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NONLINEAR MODEL OF QUASI–STATIONARY PROCESS IN CRYSTALLINE SEMICONDUCTOR

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Abstract. We consider the question of global existence and asymptotics of small, smooth, and localized solutions of a certain pseudoparabolic equation in one dimension, posed on half-line $x > 0$,

$$\begin{cases} \left(1-\partial_x^2\right) u_t = \partial_x^2 \left(u + \alpha_2 \left(|u|^{q_2} u\right)\right) + \alpha_1 |u|^{q_1} u, \; x \in \mathbb{R}^+, \; t > 0, \\ u(0,x) = u_0(x), \; x \in \mathbb{R}^+, \\ u(0,t) = h(t), \end{cases} \tag{0.1}$$

where $\alpha_i \in \mathbb{R}, q_i > 0, i = 1,2, u : R_x^+ \times R_t^+ \in \mathbb{C}$. This model is motivated by the a wave equation for media with a strong spatial dispersion, which appear in the nonlinear theory of the quasy-stationary processes in the electric media. We show that the problem (0.1) admits global solutions whose long-time behavior depend on boundary data. More precisely, we prove global existence and modified by boundary scattering of solutions.

1. Introduction

We consider the initial-boundary value problem on a half-line for the nonlinear pseudoparabolic equation

$$\begin{cases} \left(1-\partial_x^2\right) u_t = \partial_x^2 \left(u + \mathscr{N}_2(u)\right) + \mathscr{N}_1(u), \; x \in \mathbb{R}^+, \; t > 0, \\ u(0,x) = u_0(x), \; x \in \mathbb{R}^+, \\ u(0,t) = h(t), \end{cases} \tag{1.1}$$

where $u : R_x^+ \times R_t^+ \in \mathbb{C}$,

$$\mathscr{N}_1(u) = \alpha_1 |u|^{q_1} u, \; \mathscr{N}_2(u) = \alpha_2 |u|^{q_2} u, \; \alpha_i \in \mathbb{R}, \; q_i > 0, \; i = 1,2.$$

This model is motivated by the question of global existence of solutions of the wave equation for media with a strong spatial dispersion, which appear in the nonlinear theory of the quasy-stationary processes in the electric media (see subsection 1.1 for a longer discussion).

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We are interested in the initial-boundary value problem with small initial data $u(x,t)|_{t=0} = u_0(x)$ and Dirichlet boundary data $u(x,t)|_{x=0} = h(t)$ given in a suitable weighted Lebesgue spaces. We show that problem (1.1) admits global solutions whose long-time behavior depends on the boundary data.

Let the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm

$$\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}^+} |\phi(x)|^p \, dx\right)^{1/p} \text{ if } 1 \leqslant p < \infty$$

and

$$\|\phi\|_{\mathbf{L}^\infty} = ess. \sup_{x \in \mathbf{R}^+} |\phi(x)| \text{ if } p = \infty.$$

Weighted Lebesgue space is $\mathbf{L}^{p,k} = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^{p,k}} < \infty\}$, where the norm

$$\|\phi\|_{\mathbf{L}^{p,k}} = \left\|(\cdot)^k \phi(\cdot)\right\|_{\mathbf{L}^p}, \quad k \geqslant 0.$$

We now state the main result of this paper.

THEOREM 1. *Assume that the initial data* $u_0 \in \mathbf{X} = \mathbf{L}^\infty \cap \mathbf{L}^{1,1+a}$, $a \in (0,1)$, $h \in \mathbf{Y} = \mathbf{L}^{\infty,\beta}$, $\max\{q_1^{-1}, q_2^{-1}\} < \min\{\beta, 1\}$ *are sufficiently small such that*

$$\|u_0\|_{\mathbf{X}} + \|h\|_{\mathbf{Y}} \leqslant \varepsilon.$$

Then the initial-boundary value problem (1.1) has a unique global solution

$$u(x,t) \in \mathbf{C}([0,\infty); \mathbf{X}).$$

Furthermore, the solution possesses the following modified scattering behavior:

1) if $\beta > 1$, *then there exists a constant* A *such that*

$$\sup_{t>0} \langle t \rangle^{1+\gamma} \left\| u - t^{-1} A G_0(x t^{-\frac{1}{2}}) \right\|_{\mathbf{L}^\infty} \leqslant C\varepsilon,$$

where

$$A = \int_0^\infty d\tau \int_0^\infty x \left(\mathcal{N}_1(u) + \mathcal{N}_2(u) - e^{-x} \mathcal{N}_2(h(\tau))\right) dx + \int_0^\infty x u_0 dx + \int_0^\infty h(\tau) d\tau < \infty,$$

$$G_0(s) = \frac{i}{4} s e^{-\frac{s^2}{4}},$$

2) if $\beta < 1$, *then there exist a function* $\Lambda(\xi) \in \mathbf{L}^\infty$ *such that*

$$\sup_{t>0} \langle t \rangle^{\beta+\gamma} \left\| u - h(t) \Lambda(x t^{-\frac{1}{2}}) \right\|_{\mathbf{L}^\infty} \leqslant C\varepsilon,$$

where

$$\Lambda(\xi) = 1 + \xi \int_0^1 e^{-\frac{1}{4}\frac{\xi^2}{(1-z)}} (1-z)^{-\frac{3}{2}} (1-z^{\beta}) z^{-\beta} dz + \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-z^2} z^{-1} \sin z\xi \, dz,$$

3) if $\beta = 1$, *then*

$$\sup_{t>0} \langle t \rangle \left\| u - \frac{\ln t}{t} G_0(xt^{-\frac{1}{2}}) \lim_{t\to\infty} t^{-1} h(t) \right\|_{\mathbf{L}^{\infty}} \leqslant C\varepsilon.$$

REMARK 1. We have the following three cases of the influence of the boundary data $h(t) \sim t^{-\beta}$ as $t \to \infty$:

1. If $\beta > 1$ then effects of the boundary data become negligible when time tends to infinity, the solutions are said to scatter to a unperturbed asymptotic state (short range case). The decay rate of solutions is $O\left(t^{-1}\right)$ as $t \to \infty$, which is similar to the case of homogeneous boundary data.

2. If $\beta < 1$ then the influence of the boundary data becomes essential when time goes to infinity, the solutions are said to scatter to the asymptotic profile of the boundary data.

3. If $\beta = 1$ then the solutions decay as $t^{-1} \ln t$ as in the long range case (i.e. the modified scattering profile). Some resonance behavior appears: the solutions have a more slow decay rate comparing with the boundary data and unperturbed asymptotic state.

1.1. Motivation: Mathematical model of quasi-stationary process in crystalline semiconductor

Equation (1.1) is a wave equation for media with a strong spatial dispersion, which appears in the nonlinear theory of the quasy-stationary processes in the electric media (see [20]). For example, this equation describes the creation, propagation, and collapse of the so-called electric domains in semiconductors. From the mathematical standpoint, the main property of semiconductors is the fact that nonstationary processes observed in them are described by the system of quasy-stationary field equations. These equations relate the electric field E and the electric displacement D and also the electric field E and the semiconductor current density J. In the general case, the system of equations in an appropriate Cartesian coordinate system has the following form [3]:

$$\text{div} D = -4\pi e n, \ \text{rot}\, E = 0, D = E + 4\pi P, \tag{1.2}$$

$$n_t = \text{div} J + Q, J_i = \sum_{i=1}^{3} \sigma_{ij} E_j, i, j = 1, 2, 3, \tag{1.3}$$

where P is the polarization vector, σ_{ij} is the tensor of medium conductivity. Here we, as usual, have divided electrons in the semiconductor lattice into two groups: free and bound charges [21]. The term “free charges” means charges free to move over macroscopic distances. On the other side, the term “bound charges” means charges that cannot move by macroscopic distances; they only initiate the polarization of the semiconductor. The value n in Eqs. (1.2) and (1.3) has the sense of the density of free charges and the value

$$\rho = \mathrm{div} P \tag{1.4}$$

means the density of bound charge. The value Q in Eq. (1.3) appropriately defines sources or sinks of the free electron current from or into impurity centers of the semiconductor lattice, depending on the fact whether the impurity centers are donors or acceptors [3]. Under the assumption that Eqs. (1.2) and (1.3) are considered in a cylindric domain $G = R_1^+ \times (0,T), T > 0,$ the equation $\mathrm{rot} E = 0$ is equivalent to the existence of an electric field potential u, satisfying the equation

$$E = -u_x. \tag{1.5}$$

We have the well-known Debye shielding effect, which implies that ρ has form

$$\rho = \rho_0 \exp(\frac{eu}{kT_e}),$$

where T_e is the temperature of bound electrons. Thus, in our model with finite high temperature of bound electrons a good model distribution of charges ρ is

$$\rho = r(1+\varepsilon_1 u), r > 0, \varepsilon_1 = \frac{eu}{kT_e}. \tag{1.6}$$

This distribution of bound charges in a self-consistent semiconductor field leads to quasi-elastic link of main centers of the lattice of the semiconductor and bound electrons. Finally, the function $Q(u)$ in (1.3) depends on the density of sources or sinks of free electrons and has the form similar to the distributions of free and bound electrons of the lattice main centers of the semiconductor. We use the following model distribution (see, specifically, [7]):

$$Q(u) = \lambda\, |u|^{q_1} u, q_1 \geqslant 0, \tag{1.7}$$

where $\lambda < 0$ for donor impurity centers and $\lambda > 0$ for acceptor impurity centers, respectively. Obviously, $\lambda = 0$ holds in the absence of impurity centers. For the current density we assume that

$$J = e\mu n_0(u) E, \tag{1.8}$$

where in the case of strongly overheated free electrons the quasy-stationary distribution of free electrons $n_0(u)$ can be described by

$$n_0(u) = (1 + r_2\, |u|^{q_2}), q_2 \geqslant 0, r_2 > 0. \tag{1.9}$$

Substituting (1.4)-(1.8) into (1.2)-(1.3) we get the following equation for electric potential field u

$$\frac{\partial}{\partial_t}(\frac{1}{4\pi e} u_{xx} - re^{-1}\varepsilon_1 u) + (-e\mu)u_{xx} + \frac{(-e\mu)r_2}{q_2+1}\partial_x^2\left(|u|^{q_2} u\right) + \lambda\, |u|^{q_1} u = 0.$$

In the case where the electric potential is given on the boundary of the domain, we arrive at the following boundary condition: $u|_{x=0} = h(t)$, where $h(t)$ is a given function.

The phenomenon of disruption of semiconductors consists of the avalanche growth of the concentration of free charges which leads to the failure of semiconductor devices. One of the main problems of semiconductor physics consists of the search for the cause of disruption and creation semiconductor devices with parameters that would provide the needed stability in their functioning. On the other hand, a controlled growth of the concentration of free charges is the necessary condition for the creation of electromagnetic power generators. Therefore, it is necessary to produce such devices on the basis of semiconductors that, on one hand, would guarantee unlimited growth of the concentration of free charges and, on the other hand, would prevent disruption. From the mathematical point of view, the disruption of a semiconductor is described by the blow-up of a solution, i.e., the simultaneous local-in-time solvability and global-in-time unsolvability in a certain class. The obvious initial step in the study of initial-boundary value problem (1.1) is the proof of the global-in-time solvability in one or another functional class.

1.2. Previous results

For results concerning the Cauchy problem for nonlinear pseudoparabolic type equations see [4], [5],[6]. The large time asymptotics of solutions to the Cauchy problem was obtained in papers [11], [18]. Some key developments include book [20], where it was given a description of the present state of the studies of the existence/nonexistence of solutions to Cauchy problems and initial-boundary-value problems for linear and nonlinear Sobolev-type equations. For the investigation of the blow-up of solutions to various classes of nonlinear parabolic equations we also refer the reader to [14], [20]. If we take the convective type nonlinearity $|u|^{\sigma} u_x$ in Eq. (1.1) then we arrive at the damped Benjamin–Bona–Mahony equation (BBM). Many works are devoted to the study of the BBM equation (see, for example, [1], [13] and [22]).

In the case of the initial-boundary value problems (IBV- problems) there appear new difficulties comparing with the Cauchy problems. The difficulty is explained by the fact that it is necessary to take into account the boundary effects which perturb the behavior of the solutions. Comparing with the corresponding Cauchy problem, the solutions of the IBV- problem can have rapid oscillations, can converge to a self-similar profile, can grow or decay faster. In this paper we prove the global in time existence of solutions of IBV problem in the case of inhomogeneous Dirichlet boundary data. Our main goal is to evaluate the influence of the boundary data on the asymptotic behavior of solutions. In paper [12] it was proved that the solutions of the homogeneous Dirichlet problem (1.1) with $\mathcal{N}_2(u) = 0$ obtains an additional time decay comparing with the case of the Cauchy problem, due to homogeneous boundary data $h(t) = 0$. The situation is a quite different in the case of inhomogeneous boundary data (i.e. $h(t) \neq 0$). Theorem 1 shows that (1.1) admits global solutions whose long-time behavior essentially depends on the scattering properties of the boundary data. We have the following three cases of influence of the boundary data $h(t) \sim t^{-\beta}$ as $t \to \infty$. If $\beta > 1$ then effects of the boundary data become negligible when time tends to infinity. If $\beta \leqslant 1$ then the influence of the boundary data becomes essential. Some resonance behavior appears: so the solutions have a more slow decay rate comparing with the boundary data and unperturbed asymptotic state. Also there is the critical exponent $\max(q_1^{-1}, q_2^{-1}) = \min(\beta, 1)$ when (1.1) can admit global solutions whose long-time behavior is not linear, or the blow-up phenomena can occur. Our approach is based on the estimates of the integral equation in the weighted Lebesgue space. To construct the Green operator we adopt the method of book [10], based on the integral representation for sectionally analytic functions and theory of singular integro-differential equations with the Hilbert kernel and discontinuous coefficients. We will prove that the amount of the boundary data which we need to put in the problem for its well-posedness is equal to one. The integral formula is obtained by using the Laplace transform with respect to the space variable. The Laplace transform requires the boundary data $u(0,t)$ and $u_x(0,t)$ and so $u_x(0,t)$ should be determined by the given data. To achieve this we need to solve the nonlinear singular integro-differential equation with Hilbert kernel. We believe that the results of this paper could be applicable to the study of a wide class of dissipative nonlinear equations on a half-line by the use of techniques of nonlinear analysis, estimations of Green function, fixed point theorems (see [10], [16], [11]). Our approach is new and not standard, its advantage is that it can also be applied to non-integrable equations and arbitrary boundary conditions.

1.3. Notation

To state the results of the present paper we give some notations. The usual direct and inverse Laplace transformation we denote by $\mathscr{L}$ and $\mathscr{L}^{-1}$. We denote $\langle t \rangle = 1 + t, \{t\} = \frac{t}{\langle t \rangle}$. We denote by $\theta(x)$ the step function, namely $\theta(x) = 1$, for $x \geqslant 0, \theta(x) = 0$, for $x < 0$. The usual direct and inverse Laplace transformation we denote by $\mathscr{L}$ and $\mathscr{L}^{-1}$. The Fourier transform $\mathscr{F}$ and the inverse Fourier transform $\mathscr{F}^{-1}$ are defined as

$$\mathscr{F}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x)\, dx, \mathscr{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} \phi(\xi)\, d\xi.$$

Also we introduce the Fourier sine transform $\mathscr{F}_s$ and the Fourier cosine transform $\mathscr{F}_c$ as follows

$$\mathscr{F}_s\phi = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{R^+} \phi(x) \sin px dx, \quad \mathscr{F}_c\phi = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{R^+} \phi(x) \cos px dx.$$

The usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm

$$\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}^+} |\phi(x)|^p\, dx \right)^{1/p} \text{ if } 1 \leqslant p < \infty$$

and

$$\|\phi\|_{\mathbf{L}^\infty} = ess. \sup_{x \in \mathbf{R}^+} |\phi(x)| \text{ if } p = \infty.$$

Weighted Lebesgue space is $\mathbf{L}^{p,k} = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^{p,k}} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^{p,k}} = \left\| \langle \cdot \rangle^k \phi(\cdot) \right\|_{\mathbf{L}^p}, \ k \geqslant 0$.

Weighted Sobolev space is

$$\mathbf{H}^{m,k}=\left\{\phi\in\mathbf{S}':\|\phi\|_{\mathbf{H}^{m,k}}\equiv\|(1+\partial)^m\phi\|_{\mathbf{L}^{2,k}}<\infty\right\},$$

where $m,k\in\mathbb{R}^+$. The usual Sobolev space is $\mathbf{H}^m=\mathbf{H}^{m,0}$, so the index 0 we omit if it does not cause a confusion. Also $\|\varphi\|_{\mathbf{L}^\infty}=\|\varphi\|_\infty$, $\|\varphi\|_{\mathbf{L}^2}=\|\varphi\|$. Different positive constants we denote by the same letter C.

We organize the rest of our paper as follows. In Section 2 we obtain some preliminary estimates for the Green operator and prove the well-posedness of the linearized problem (1.1). Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary lemmas

We consider the linear initial boundary-value problem on a half-line

$$\begin{cases}\left(1-\partial_x^2\right)u_t=u_{xx}+f(x,t),\ x\in\mathbf{R}^+,\ t>0,\\ u(x,0)=u_0(x),\ x\in\mathbf{R}^+,\ u(0,t)=h(t).\end{cases}\tag{2.1}$$

Denote by $\mathscr{G}(t)$ and $\mathscr{H}(t)$

$$\begin{aligned}\mathscr{G}(t)\phi&=\mathscr{F}_s\frac{e^{-K(z)t}}{1+z^2}\mathscr{F}_s\phi,\\ \mathscr{H}h&=e^{-x}h(t)+\mathscr{F}_s\frac{z}{(1+z^2)^2}\int_0^{+\infty}e^{-K(z)(t-\tau)}h(\tau)d\tau,\\ K(z)&=\frac{z^2}{z^2+1}.\end{aligned}\tag{2.2}$$

THEOREM 2. *Let*

$$u_0\in\mathbf{L}^1\left(\mathbf{R}^+\right),h(t)\in\mathbf{L}^1\left(\mathbf{R}^+\right),f(x,t)\in\mathbf{C}^0(\mathbf{R}^+,\mathbf{L}^1)$$

Then solution of (2.1) has the following form

$$u(x,t)=\theta(x)\left(\mathscr{G}(t)\left(1-\partial_x^2\right)u_0+\mathscr{H}(t)h+\int_0^t\mathscr{G}(t-\tau)f(\tau)d\tau\right).\tag{2.3}$$

Proof. To derive an integral representation for the solutions of problem (2.1) we suppose that there exists a solution $u(x,t)$ of problem (2.1). Applying the Laplace transformation with respect to time and space variables to problem (2.1) we find for Re $p>0$, Re$\xi>0$

$$\widehat{\widehat{u}}(p,\xi)=\frac{1}{(K(p)+\xi)(1-p^2)}\left(\widehat{u_0}(p)-\widehat{u}_{0xx}(p)+H(p,\xi)+X(\xi)+\widehat{\widehat{f}}(p,\xi)\right),\tag{2.4}$$

where

$$K(p)=\frac{p^2}{p^2-1},\ H(p,\xi)=-(\xi+1)p\widehat{h}(\xi),\ X(\xi)=-(\xi+1)\widehat{u}_x(0,\xi).\tag{2.5}$$

Here the functions $\widehat{\widehat{u}}(p,\xi),\widehat{h}(\xi)$ and $\widehat{\widehat{f}}(p,\xi)$ are the Laplace transforms for $u(x,t)$, $h(t)$ and $f(x,t)$ with respect to time and space , respectively. We will find the function $X(\xi)$ using the analytic properties of the function $\widehat{\widehat{u}}$ in the right-half complex planes Re $p>0$ and Re $\xi>0$. There exist one function $\phi(\xi)$ such that $K(\phi(\xi))=-\xi$ and Re$\phi(\xi)>0$ for Re$\xi>0$. Since $\widehat{\widehat{u}}(p,\xi)$ analytic for Re$p>0$ we need to put the following condition

$$X(\xi)=-\widehat{u}_0(\phi)+\widehat{u}_{0xx}(\phi)-H(\phi,\xi)-\widehat{\widehat{f}}(\phi,\xi). \tag{2.6}$$

Thus there exist $u_x(0,t)$ such that (2.6) is valid. Note the fundamental importance of the proven fact, that the solution $\widehat{\widehat{u}}$ constitutes an analytic function in Re $z>0$. Therefore taking the inverse Laplace transform of (2.4) with respect to time and space variables we obtain

$$\begin{aligned} u(x,t)=&-\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}d\xi e^{\xi t} \\ &\times\int_{-i\infty}^{i\infty}\frac{e^{px}}{(K(p)+\xi)(1-p^2)}\left(\widehat{u}_0(p)-\widehat{u}_{0xx}(p)+H(p,\xi)+X(\xi)+\widehat{\widehat{f}}(p,\xi)\right), \end{aligned} \tag{2.7}$$

where $X(\xi)$ is satisfied (2.6). Note that under condition (2.6) for $x>0$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-px}}{(K(p)+\xi)(1-p^2)}\left(\widehat{u}_0(p)-\widehat{u}_{0xx}(p)+H(p,\xi)+X(\xi)+\widehat{\widehat{f}}(p,\xi)\right)=0.$$

Therefore we rewrite (2.7) in the form for $x>0$

$$\begin{aligned} u(x,t)=&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)} \\ &\times\left(\widehat{u}_0(p)-\widehat{u}_{0xx}(p)+H(p,\xi)+\widehat{\widehat{f}}(p,\xi)\right) \\ &+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}X(\xi), \end{aligned} \tag{2.8}$$

where $X(\xi)=(\xi+1)\widehat{u_x}(0,\xi)$. Since

$$\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)} \text{ is an even function,}$$

we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}X(\xi)=0.$$

As a consequence via (2.8),

$$u(x,t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}$$

$$\times\left(\widehat{u}_0(p)-\widehat{u}_{0xx}(p)+H(p,\xi)+\widehat{\widehat{f}}(p,\xi)\right). \tag{2.9}$$

Via Jordan Lemma taking residue in the point $\xi=-K(p)$ we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}\\&\times(\widehat{u}_0(p)-\widehat{u}_{0xx}(p))\\=&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{-K(p)t}\frac{e^{px}-e^{-px}}{1-p^2}(\widehat{u}_0(p)-\widehat{u}_{0xx}(p)).\end{aligned} \tag{2.10}$$

By the definition (2.5) $H(p,\xi)=-(\xi+1)p\widehat{h}(\xi)$. Therefore we rewrite

$$I=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}H(p,\xi)$$

in the following form

$$\begin{aligned}I&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}(\xi+1)ph(\xi)\\&=\int_0^t h(\tau)H_1(t-\tau)d\tau,\end{aligned} \tag{2.11}$$

where for $s>0$

$$\begin{aligned}H_1(s)=&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi s}(\xi\pm K(p)+1)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{(e^{px}-e^{-px})p}{(K(p)+\xi)(1-p^2)}\\=&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi s}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{(e^{px}-e^{-px})p}{1-p^2}\\&+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi s}(-K(p)+1)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{(e^{px}-e^{-px})p}{(K(p)+\xi)(1-p^2)}\\=&e^{-x}\delta(s)+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{-K(p)s}\frac{(e^{px}-e^{-px})p}{1-p^2}.\end{aligned}$$

Also we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\xi e^{\xi t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dp\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}\widehat{\widehat{f}}(p,\xi)\\&\qquad=\int_0^t d\tau\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{-K(p)(t-\tau)}\frac{e^{px}-e^{-px}}{(K(p)+\xi)(1-p^2)}\widehat{f}(p,\tau).\end{aligned} \tag{2.12}$$

Thus substituting (2.10)-(2.12) into (2.9) we obtain (2.3). Theorem 2 is proved.

We introduce operator $\mathscr{G}_0(t)$

$$\mathscr{G}_0(t)\phi=\int_0^{+\infty}\widetilde{G}(x,y,t)\phi(y)dy,$$

where the kernel

$$\widetilde{G}(x,y,t) = \frac{1}{2\pi\sqrt{t}} \int_{-\infty}^{\infty} e^{-z^2} (e^{iz(x-y)t^{-\frac{1}{2}}} - e^{iz(x+y)t^{-\frac{1}{2}}}) dz.$$

From [10] we get the following

LEMMA 1. *Let* $\phi \in \mathbf{L}^r$

$$\|\mathscr{G}_0(t)\phi\|_{\mathbf{L}^q} \leqslant C\langle t\rangle^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\phi\|_{\mathbf{L}^r},$$

is true for all $t > 0$, $1 \leqslant q \leqslant \infty, 1 \leqslant r \leqslant \infty$. *Furthermore we assume that* $\phi \in \mathbf{L}^{1,1+a}$, *then the estimates*

$$\left\|(\cdot)^b \mathscr{G}_0(t)\phi\right\|_{\mathbf{L}^q} \leqslant Ct^{-\frac{1}{2}+\frac{1}{2q}+\frac{b-a}{2}} \|\phi\|_{\mathbf{L}^{1,a}},$$

$$\left\|(\cdot)^b \left(\mathscr{G}_0(t)\phi - \frac{1}{t}\vartheta G_0\left(xt^{-\frac{1}{2}}\right)\right)\right\|_{\mathbf{L}^q} \leqslant Ct^{-1+\frac{1}{2q}+\frac{b-a}{2}} \|\phi\|_{\mathbf{L}^{1,1+a}}$$

is valid for all $t > 0$, *where* $q = 1, \infty$, $b \in [0, 1+a]$ *and*

$$\vartheta = \int_0^{+\infty} x\phi(x)\,dx, G_0(s) = \frac{i}{4} se^{-\frac{s^2}{4}}.$$

LEMMA 2.

$$\left\|\mathscr{F}_s e^{-K(z)t} \mathscr{F}_s \phi\right\|_{\mathbf{L}^r} \leqslant C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{r_1}-\frac{1}{r}\right)} \|\phi\|_{\mathbf{L}^{r_1}} + e^{-t}\|\phi\|_{\mathbf{L}^r}, \tag{2.13}$$

$$\left\|(\cdot)^b \mathscr{F}_s e^{-K(z)t} \mathscr{F}_s \phi\right\|_{\mathbf{L}^q} \leqslant Ct^{-\frac{1}{2}+\frac{1}{2q}+\frac{b-a}{2}} \|\phi\|_{\mathbf{L}^{1,a}} + e^{-t}\|\phi\|_{\mathbf{L}^{q,b}}, \tag{2.14}$$

$$\left\|\mathscr{F}_s e^{-K(z)t} \mathscr{F}_s \phi - \frac{1}{t}\vartheta G_0(t)\right\|_{\mathbf{L}^\infty} \leqslant Ct^{-1-\frac{a}{2}} \|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-t}\|\phi\|_{\mathbf{L}^\infty} \tag{2.15}$$

are valid for all $t > 0$, *where* $1 \leqslant r \leqslant r_1 \leqslant \infty$, $0 \leqslant b \leqslant a$,

$$\vartheta = \int_0^\infty x\phi(x)dx.$$

Proof. Note that $\mathscr{F}_s e^{-K(z)t} \mathscr{F}_s \phi$ can be represented as

$$\mathscr{F}_s e^{-K(z)t} \mathscr{F}_s \phi = \mathscr{G}_0(t)\phi + e^{-t}\phi + \mathscr{R}_1(t)\phi, \tag{2.16}$$

where the remainder

$$\widehat{R}_1(z,t) = e^{-K(z)t} - e^{-z^2 t} - e^{-t}. \tag{2.17}$$

From Lemma 1 the operator $\mathscr{G}_0(t)$ satisfies the estimates of the Lemma. Now we estimate the remainder $\mathscr{R}_1(t)$. We represent

$$\widehat{R}_1(z,t) = e^{-K(z)t}\left(1 - e^{-z^2 t + K(z)t}\right) - e^{-t}$$

for all $|z| \leqslant 1$, and

$$\widehat{R}_1(z,t) = -e^{-tz^2} + e^{-t}\left(e^{(1-K(z))t} - 1\right)$$

for all $|z| \geqslant 1$, then we see that

$$\begin{aligned} |R_1(s,t)| &\leqslant C\left\langle s\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-2}\langle t\rangle^{-\frac{1}{2}-1} + C\langle s\rangle^{-2}\langle t\rangle^{2} e^{-t} \\ &\leqslant C\left\langle s\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-2}\langle t\rangle^{-\frac{1}{2}-1} \end{aligned}$$

for all $s \in \mathbf{R},\ t > 0$. Applying this estimate by the Young inequality we find

$$\|\mathscr{R}_1(t)\phi\|_{\mathbf{L}^r} \leqslant C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-1}\|\phi\|_{\mathbf{L}^q} \tag{2.18}$$

for all $1 \leqslant q \leqslant r \leqslant \infty$ and

$$\|\mathscr{R}_1(t)\phi\|_{\mathbf{L}^{1,w}} \leqslant C\langle t\rangle^{-1}\left(\langle t\rangle^{\frac{w}{2}}\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,w}}\right)$$

for all $t > 0$. As consequence via (2.16) Lemma 2 is proved.

We now collect some preliminary estimates of the Green operator $\mathscr{G}(t)$ defined by (2.2), in the norms $\|\phi\|_{\mathbf{L}^r}$ and $\|\phi\|_{\mathbf{L}^{1,1+w}}$, where $w \in (0,1)$, $1 \leqslant r \leqslant \infty$.

LEMMA 3. *Suppose that the function $\phi \in \mathbf{L}^\infty(\mathbf{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbf{R}^+)$, where $a \in (0,1)$. Then the estimates*

$$\left\|\mathscr{G}(t)\phi_x^{(n)}\right\|_{\mathbf{L}^r} \leqslant C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{r_1}-\frac{1}{r}\right)}\left(\|\phi\|_{\mathbf{L}^{r_1}} + n\|\phi\|_{\mathbf{L}^\infty}\right) + e^{-t}\|\phi\|_{\mathbf{L}^r}, \tag{2.19}$$

$$\left\|(\cdot)^b\mathscr{G}(t)\phi_x^{(n)}\right\|_{\mathbf{L}^r} \leqslant Ct^{-\frac{1}{2}+\frac{1}{2r}+\frac{b-a}{2}}\left(\|\phi\|_{\mathbf{L}^{1,a}} + n\|\phi\|_{\mathbf{L}^\infty}\right) + e^{-t}\|\phi\|_{\mathbf{L}^{r,b}}, \tag{2.20}$$

$$\left\|\mathscr{G}(t)\phi_x^{(n)} - \widetilde{\vartheta}(n)t^{-1}G_0\left(xt^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^\infty} \leqslant Ct^{-1-\frac{a}{2}}\left(\|\phi\|_{\mathbf{L}^{1,1+a}} + n\|\phi\|_{\mathbf{L}^\infty}\right) \tag{2.21}$$

are valid for all $t > 0$, where $1 \leqslant r \leqslant r_1 \leqslant \infty$, $0 \leqslant b \leqslant a$, $n = 0,2$

$$\widetilde{\vartheta}(n) = \int_0^\infty x\left(\phi(x) - \frac{n}{2}e^{-x}\phi(0)\right)dx.$$

Proof. Integrating by parts we get

$$\begin{aligned} &\mathscr{F}_s\phi_{xx} \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin zx\, d\phi_x = \sqrt{\frac{2}{\pi}}\left(\sin zx\phi_x|_0^\infty - z\int_0^\infty \cos zx\, d\phi(x)\right) \\ &= \sqrt{\frac{2}{\pi}}\left(-z\phi\cos zx|_0^\infty - z^2\int_0^\infty \phi(x)\sin zx\, dx\right). \end{aligned}$$

and as a consequence for $n=0,2$

$$\begin{aligned}\mathscr{G}(t)\phi_{xx} &= -\mathscr{F}_s e^{-K(z)t}\frac{z^2}{1+z^2}\mathscr{F}_s\phi+\phi(0)\mathscr{F}_s e^{-K(z)t}\frac{z}{1+z^2}\\ &= -\mathscr{F}_s e^{-K(z)t}\mathscr{F}_s\phi+\mathscr{F}_s e^{-K(z)t}\frac{1}{1+z^2}\mathscr{F}_s\phi+\phi(0)\mathscr{F}_s e^{-K(z)t}\frac{z}{1+z^2}.\end{aligned} \tag{2.22}$$

Since for $x>0$ $\frac{z}{1+z^2}=\mathscr{F}_s e^{-x}$ from (2.22) we obtain

$$\mathscr{G}(t)\phi_x^{(n)}=\mathscr{F}_s e^{-K(z)t}\frac{1}{1+z^2}\mathscr{F}_s\phi-\frac{n}{2}\mathscr{F}_s e^{-K(z)t}\mathscr{F}_s\left(\phi-e^{-x}\phi(0)\right).$$

We have

$$\mathscr{F}_s\frac{1}{1+z^2}\mathscr{F}_s\phi=\int_0^x e^{-(x-y)}\phi(y)dy=\phi_1(x)$$

and therefore

$$\begin{aligned}\mathscr{G}(t)\phi_x^{(n)} &= \mathscr{F}_s e^{-K(z)t}\mathscr{F}_s\phi_1\\ \phi_1 &= \int_0^x e^{-(x-y)}\phi(y)dy-\frac{n}{2}\left(\phi-e^{-x}\phi(0)\right).\end{aligned}$$

Applying the Young inequality we find for $q\geqslant 1, w\geqslant 0$

$$\|\phi_1\|_{\mathbf{L}^{q,w}}\leqslant C\left(\|\phi\|_{\mathbf{L}^{q,w}}+n\|\phi\|_{\mathbf{L}^\infty}\right)$$

and as a consequence via Lemma 2 we prove Lemma 3.

LEMMA 4. *Let $h\in\mathbf{Y}=\mathbf{H}_\infty^{0,\beta}(\mathbb{R}^+)$, $\beta>0$. Then the following estimates are true:*

$$\begin{aligned}\max\left(\langle t\rangle^{-\beta+\gamma},\langle t\rangle^{\frac{-b}{2}},\langle t\rangle^{\frac{-b}{2}+1-\beta}\right)\|\mathscr{H}(t)h\|_{\mathbf{L}_1^{0,1+b}} &\leqslant C\|h\|_{\mathbf{Y}}, a\geqslant 0,\\ \max\left(\langle t\rangle^{\beta},\langle t\rangle\right)\|\mathscr{H}(t)h\|_{\mathbf{L}^\infty} &\leqslant C\|h\|_{\mathbf{Y}},\beta\neq 1,\\ \langle t\rangle\ln t\|\mathscr{H}(t)h\|_{\mathbf{L}^\infty} &\leqslant C\|h\|_{\mathbf{Y}},\beta=1.\end{aligned}$$

Moreover

$$\begin{aligned}\mathscr{H}(t)h &= h(t)\Lambda(xt^{-\frac{1}{2}})+(1-e^{-x})h(t)+t^{-\beta-\gamma}O\left(\|h\|_{\mathbf{H}_\infty^{0,\beta}}\right),\beta<1, \qquad (2.23)\\ \mathscr{H}(t)h &= \frac{1}{t}G_0(xt^{-\frac{1}{2}})\int_0^\infty h(\tau)d\tau+t^{-1-\gamma}O\left(\|h\|_{\mathbf{H}_\infty^{0,1+\gamma}}\right),\beta>1,\\ \mathscr{H}(t)h &= \frac{\ln t}{t}G_0(xt^{-\frac{1}{2}})\lim_{t\to\infty}t^{-1}h(t)+t^{-1}O\left(\|h\|_{\mathbf{H}_\infty^{0,1}}\right),\beta=1,\end{aligned}$$

where $\Lambda(\xi)\in\mathbf{L}^\infty$ given by

$$\Lambda(\xi)=\xi\int_0^1 e^{-\frac{1}{4}\frac{\xi^2}{(1-z)}}(1-z)^{-\frac{3}{2}}(1-z^\beta)z^{-\beta}dz+\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-z^2}z^{-1}\sin z\xi\, dz.$$

Proof. We follow the method of papers [17] and [15]. Since

$$\mathscr{F}_s xe^{-x} = \frac{z}{(1+z^2)^2}$$

we split $\mathscr{H}h$ in the form

$$\begin{aligned}\mathscr{H}h &= \int_0^t d\tau h(\tau)\mathscr{F}_s e^{-K(z)(t-\tau)}\mathscr{F}_s xe^{-x} \\ &= \int_0^t d\tau h(\tau)\mathscr{G}(t-\tau)\phi(x),\ \phi(x) \\ &= xe^{-x}.\end{aligned} \tag{2.24}$$

Applying estimate (2.20) of Lemma 2 we have for $b \geqslant 0, \beta \neq 1$

$$\begin{aligned}\|\mathscr{H}h\|_{\mathbf{L}_1^{0,1+b}} &\leqslant C\|h\|_{\mathbf{H}_\infty^{0,\beta}} \int_0^t (t-\tau)^{-1}\left\|G_0(x(t-\tau)^{-\frac{1}{2}})\right\|_{\mathbf{L}^{1,1+b}} \langle\tau\rangle^{-\beta} d\tau \\ &\quad + \int_0^t (t-\tau)^{-\frac{1}{2}}\langle\tau\rangle^{-\beta} d\tau \\ &\leqslant C\|h\|_{\mathbf{H}_\infty^{0,\beta}} \int_0^t (t-\tau)^{\frac{b}{2}}\langle\tau\rangle^{-\beta} d\tau \leqslant C\|h\|_{\mathbf{H}_\infty^{0,\beta}} \langle t\rangle^{\frac{b}{2}} \max(\langle t\rangle^{1-\beta}, 1).\end{aligned} \tag{2.25}$$

Note that in the case of $\beta = 1$

$$\|\mathscr{H}h\|_{\mathbf{L}_1^{0,1+b}} \leqslant C\|h\|_{\mathbf{H}_\infty^{0,\beta}} \langle t\rangle^{\frac{b}{2}+\gamma}. \tag{2.26}$$

Now we prove the asymptotic of operator $\mathscr{H}(t)$.

1. Firstly we consider the case of $\beta < 1$. Via formula (2.16) from Lemma 2

$$\mathscr{H}(t)h = \mathscr{R}_0 h + \mathscr{H}_0(t)h + \mathscr{R}_1 h, \tag{2.27}$$

where

$$\mathscr{R}_0 = h(t)\int_0^t d\tau \mathscr{F}_s e^{-K(z)(t-\tau)} \frac{z}{(1+z^2)^2},$$

$$\begin{aligned}\mathscr{H}_0(t)h &= \vartheta \int_0^t (t-\tau)^{-1} e^{-\frac{x^2}{(t-\tau)}} \frac{x}{(t-\tau)^{\frac{1}{2}}}(h(\tau)-h(t))d\tau, \\ \vartheta &= \int_0^\infty x^2 e^{-x} dx = 2,\end{aligned}$$

where the remainder

$$\mathscr{R}_1(t)h = \int_0^t d\tau(h(\tau)-h(t))\int_0^{+\infty}(R_1(x-y,t)-R_1(x+y,t))ye^{-y}dy,$$

with a kernel defined by (2.17). Integrating by parts and using $\mathscr{F}_s \frac{1}{z(1+z^2)} = 1 - e^{-x}$, $x > 0$,

$$\mathscr{R}_0 = h(t)\mathscr{F}_s\left(e^{-K(z)t} - 1\right)\frac{1}{z(1+z^2)}$$
$$= h(t)\left(\mathscr{F}_s e^{-K(z)t}\frac{1}{z} + \mathscr{F}_s e^{-K(z)t}\mathscr{F}_s e^{-x} + (e^{-x} - 1)\right).$$

Applying (2.16) we prove

$$\left\|\mathscr{F}_s e^{-K(z)t}\mathscr{F}_s e^{-x}\right\|_{\mathbf{L}^\infty} \leqslant C\langle t\rangle^{-1} \tag{2.28}$$

and

$$\mathscr{F}_s e^{-K(z)t}\frac{1}{z} = \mathscr{F}_s e^{-z^2}\frac{1}{z}(\xi) + O(t^{-\gamma}). \tag{2.29}$$

Changing $\tau = tz$ we get

$$\mathscr{H}_0(t)h = \int_0^t (t-\tau)^{-1} e^{-\frac{x^2}{(t-\tau)}} \frac{x}{(t-\tau)^{\frac{1}{2}}}(h(\tau) - h(t))\,d\tau$$
$$= \frac{x}{\sqrt{t}}\int_0^1 e^{-\frac{1}{4}\frac{x^2}{t(1-z)}}(1-z)^{-\frac{3}{2}}(h(tz) - h(t))\,dz$$
$$= \xi\int_0^1 e^{-\frac{1}{4}\frac{\xi^2}{(1-z)}}(1-z)^{-\frac{3}{2}}(h(tz) - h(t))\,dz,\ \xi = xt^{-\frac{1}{2}}.$$

Since $h(t) = At^{-\beta} + t^{-\beta-\gamma}O(\|h\|_{\mathbf{H}_\infty^{0,\beta+\gamma}}), \gamma > 0$ we get

$$\mathscr{H}_0(t)h = \xi h(t)\int_0^1 e^{-\frac{1}{4}\frac{\xi^2}{(1-z)}}(1-z)^{-\frac{3}{2}}(1-z^\beta)z^{-\beta}\,dz + t^{-\beta-\gamma}O(\|h\|_{\mathbf{H}_\infty^{0,\beta+\gamma}}), \tag{2.30}$$

$\xi = xt^{-\frac{1}{2}}$. From (2.17) by the same way we prove

$$\|\mathscr{R}_1 h\| \leqslant C\|h\|_{\mathbf{H}_\infty^{0,\beta}}\int_0^t d\tau(t-\tau)^{-1-\frac{1}{2}}\left(\tau^{-\beta} - t^{-\beta}\right) = \|h\|_{\mathbf{H}_\infty^{0,\beta}} O(t^{-\beta-\frac{1}{2}}). \tag{2.31}$$

From (2.27) via (2.28)-(2.31) we prove (2.23) for $\beta < 1$. Also we prove

$$\|\mathscr{H}(t)h\|_{\mathbf{H}_\infty^{0,\beta}} \leqslant \|h\|_{\mathbf{H}_\infty^{0,\beta}}. \tag{2.32}$$

Now we consider the case of $\beta > 1$. We represent

$$\mathscr{H}(t)h = t^{-1}G_0(xt^{-\frac{1}{2}})\int_0^{\frac{t}{2}} h(\tau)d\tau + \int_0^{\frac{t}{2}} \mathscr{R}(t-\tau)h(\tau)d\tau \tag{2.33}$$
$$+ \int_{\frac{t}{2}}^t H(x,t-\tau)h(\tau)d\tau,$$

where

$$H(x,t) = \mathscr{F}_s e^{-K(z)t} \mathscr{F}_s x e^{-x},$$
$$\mathscr{R}(t-\tau) = H(x,t-\tau) - t^{-1} G_0(xt^{-\frac{1}{2}}).$$

By the same way as in Lemma 2 we prove

$$\|H(x,t)\|_{\mathbf{L}^\infty} \leqslant Ct^{-1+\gamma}, \|\mathscr{R}(t)\|_{\mathbf{L}^\infty} \leqslant C(t)^{-1-\gamma}. \tag{2.34}$$

Applying (2.34) into (2.33) we obtain for $\beta > 1$

$$\mathscr{H}(t)h = t^{-1}G_0(xt^{-\frac{1}{2}}) \int_0^\infty h(\tau)d\tau + t^{-1-\gamma}O(t^{-\beta-\gamma}) \|h\|_{\mathbf{H}_\infty^{0,1+\gamma}}. \tag{2.35}$$

Note that by the same way we can prove that

$$\mathscr{H}(t)h = t^{-1}\ln t G_0(xt^{-\frac{1}{2}}) \lim_{t\to\infty} t^{-1}h(t) + t^{-1}O(t^{-\beta-\gamma}) \|h\|_{\mathbf{H}_\infty^{0,1+\gamma}}$$

Also we have

$$\|\mathscr{H}(t)h\|_{\mathbf{H}_\infty^{0,1}} \leqslant \langle t\rangle^{-1+\gamma} \|h\|_{\mathbf{H}_\infty^{0,1}}. \tag{2.36}$$

Via (2.26), (2.32), (2.34) and (2.36) the Lemma is proved.

3. Proof of Theorem 1

We rewrite the initial-boundary value problem (1.1) as the following integral equation

$$u(t) = \mathscr{G}(t)\left(1-\partial_x^2\right)u_0 - \int_0^t \mathscr{G}(t-\tau)\mathscr{N}(u(\tau))d\tau + e^{-x}h(t) + \mathscr{H}h, \tag{3.1}$$

where the Green operator $\mathscr{G}$ of the corresponding linear problem and for $q_1 \geqslant 0, q_2 \geqslant 0, \alpha_i \in \mathbb{C}, i = 1,2$

$$\mathscr{N}(u) = \mathscr{N}_1(u) + \partial_x^2 \mathscr{N}_2(u), \tag{3.2}$$
$$\mathscr{N}_1(u) = \alpha_1 |u|^{q_1} u, \ \ \mathscr{N}_2(u) = \alpha_2 |u|^{q_2} u.$$

Denote

$$\|g\|_{\mathbf{Z}} = \left(\|g(t)\|_{\mathbf{L}^\infty} + \|g(t)\|_{\mathbf{H}_1^{0,1+a}}\right), \|h\|_{\mathbf{Y}} = \left\|\langle t\rangle^\beta h\right\|_{\mathbf{L}^\infty}$$

and $\sigma = \min(\beta,1), \beta > \max\left(\frac{1}{q_1};\frac{1}{q_2}\right), a > 0$

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left(\langle t\rangle^\sigma \|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{b}{2}+\sigma-1}\|\phi(t)\|_{\mathbf{L}^{1,1+b}}\right),$$

where $b \in [0,a]$

$$\sigma = \begin{cases} \min(\beta,1), \beta \neq 1, \\ 1-\gamma, \beta = 1. \end{cases}$$

Note that the $\mathbf{L}^1$ - norm is estimated by the norm $\mathbf{X}$

$$\begin{aligned}\|\phi(t)\|_{\mathbf{L}^1} &= \int_0^{\langle t\rangle^\sigma} |\phi(x,t)|\,dx + \int_{\langle t\rangle^\sigma}^{+\infty} |1+x|^{-1-\alpha}|x|^{1+\alpha}|\phi(x,t)|\,dx \\ &\leqslant C\langle t\rangle^\sigma \|\phi(t)\|_{\mathbf{L}^\infty} + C\langle t\rangle^{-\frac{b}{2}+\sigma-1}\|\phi(t)\|_{\mathbf{L}^{1,1+a}} \leqslant C\|\phi\|_{\mathbf{X}}.\end{aligned} \tag{3.3}$$

The local existence in the function space $\mathbf{X}_T$ can be proved by a standard contraction mapping principle. We state it without a proof.

THEOREM 3. *Let*

$$u_0 \in \mathbf{Z} = \mathbf{L}^\infty \cap \mathbf{L}^{1,1+a},\ a\in(0,1),\ h\in\mathbf{Y} = \mathbf{L}^{\infty,\beta},\ \beta > \max(q_1^{-1}, q_2^{-1}).$$

Then for some $T>0$ there exists an unique solution $u\in\mathbf{C}([0,T];\mathbf{Z})$ to problem (1.1) with the estimate $\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}$.

Let us prove that the existence time T can be extended to infinity which then yields the result of Theorem 1. By contradiction, we assume that there exist a minimal time $T>0$ such that the a-priori estimate $\|u\|_{\mathbf{X}_T} < \sqrt{\varepsilon}$ does not hold, namely, we have $\|u\|_{\mathbf{X}_T} \leqslant \sqrt{\varepsilon}$. From Lemmas 3 and 4 we get the following estimates

$$\|\mathscr{G}(t)u_0\|_{\mathbf{X}} \leqslant C\|u_0\|_{\mathbf{X}},\ \|\mathscr{H}(t)h\|_{\mathbf{X}} \leqslant C\|h\|_{\mathbf{Y}}. \tag{3.4}$$

Also from Lemma 3 we get

$$\begin{aligned}&\left\|\int_0^t \mathscr{G}(t-\tau)\mathscr{N}(\tau)\,d\tau\right\|_{\mathbf{L}^{1,1+a}} \\ &\leqslant C\int_0^t (t-\tau)^{\frac{a}{2}}\left(\|\mathscr{N}_1(\tau)\|_{\mathbf{L}^{1,1}} + \|\mathscr{N}_2(\tau)\|_{\mathbf{L}^{1,1}} + \left\|e^{-x}\mathscr{N}_2(h(\tau))\right\|_{\mathbf{L}^{1,1}}\right)d\tau.\end{aligned} \tag{3.5}$$

Since

$$\begin{aligned}&\|\mathscr{N}_1(\tau)\|_{\mathbf{L}^{1,1}} + \|\mathscr{N}_2(\tau)\|_{\mathbf{L}^{1,1}} + \left\|e^{-x}\mathscr{N}_2(h(\tau))\right\|_{\mathbf{L}^{1,1}} \\ &\leqslant C\|u\|_{\mathbf{L}^\infty}^{q_1}\|u\|_{\mathbf{L}^{1,1}} + C\|u\|_{\mathbf{L}^\infty}^{q_2}\|u\|_{\mathbf{L}^{1,1}} + Ct^{-q_2\beta}\|h(\tau)\|_{\mathbf{Y}}^3 \\ &\leqslant C\langle t\rangle^{-q_1\sigma+1}\|u\|_{\mathbf{X}}^{q_1+1} + C\langle t\rangle^{-q_2\sigma+1}\left(\|u\|_{\mathbf{X}}^{q_2+1} + \|h(\tau)\|_{\mathbf{Y}}^{q_2+1}\right),\end{aligned}$$

along to (3.5) and (3.2) we prove

$$\begin{aligned}&\left\|\int_0^t \mathscr{G}(t-\tau)\mathscr{N}(\tau)\,d\tau\right\|_{\mathbf{L}^{1,1+a}} \\ &\leqslant C\left(\|u\|_{\mathbf{X}}^{q+1} + \|h\|_{\mathbf{Y}}^{q+1}\right) \\ &\quad\times\left(\int_0^t (t-\tau)^{-\frac{1}{2}}\langle\tau\rangle^{-2\sigma+\frac{a}{2}+1-\sigma}\,d\tau + \int_0^t (t-\tau)^{\frac{a}{2}}\langle\tau\rangle^{-3\sigma+1}\,d\tau\right) \\ &\leqslant C\left(\|u\|_{\mathbf{X}}^{q+1} + \|h\|_{\mathbf{Y}}^{q+1}\right)\langle t\rangle^{\frac{a}{2}+1-\sigma},\ q=\min(q_1,q_2).\end{aligned} \tag{3.6}$$

Now we estimate $\mathbf{L}^{\infty}-$ norm of the solution $u(x,t)$. From Lemma 2 we get

$$\begin{aligned} &\left\|\int_0^t \mathscr{G}(t-\tau)\mathscr{N}(\tau)\,d\tau\right\|_{\mathbf{L}^{\infty}} \\ &\leqslant C\int_0^{\frac{t}{2}}(t-\tau)^{-1}\left(\|\mathscr{N}_1(\tau)\|_{\mathbf{L}^{1,1}}+\|\mathscr{N}_2(\tau)\|_{\mathbf{L}^{1,1}}+\left\|e^{-x}\mathscr{N}_2(h(\tau))\right\|_{\mathbf{L}^{1,1}}\right)d\tau \\ &\quad +C\int_{\frac{t}{2}}^{t}\left(\|\mathscr{N}_1(\tau)\|_{\mathbf{L}^{\infty}}+\|\mathscr{N}_2(\tau)\|_{\mathbf{L}^{\infty}}+\left\|e^{-x}\mathscr{N}_2(h(\tau))\right\|_{\mathbf{L}^{\infty}}\right)d\tau. \end{aligned} \tag{3.7}$$

Since for $q=\min(q_1,q_2)$

$$\begin{aligned} &\|\mathscr{N}_1(\tau)\|_{\mathbf{L}^{\infty}}+\|\mathscr{N}_2(\tau)\|_{\mathbf{L}^{\infty}}+\left\|e^{-x}\mathscr{N}_2(h(\tau))\right\|_{\mathbf{L}^{\infty}} \\ &\leqslant C\|u\|_{\mathbf{L}^{\infty}}^{q+1}+Ct^{-(q+1)\beta}\|h\|_{\mathbf{Y}}^{3}\leqslant C\langle t\rangle^{-(q+1)\sigma}\left(\|u\|_{\mathbf{X}}^{q+1}+\|h(\tau)\|_{\mathbf{Y}}^{q+1}\right) \end{aligned}$$

via (3.7) we obtain for $\sigma>\frac{1}{q}$

$$\begin{aligned} &\left\|\int_0^t \mathscr{G}(t-\tau)\mathscr{N}(\tau)\,d\tau\right\|_{\mathbf{L}^{\infty}} \\ &\leqslant C\left(\|u\|_{\mathbf{X}}^{q+1}+\|h(\tau)\|_{\mathbf{Y}}^{q+1}\right)\int_0^{\frac{t}{2}}(t-\tau)^{-1}\langle\tau\rangle^{-q\sigma}\,d\tau+\int_{\frac{t}{2}}^{t}\langle\tau\rangle^{-(q+1)\sigma}\,d\tau \\ &\leqslant C\left(\|u\|_{\mathbf{X}}^{q+1}+\|h(\tau)\|_{\mathbf{Y}}^{q+1}\right)\langle t\rangle^{-\sigma}. \end{aligned} \tag{3.8}$$

Via (3.4), (3.7), (3.8) along to (3.1) we get that

$$\|u\|_{\mathbf{X}_T}\leqslant C\varepsilon<\sqrt{\varepsilon},$$

which implies the desired contradiction. Thus there exists a unique global solution $u\in\mathbf{C}\left([0,\infty);\mathbf{L}^{1,1+a}\bigcap\mathbf{L}^{\infty}\right)$ of (1.1) with the time decay estimate

$$\sup_{t\geqslant 1}\langle t\rangle^{\sigma}\|u(t)\|_{\infty}\leqslant C.$$

We now prove the asymptotics of solutions. Via Lemma 2 we have

$$\int_0^t \mathscr{G}(t-\tau)\mathscr{N}(\tau)\,d\tau=t^{-1}\theta G_0(xt^{-\frac{1}{2}})+\mathscr{R},$$

where for $q=\min(q_1,q_2)$

$$\theta=\int_0^{\infty}d\tau\int_0^{\infty}x\left(\mathscr{N}_1(u)+\mathscr{N}_2(u)-e^{-x}\mathscr{N}_2(h(\tau))\right)dx\leqslant C\int_0^{+\infty}\langle\tau\rangle^{-q\sigma}\,d\tau\leqslant\infty,$$

$$\mathscr{R}=\int_0^{\frac{t}{2}}\left(\mathscr{G}(t-\tau)-\mathscr{G}_0(t-\tau)\right)\mathscr{N}(\tau)\,d\tau+\int_0^{\frac{t}{2}}\left(\mathscr{G}_0(t-\tau)-\mathscr{G}_0(t)\right)\mathscr{N}(\tau)\,d\tau$$

$$+\int_{\frac{t}{2}}^{t} \mathscr{G}(t-\tau)\mathscr{N}(\tau)\,d\tau$$
$$= t^{-1-\frac{a}{2}}\left(\|u\|_{\mathbf{X}}^{q+1} + \|h\|_{\mathbf{Y}}^{q+1}\right).$$

Therefore via Lemmas 3 and 4 we get

$$u = t^{-1}AG_0(xt^{-\frac{1}{2}}) + \varepsilon^{q+1}t^{-1-\gamma}, \beta > 1,$$
$$A = \theta + \int_0^\infty xu_0dx + \int_0^\infty h(\tau)d\tau < \infty.$$

In the case of $\beta \in \left(\frac{1}{q}, 1\right)$ we have

$$u = h(t)\Lambda(xt^{-\frac{1}{2}}) + t^{-\beta-\gamma}\varepsilon^{q+1}, \beta < 1,$$
$$\Lambda(\xi) = 1 + \xi\int_0^1 e^{-\frac{1}{4}\frac{\xi^2}{(1-z)}}(1-z)^{-\frac{3}{2}}(1-z^\beta)z^{-\beta}dz$$
$$+\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-z^2}z^{-1}\sin z\xi\,dz$$

For $\beta = 1$ we obtain

$$u = \frac{\ln t}{t}G_0(xt^{-\frac{1}{2}})\lim_{t\to\infty}t^{-1}h(t) + t^{-1}\varepsilon^{q+1}.$$

Theorem1 is proved.

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