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OSCILLATION OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH SUB–LINEAR NEUTRAL TERM

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Abstract. In this paper the authors established sufficient conditions for the oscillation of all solutions of a nonlinear differential equation

$$\left(a(t)\left(x(t)+p(t)x^{\alpha}\left(\tau(t)\right)\right)'\right)'+q(t)x^{\beta}\left(\sigma(t)\right)=0,\ t\geqslant t_0,$$

where α and β are ratio of odd positive integers. The results obtained here extend and improve some of the existing results. Examples are included to illustrate the importance of the results.

1. Introduction

In this paper, we are concerned with the second order differential equation of the form

$$\left(a(t)\left(x(t)+p(t)x^{\alpha}\left(\tau(t)\right)\right)'\right)'+q(t)x^{\beta}\left(\sigma(t)\right)=0,\ t\geqslant t_0>0, \tag{1.1}$$

subject to the following conditions:

(H_1) $0<\alpha\leqslant 1$, and β are ratio of odd positive integers;

(H_2) $a\in C'\left([t_0,\infty),(0,\infty)\right)$, $p,q\in C\left([t_0,\infty),[0,\infty)\right)$ and q is not eventually zero on $[t_*,\infty)$ for $t_*\geqslant t_0$;

(H_3) $\tau\in C'\left([t_0,\infty),\mathbb{R}\right)$, $\sigma\in C'\left([t_0,\infty),\mathbb{R}\right)$, $\tau(t)\leqslant t$, $\sigma(t)\leqslant t$, $\sigma'(t)>0$ and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$.

By a solution of equation (1.1), we mean a function $x\in C\left([T_x,\infty),\mathbb{R}\right)$, $T_x\geqslant t_0$, which has the property $a(t)\left(x(t)+p(t)x^{\alpha}\left(\tau(t)\right)\right)'\in C'\left([T_x,\infty),\mathbb{R}\right)$ and satisfies equation (1.1) on $[T_x,\infty)$. We consider only these solutions x of equation (1.1) which satisfy $\sup\{|x(t)|:t\geqslant T\}>0$ for all $T\geqslant T_x$, and assume that the equation (1.1) possess such solutions. As usual a solution of equation (1.1) is called oscillatory if it has a zero on $[T,\infty)$ for all $T\geqslant T_x$; otherwise it is called non-oscillatory.

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As indicated by Hale [8] and others, neutral differential equations having a nonlinearity in the neutral term arise in many applications. We choose to examine the oscillatory behavior of solutions of the equation (1.1) since similar properties for neutral differential equations having linear neutral term studied in many papers, see for example ([1], [2], [4]-[7], [9]-[17], [18], [19]) and the references cited therein.

Recently in [3], the authors considered the equation (1.1) with $\beta = 1$, and obtained sufficient conditions for the oscillation of all solutions of equation (1.1) under the condition

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty. \tag{1.2}$$

Motivated by this observation, in this paper we extend the results obtained in [3] for the case $\beta \geqslant 1$ and $0 < \beta < 1$. In Section 2, we present sufficient conditions for the oscillation of all solutions of equation (1.1) and in Section 3, we provide three examples to illustrate the main results.

2. Oscillation results

In this section, we present sufficient conditions for the oscillation of all solutions of equation (1.1). In what follows, all functional inequalities are assumed to hold for all t large enough. Further, we can only deal with the positive solutions of equation (1.1) since the proof of the other case is similar. In the following, for convenience we denote

$$Z(t) = x(t) + p(t)x^{\alpha}\big(\tau(t)\big),$$

and

$$R(t) = \int_{t_1}^{t} \frac{1}{a(s)} ds.$$

THEOREM 1. *Assume* $(H_1)-(H_3)$ *and* (1.2) *hold. If* $\beta \geqslant 1$, *and there exists a positive nondecreasing function* $\rho \in C'\big([t_0,\infty),\mathbb{R}\big)$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\left(1 - \frac{p\big(\sigma(s)\big)}{M^{1-\alpha}}\right)^{\beta} \rho(s)q(s) - \frac{a\big(\sigma(s)\big)\big(\rho'(s)\big)^2}{4\beta M^{\beta-1}\rho(s)\sigma'(s)} \right] ds = \infty \tag{2.1}$$

holds for all constants $M > 0$, *and all* $t_1 \geqslant t_0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution x, that is, there exists a $t_1 \geqslant t_0$ such that $x(t) > 0$, $x\big(\tau(t)\big) > 0$, and $x\big(\sigma(t)\big) > 0$ for all $t \geqslant t_1$. From equation (1.1) and condition (1.2) one can easily obtain that

$$Z(t) > 0,\ Z'(t) > 0,\ \big(a(t)Z'(t)\big)' \leqslant 0,\ t \geqslant t_1. \tag{2.2}$$

Since $\sigma(t) \leqslant t$, we have from (2.2)

$$a(t)Z'(t) \leqslant a\big(\sigma(t)\big)Z'\big(\sigma(t)\big),\ t \geqslant t_1. \tag{2.3}$$

From $Z'(t) > 0$, there exists a constant $M > 0$ such that $Z(t) \geqslant M$ for all t large enough. From the definition of Z, we have

$$x(t) = Z(t) - p(t)x^{\alpha}\big(\tau(t)\big) \geqslant Z(t) - p(t)Z^{\alpha}\big(\tau(t)\big) \geqslant \left(1 - \frac{p(t)}{M^{1-\alpha}}\right)Z(t). \tag{2.4}$$

Define

$$\omega(t) = \rho(t)\frac{a(t)Z'(t)}{Z^{\beta}\big(\sigma(t)\big)},\ t \geqslant t_1. \tag{2.5}$$

Then $\omega(t) > 0$ for $t \geqslant t_1$, and

$$\omega'(t) = \rho'(t)\frac{a(t)Z'(t)}{Z^{\beta}\big(\sigma(t)\big)} + \rho(t)\frac{\big(a(t)Z'(t)\big)'}{Z^{\beta}\big(\sigma(t)\big)} - \beta\rho(t)\frac{a(t)Z'(t)Z'\big(\sigma(t)\big)\sigma'(t)}{Z^{\beta+1}\big(\sigma(t)\big)}. \tag{2.6}$$

From equation (1.1), (2.3), (2.4), and (2.6) we obtain

$$\omega'(t) \leqslant -\rho(t)q(t)\left(1 - \frac{p\big(\sigma(t)\big)}{M^{1-\alpha}}\right)^{\beta} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\beta M^{\beta-1}\sigma'(t)\omega^2(t)}{\rho(t)a\big(\sigma(t)\big)}.$$

Thus, we have

$$\omega'(t) \leqslant -\rho(t)q(t)\left(1 - \frac{p\big(\sigma(t)\big)}{M^{1-\alpha}}\right)^{\beta} + \frac{a\big(\sigma(t)\big)\big(\rho'(t)\big)^2}{4\beta M^{\beta-1}\rho(t)\sigma'(t)}.$$

Integrating the last inequality from t_1 to t, we obtain

$$\int_{t_1}^{t}\left[\rho(s)q(s)\left(1 - \frac{p\big(\sigma(s)\big)}{M^{1-\alpha}}\right)^{\beta} - \frac{a\big(\sigma(s)\big)\big(\rho'(s)\big)^2}{4\beta M^{\beta-1}\rho(s)\sigma'(s)}\right]ds < \omega(t_1)$$

which contradicts condition (2.1) as $t \to \infty$. This completes the proof.

REMARK 1. Note that when $\beta = 1$ and $0 < \alpha < 1$ then Theorem 1 improves Theorem 2.1 of [3] in the sense that we need $p\big(\sigma(t)\big) < M^{1-\alpha}$ where as in [3], one requires that $p\big(\sigma(t)\big) < M/(\alpha 2^{1-\alpha}M + 2^{1-\alpha} - 1) < M^{1-\alpha}$. Further when $\alpha = \beta = 1$, then Theorem 1 reduces to Theorem 2.1 of [3].

Next we present an oscillation criterion for equation (1.1) when $0 < \beta < 1$.

THEOREM 2. *Let* $(H_1)-(H_3)$ *and* (1.2) *hold. If* $0<\beta<1$, *and*

$$\int_{t_1}^{\infty} q(t)R^{\beta}\left(\sigma(t)\right)\left(1-\frac{p\left(\sigma(t)\right)}{M^{1-\alpha}}\right)^{\beta} dt=\infty \tag{2.7}$$

hold for all $M>0$, *and all* $t_1 \geqslant t_0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that equation (1.1) has an eventually positive solution x, that is, there exists a $t_1 \geqslant t_0$ such that $x(t)>0$, $x\big(\tau(t)\big)>0$, and $x\big(\sigma(t)\big)>0$ for all $t \geqslant t_1$. From the equation (1.1) and condition (1.2) we have (2.2). From (2.2) we have

$$Z\big(\sigma(t)\big)=Z(t_1)+\int_{t_1}^{\sigma(t)} \frac{a(s)Z'(s)}{a(s)}ds \geqslant a\big(\sigma(t)\big)Z'\big(\sigma(t)\big)R\big(\sigma(t)\big). \tag{2.8}$$

By equation (1.1), (2.4) and (2.8), we have

$$\big(a(t)Z'(t)\big)'+q(t)\left(1-\frac{p\big(\sigma(t)\big)}{M^{1-\alpha}}\right)^{\beta} R^{\beta}\left(\sigma(t)\right)\Big(a\big(\sigma(t)\big)Z'\big(\sigma(t)\big)\Big)^{\beta} \leqslant 0. \tag{2.9}$$

Let $\omega(t)=a(t)Z'(t)$. Then $\omega(t)>0$, and from (2.9) we obtain

$$\omega'(t)+q(t)\left(1-\frac{p\big(\sigma(t)\big)}{M^{1-\alpha}}\right)^{\beta} R^{\beta}\left(\sigma(t)\right)\omega^{\beta}\left(\sigma(t)\right) \leqslant 0,\ t \geqslant t_1. \tag{2.10}$$

By condition (2.7) and Theorem 3.9.3 of [5], the inequality (2.10) has no eventually positive solution. This contradiction completes the proof.

Finally in this section we present oscillation criteria for equation (1.1) when $\sigma(t)=t-\delta$ where $\delta>0$, and $\beta \geqslant 1$.

THEOREM 3. *Let* $(H_1)-(H_3)$ *and* (1.2) *be hold. If* $\beta>1$, $\sigma(t)=t-\delta$, $\delta>0$, *and*

$$\liminf_{t\to\infty} \beta^{-t/\delta}\log(q(t)\left(1-\frac{p\big(t-\delta\big)}{M^{1-\alpha}}\right)^{\beta} R^{\beta}\left(t-\delta\right))>0 \tag{2.11}$$

hold for all $M>0$, *and all* $t_1 \geqslant t_0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2, we have from (2.10) with $\sigma(t)=t-\delta$ that $\omega(t)>0$, and

$$\omega'(t)+q(t)\left(1-\frac{p\big(t-\delta\big)}{M^{1-\alpha}}\right)^{\beta} R^{\beta}\left(t-\delta\right)\omega^{\beta}\left(t-\delta\right) \leqslant 0. \tag{2.12}$$

By Lemma 2.2 of [15], we have $\omega(t)$ is a positive solution of the equation

$$\omega'(t)+q(t)\left(1-\frac{p(t-\delta)}{M^{1-\alpha}}\right)^{\beta}R^{\beta}(t-\delta)\omega^{\beta}(t-\delta)=0.$$

By condition (2.11) and Corollary 1.2 of [15], we see that the last equation has no positive solution. This contradiction completes the proof.

THEOREM 4. *Let* $(H_1)-(H_3)$ *and* (1.2) *be hold. If* $\beta=1$, $\sigma(t)=t-\delta$, *and*

$$\liminf_{t\to\infty}\int_{t-\delta}^{t} q(s)\left(1-\frac{p(s-\delta)}{M^{1-\alpha}}\right)R(s-\delta)ds>\frac{1}{e} \tag{2.13}$$

hold for all $M>0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3, we have (2.12). Since $\beta=1$, the inequality (2.12) becomes

$$\omega'(t)+q(t)\left(1-\frac{p(t-\delta)}{M^{1-\alpha}}\right)R(t-\delta)\omega(t-\delta)\leqslant 0. \tag{2.14}$$

By condition (2.13) and Theorem 2.1.1 of [5], the inequality (2.14) has no positive solution. This contradiction completes the proof.

3. Examples

In this section, we present three examples to illustrate the main results.

EXAMPLE 1. Consider a second order neutral differential equation

$$\left(t\left(x(t)+\frac{1}{t}x^{1/3}(\tau(t))\right)'\right)'+\lambda x^{3}(\sigma(t))=0,\ t\geqslant 1, \tag{3.1}$$

where $\lambda>0$ is a constant. By taking $\rho(t)=1$, we see that by Theorem 1 every solution of equation (3.1) is oscillatory.

EXAMPLE 2. Consider a second order neutral differential equation

$$\left(t\left(x(t)+\frac{1}{t}x^{1/3}(t/2)\right)'\right)'+tx^{3}(t/2)=0,\ t\geqslant 1. \tag{3.2}$$

Since $R(t)=\log t-\log t_1$,then by Theorem 2 every solution of equation (3.2) is oscillatory.

EXAMPLE 3. Consider a second order neutral differential equation

$$\left(x(t)+\frac{1}{t}x^{1/3}(t/3)\right)''+tx(t-1)=0,\ t\geqslant 1. \tag{3.3}$$

Since $R(t)=t-t_1$, by Theorem 4 every solution of equation (3.3) is oscillatory.

REMARK 2. One can easily see that the results reported in [1]-[7], [9]-[17], [18], [19] cannot be applied to equations (3.1)-(3.3). In particular the Theorem 2.1 of [3] cannot be applied to equations (3.1) and (3.2) since $\beta\neq 1$. Thus the results obtained here extend and improve many of the existing results reported in the literature.

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