}{4}\int \eta^2\eta_x\Psi(\eta)dx.\end{aligned} \tag{3.11}$$

Observe that

$$-A_\gamma := \Big(-\frac{1}{8} + \frac{K_\gamma}{2} - 2\gamma\Big) \leqslant 0 \quad \text{if and only if} \quad \gamma \leqslant \frac{5}{24}, \tag{3.12}$$

$$-A_{\gamma,\delta_1} := \Big(-\frac{15\delta_1}{4} + \frac{K_\gamma}{6} - \frac{\gamma}{6}\Big) \leqslant 0 \quad \text{if and only if} \quad \gamma \leqslant \frac{1}{30} + \frac{9}{2}\delta_1, \tag{3.13}$$

and

$$-B_\gamma := K_\gamma + \gamma \leqslant 0 \quad \text{if and only if} \quad \gamma \leqslant \frac{1}{42}. \tag{3.14}$$

On the other hand the hypotheses (3.4) and inequalities (3.12)-(3.14) imply that

$$\frac{1}{2}\partial_t\int[\eta-\frac{1}{6}\eta_{xx}+\delta_1\eta_{xxxx}]^2dx \leqslant A_\gamma L_1\int(\eta_x)^2dx+A_{\gamma,\delta_1}L_1\int(\eta_{xx})^2dx + \delta_1 B_\gamma L_1\int(\eta_{xxx})^2dx+\frac{3}{4}L_2^2\int\big|\eta_x\Psi(\eta)\big|dx. \tag{3.15}$$

Let

$$\mathscr{X}(t):=\|\Psi(\eta)\|_{L^2}^2=\int\Big(\eta-\frac{1}{6}\eta_{xx}+\delta_1\eta_{xxxx}\Big)^2dx=\int\Big(1+\frac{1}{6}\xi^2+\delta_1\xi^4\Big)^2|\widehat{\eta}(\xi)|^2d\xi.$$

Note that

$$\mathscr{X}(t)\sim_{\delta_1}\|\eta\|_{H^4}^2.$$

As $|\xi|\leqslant 3(1+\xi^2/6)$ and $|\xi|^3\leqslant 3(\xi^2/6)+\xi^4/2=3(\xi^2/6)+\frac{1}{2\delta_1}\big(\delta_1\xi^4\big)$, one has that

$$\int(\eta_x)^2dx\leqslant 9\mathscr{X}(t),\quad \int(\eta_{xx})^2dx\leqslant 36\mathscr{X}(t),\quad \int(\eta_{xxx})^2dx\leqslant\Big(9+\frac{1}{4\delta_1^2}\Big)\mathscr{X}(t).$$

Also using Cauchy-Schwartz inequality, we have

$$\int\big|\eta_x\Psi(\eta)\big|\leqslant 3\mathscr{X}(t).$$

Now, using these inequalities, one obtains from (3.15) that

$$\frac{1}{2}\partial_t\mathscr{X}(t)\leqslant K_0\mathscr{X}(t), \tag{3.16}$$

where

$$K_0=9A_\gamma L_1+36A_{\gamma,\delta_1}L_1+\delta_1B_\gamma L_1\Big(9+\frac{1}{4\delta_1^2}\Big)+\frac{9}{4}L_2^2.$$

An application of the Gronwell's inequality implies

$$\mathscr{X}(t)\leqslant\mathscr{X}(0)e^{2K_0t},$$

for any $t\in[0,T)$ and therefore

$$\|\eta(t)\|_{H^4}^2\leqslant\|\eta(0)\|_{H^4}^2e^{2K_0t}.$$

From this last inequality, one can conclude the proof of the theorem. □

As our solution satisfies $\sup_{x\in\mathbb{R}}|\eta(x,t)|=\infty$ we concludes that $\eta(x,t)$ have blow-up in H^4.

4. Exact traveling waves solutions for an equation of Benney-Lin type

The main result in this section is

THEOREM 5. *An exact traveling wave solution of* (1.3) *is*

$$u(x,t) = -\frac{1}{\left(C_{k,\lambda_1} + C_3\cosh(kx-\omega t) - C_3\sinh(kx-\omega t)\right)^4} + \alpha_1,$$

where

$$C_{k,\lambda_1} = \frac{1}{2k}\left(\frac{1}{105\lambda_1}\right)^{1/4}$$

and $k,\omega,\lambda_1,\alpha_1$ *are given in* (4.6) *and* C_3 *is an integration constant.*

Proof. Travelling wave solutions to (1.3) are sought by taking

$$u(x,t) = \phi(\xi) \tag{4.1}$$

where

$$\xi = kx - wt \tag{4.2}$$

and k and w are constants to be determined.

Substituting into (1.3) we have

$$-w\phi_\xi + \lambda_1 k^5\phi_{\xi\xi\xi\xi\xi\xi} + \lambda_2 k^4\phi_{\xi\xi\xi\xi\xi} + k^3\phi_{\xi\xi\xi} + \lambda_3 k^2\phi_{\xi\xi} + \frac{k}{2}(\phi^2)_\xi = 0.$$

Thus, integrating yields

$$-w\phi + \frac{k}{2}\phi^2 + \lambda_1 k^5\phi_{\xi\xi\xi\xi\xi} + \lambda_2 k^4\phi_{\xi\xi\xi\xi} + k^3\phi_{\xi\xi} + \lambda_3 k^2\phi_\xi = C, \tag{4.3}$$

where C is the constant of integration and $u(x,t) = \phi(\xi)$. Equation (4.3) cannot be integrated directly, hence for to solve (4.3) we used the same idea given in [24, 30]. Indeed, we consider the Ince transformation method (ref. [18], pp. 333-334)

$$\phi(\xi) = r(e^\xi)e^{4\xi} + \alpha_1 \tag{4.4}$$

to reduce (4.3) to a directly integrable differential equation for r. Replacing (4.4) into (4.3) and performing straightforward calculations we obtain that (where the first constant of integration is taken to be $C = 0$)

$$\lambda_1 k^4 r'''' + \frac{1}{2}r^2 = 0 \quad \Longleftrightarrow \quad r'''' = \frac{-1}{2\lambda_1 k^4}r^2 \tag{4.5}$$

$$\begin{aligned} k &= \frac{-22}{179\lambda_2}, \qquad \lambda_1 = \frac{179}{484}\lambda_2^2, \qquad \lambda_3 = \frac{14036}{32041\lambda_2}, \\ \alpha_1 &= \frac{813120}{5735339\lambda_2^2} \qquad \text{and} \qquad w = \frac{-8944320}{1026625681\lambda_2^3}. \end{aligned} \tag{4.6}$$

By integration (ref. [29], pp. 644), the equation (4.5) reduce to

$$2\,r'\,r''' - (r'')^2 = \frac{-1}{3\,\lambda_1\,k^4}\,r^3 + \frac{4}{3}C_1 \tag{4.7}$$

where C_1 is an arbitrary constant. The substitution

$$z(r) = (r')^{3/2} \Longleftrightarrow r'_\xi = [z(r(\xi))]^{2/3} \tag{4.8}$$

leads to a second order equation

$$z''_r = \left(\frac{-1}{4\,\lambda_1\,k^2}\,r^3 + C_2\right) z^{-5/3}. \tag{4.9}$$

With loss of generality, we consider the case $C_2 = 0$ which correspond to the Emden-Fowler equation. Then from (4.9) we obtain

$$z''_r = \frac{-1}{4\,\lambda_1\,k^4}\,r^3\,z^{-5/3}. \tag{4.10}$$

Our goal is to get a real solution to equation (1.3). For this, let us consider the following change of variable

$$r = -s \qquad \text{and} \qquad \phi(s) = z(r). \tag{4.11}$$

Replacing (4.11) into (4.10) we obtain

$$\phi''_s = \frac{1}{4\,\lambda_1\,k^4}\,s^3\,\phi^{-5/3}, \tag{4.12}$$

where the solution is given by (ref. [17])

$$\phi(s) = \left(\frac{16}{105\,\lambda_1\,k^4}\right)^{3/8} s^{15/8}. \tag{4.13}$$

Replacing (4.13) into (4.8) and taking in account (4.11) we obtain, after performing straightforward calculations, that

$$r(e^\xi) = -\left[\frac{1}{2\,k}\left(\frac{1}{105\,\lambda_1}\right)^{1/4} e^\xi + C_3\right]^{-4} \tag{4.14}$$

satisfies (4.5). Combining (4.4) and (4.14) the resulting solutions for u is

$$\begin{aligned} u(x,t) &= -\left[C_{k,\lambda_1}\, e^{kx-wt} + C_3\right]^{-4} e^{4kx-4wt} + \alpha_1, \\ &= -\frac{1}{\left[C_{k,\lambda_1} + C_3\, e^{-(kx-wt)}\right]^4} + \alpha_1, \\ &= -\frac{\operatorname{sech}^4(kx-\omega t)}{[C_3 - C_3 \tanh(kx-\omega t) + C_{k,\lambda_1} \operatorname{sech}(kx-\omega t)]^4} + \alpha_1 \end{aligned} \tag{4.15}$$

where

$$C_{k,\lambda_1} = \frac{1}{2k}\left(\frac{1}{105\,\lambda_1}\right)^{1/4}$$

and k, w, λ_1, α_1 are given in (4.6) and C_3 is an integration constant. □

REMARK 2. We have the following remarks:

(i) If we define

$$C_k^+ = \begin{cases} 1, & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{and} \quad C_k^- = \begin{cases} 1, & k < 0, \\ 0, & k > 0, \end{cases} \tag{4.16}$$

then

$$\lim_{x \to \pm\infty} u(x,t) = \alpha_1 - \frac{C_k^{\pm}}{C_{k,\lambda_1}^4}. \tag{4.17}$$

From (4.15) we have

- If $x \to -\infty$, then $C_3\, e^{-(\kappa x - wt)} \longrightarrow 0\ (\kappa < 0)$. Hence,

$$u(x,t) \equiv u_- \longrightarrow -\frac{1}{C_{k,\lambda_1}^4} + \alpha_1. \tag{4.18}$$

- If $x \to +\infty$, then $C_3\, e^{-(\kappa x - wt)} \longrightarrow \infty\ (\kappa < 0)$. Hence,

$$u(x,t) \equiv u_+ \quad \longrightarrow \quad \alpha_1. \tag{4.19}$$

The function $(u_- \neq u_+)$

$$u(x,t) = \begin{cases} u_-, & x \to -\infty, \\ u_+, & x \to +\infty, \end{cases}$$

is called the shock wave connecting u_- to u_+ and w the corresponding shock speed. Moreover, it satisfies the Rankine-Hugoniot relation

$$w = \frac{\kappa}{2}\,(u_+ - u_-) \tag{4.20}$$

(ii) When $\lambda_2 = 0$ and $\lambda_3 = 0$, the whole shock structure disappears. This is a direct consequence of the very delicate balance needed between a solitary wave (Korteweg de Vries Kawahara) and a shock wave (Benney-Lin) to form combined solution.

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