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# CHARACTERIZATION OF A CLASS OF SECOND ORDER NEUTRAL IMPULSIVE SYSTEMS VIA PULSATILE CONSTANT

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*Abstract.* In this work, we study the oscillation and nonoscillation properties of a class of second order neutral impulsive differential equations with constant coefficients and constant delays by using pulsatile constant. Also, an attempt is made to extend the constant coefficient results to variable coefficient equations.

## 1. Introduction

Consider

$$(y(t)-ry(t-\tau))''+qy(t-\sigma)=0,\ \ t\neq\tau_k,\ \ k\in\mathbb{N} \tag{1.1}$$

$(E)$

$$\Delta(y(\tau_k)-ry(\tau_k-\tau))'+py(\tau_k-\sigma)=0,\ \ k\in\mathbb{N}, \tag{1.2}$$

where $\tau>0$, $\sigma\geqslant 0$ are real constants, $r\in\mathbb{R}\setminus\{0\}$, $p$, $q\in\mathbb{R}$ and $\tau_k$, $k\in\mathbb{N}$ with $\tau_1<\tau_2<\ldots<\tau_k<\ldots$ and $\lim\limits_{k\to\infty}\tau_k=+\infty$ are fixed moments of impulse effect with the property $\max\{\tau_{k+1}-\tau_k\}<+\infty$, $k\in\mathbb{N}$. For $(E)$, $\Delta$ is the difference operator defined by

$$\Delta(y(\tau_k)-ry(\tau_k-\tau))'=y'(\tau_k+0)-ry'(\tau_k-\tau+0)-y'(\tau_k-0)+ry'(\tau_k-\tau-0);$$
$$y'(\tau_k-0)=y'(\tau_k)\ \text{ and }\ y'(\tau_k-\tau-0)=y'(\tau_k-\tau),\ \ k\in\mathbb{N}.$$

The objective of this work is to study $(E)$ and establish conditions for oscillation and nonoscillation of solutions of $(E)$ subject to its associated characteristic equation. We may expect the possible solutions of $(E)$ as

$$y(t)=e^{-\lambda t}A^{i(t_0,t)},\quad t_0\geqslant\rho=\max\{\tau,\sigma\}, \tag{1.3}$$

where $i(t_0,t)=k$ = number of impulses $\tau_k$, $k\in\mathbb{N}$ and $A\neq 0$ is a real number which is called as the pulsatile constant. A close observation reveals that $y(t)=C_1e^{-\lambda t}$ is a

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possible solution of (1.1) when $(E)$ is without impulses and $y(n) = C_2A^n$ is a possible solution of (1.2) when $i(t_0,t) = n$ and the impulses are the discrete values only ($\because$ in case (1.2), $\lambda = 0$). Therefore, (1.3) seems to be the possible choice of solution of $(E)$.

Cheng and Chu [5] have studied the oscillatory and asymptotic behaviour of solutions of the impulsive system $(E)$ with the restrictions $r > 0$ and $\tau, \sigma < 0$. It is noticed in their study that the authors have restricted the solutions also. But, our study deals with the characteristic equation of $(E)$ and their roots. In [1], Bainov and Dimitrova have considered

$$\begin{aligned} &(r(t)y'(t))' + f(t, y(t), y(t-\sigma)) = 0, \;\; t \neq \tau_k, \;\; k \in \mathbb{N}, \\ &\Delta(r_k y'(\tau_k)) + g(y(\tau_k), y(\tau_k - \sigma)) = 0, \;\; k \in \mathbb{N} \end{aligned} \tag{1.4}$$

and studied the oscillatory character of the solutions of the system. They have established the sublinear and superlinear oscillation criteria for the impulsive system (1.4). It is observed that the study of (1.4) is easier than the study of $(E)$ subject to its characteristic equations. Unlike the methods in [1], we encounter here the linearized oscillation for the highly nonlinear impulsive system

$$\begin{aligned} &(y(t) - r(t)g(y(t-\tau)))'' + q(t)f(y(t-\sigma)) = 0, \;\; t \neq \tau_k, \;\; k \in \mathbb{N}, \\ &\Delta(y(\tau_k) - r_k g(y(\tau_k - \tau)))' + p(\tau_k)f(y(\tau_k - \sigma)) = 0, \;\; k \in \mathbb{N}. \end{aligned}$$

For more details about the theory of impulsive differential equations we refer the monographs [8] and [9] and some works (for e. g. [2]–[4], [6], [7], [10]–[15]) to the readers.

DEFINITION 1.1. A function $y : [-\rho, +\infty) \to \mathbb{R}$ is said to be a solution of $(E)$ with initial function $\phi \in C([-\rho, 0], \mathbb{R})$, if $y(t) = \phi(t)$ for $t \in [-\rho, 0]$, $y \in PC(\mathbb{R}_+, \mathbb{R})$, $z(t) = y(t) + p(t)y(t-\tau)$ is twice continuously differentiable for $t \in \mathbb{R}_+$ and $y(t)$ satisfies $(E)$ for all sufficiently large $t \geqslant 0$, where $\rho = \max\{\tau, \sigma\}$ and $PC(\mathbb{R}_+, \mathbb{R})$ is the set of all functions $U : \mathbb{R}_+ \to \mathbb{R}$ which are continuous for $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k \in \mathbb{N}$, continuous from the left-side for $t \in \mathbb{R}_+$, and have discontinuity of the first kind at the points $\tau_k \in \mathbb{R}_+$, $k \in \mathbb{N}$.

DEFINITION 1.2. A nontrivial solution $y(t)$ of $(E)$ is said to be nonoscillatory, if there exists a point $t_0 \geqslant 0$ such that $y(t)$ has a constant sign for $t \geqslant t_0$. Otherwise, the solution $y(t)$ is said to be oscillatory.

DEFINITION 1.3. A solution $y(t)$ of $(E)$ is said to be regular, if it is defined on some interval $[T_y, +\infty) \subset [t_0, +\infty)$ and

$$\sup\{|y(t)| : t \geqslant T_y\} > 0$$

for every $T_y \geqslant T$. A regular solution $y(t)$ of $(E)$ is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $y(t) > 0$ $(y(t) < 0)$, for $t \geqslant t_1$.

## 2. Main results

In this section, we study the oscillatory and nonoscillatory behaviour of solutions of $(E)$ through its associated characteristic equation provided (1.3) holds.

THEOREM 2.1. *Let $\tau > \sigma > 0$ and $p \neq 0 \neq q$. Then $(E)$ admits an oscillatory solution in the exponential impulsive form (1.3) if and only if the algebraic equation*

$$\lambda^2\left(1-\frac{p}{q}\lambda\right)^{n_1} - r\lambda^2 e^{\lambda\tau}\left(1-\frac{p}{q}\lambda\right)^{n_1-n_2} + qe^{\lambda\sigma} = 0 \tag{2.1}$$

*has at least one real root $\lambda$ with $\lambda > \frac{q}{p}$ for $pq > 0$ and $\lambda < \frac{q}{p}$ for $pq < 0$, where $i(t-\sigma,t) = n_1 > 0$ is a constant and $i(t-\tau,t) = n_2 =$ number of impulses between $t-\tau$ and $t$.*

*Proof.* Let $y(t)$ be a regular nontrivial solution of the system $(E)$ such that $y(t) = e^{-\lambda t}A^{i(t_0,t)}$, $t > t_0 > \rho$. Then (1.1) becomes

$$\lambda^2 e^{-\lambda t}A^{i(t_0,t)} - r\lambda^2 e^{-\lambda(t-\tau)}A^{i(t_0,t-\tau)} + qe^{-\lambda(t-\sigma)}A^{i(t_0,t-\sigma)} = 0,$$

that is,

$$qe^{\lambda\sigma} + \lambda^2 A^{i(t_0,t)-i(t_0,t-\sigma)} - r\lambda^2 e^{\lambda\tau}A^{i(t_0,t-\tau)-i(t_0,t-\sigma)} = 0. \tag{2.2}$$

Indeed, $i(t_0,t) - i(t_0,t-\sigma) = i(t-\sigma,t) = n_1$ and

$$i(t_0,t-\tau) - i(t_0,t-\sigma) = -i(t-\tau,t-\sigma) = -[i(t-\tau,t) - i(t-\sigma,t)] = n_1 - n_2$$

implies that

$$\lambda^2 A^{n_1} - r\lambda^2 e^{\lambda\tau}A^{n_1-n_2} + qe^{\lambda\sigma} = 0 \tag{2.3}$$

due to (2.2). Once again we use (1.3) in (1.2) to obtain a relation of the form

$$y'(\tau_k+0) - ry'(\tau_k-\tau+0) - y'(\tau_k-0) + ry'(\tau_k-\tau-0) + py(\tau_k-\sigma) = 0,$$

that is,

$$\begin{aligned}&-\lambda e^{-\lambda\tau_k}A^{i(t_0,\tau_k+0)} + r\lambda e^{-\lambda(\tau_k-\tau)}A^{i(t_0,\tau_k-\tau+0)} + \lambda e^{-\lambda\tau_k}A^{i(t_0,\tau_k-0)}\\&-r\lambda e^{-\lambda(\tau_k-\tau)}A^{i(t_0,\tau_k-\tau-0)} + pe^{-\lambda(\tau_k-\sigma)}A^{i(t_0,\tau_k-\sigma)} = 0.\end{aligned}$$

We may note that $i(t_0,\tau_k+0) - i(t_0,\tau_k-0) = 1$. Hence, the last inequality becomes

$$\begin{aligned}&-\lambda A^{1+i(t_0,\tau_k-0)} + r\lambda e^{\lambda\tau}A^{1+i(t_0,\tau_k-\tau-0)} + \lambda A^{i(t_0,\tau_k-0)}\\&-r\lambda e^{\lambda\tau}A^{i(t_0,\tau_k-\tau-0)} + pe^{\lambda\sigma}A^{i(t_0,\tau_k-\sigma)} = 0,\end{aligned}$$

that is,

$$-\lambda(A-1)A^{i(t_0,\tau_k)} + r\lambda(A-1)e^{\lambda\tau}A^{i(t_0,\tau_k-\tau)} + pe^{\lambda\sigma}A^{i(t_0,\tau_k-\sigma)} = 0.$$

Therefore,

$$-\lambda(A-1)A^{i(t_0,\tau_k)-i(t_0,\tau_k-\sigma)}+r\lambda(A-1)e^{\lambda\tau}A^{i(t_0,\tau_k-\tau)-i(t_0,\tau_k-\sigma)}+pe^{\lambda\sigma}=0. \tag{2.4}$$

Using the fact

$$i(t_0,\tau_k)-i(t_0,\tau_k-\sigma)=i(\tau_k-\sigma,\tau_k)=n_1$$

and

$$i(t_0,\tau_k-\tau)-i(t_0,\tau_k-\sigma)=-i(\tau_k-\tau,\tau_k-\sigma)=-[i(\tau_k-\tau,t)-i(\tau_k-\sigma,t)]=n_1-n_2,$$

we obtain from (2.4) that

$$-\lambda(A-1)A^{n_1}+r\lambda(A-1)e^{\lambda\tau}A^{n_1-n_2}+pe^{\lambda\sigma}=0. \tag{2.5}$$

If we choose $A=1-\frac{p}{q}\lambda$, then it is easy to verify that (2.5) reduces to (2.3). Consequently, (2.3) is same as (2.1). Moreover, (2.1) is the required characteristic equation for $(E)$. Ultimately, if $y(t)$ is an oscillatory solution of $(E)$ with the pulsatile constant $A=1-\frac{p}{q}\lambda<0$, where $\lambda>\frac{q}{p}$ for $pq>0$ and $\lambda<\frac{q}{p}$ for $pq<0$, then $\lambda$ satisfies the characteristic equation (2.1). Conversely, consider the characteristic equation (2.1) and assume that $\lambda=\lambda^*$ is the real root of (2.1) with $\lambda^*>\frac{q}{p}$ for $pq>0$ and $\lambda^*<\frac{q}{p}$ for $pq<0$. Then $(E)$ admits an oscillatory solution $y(t)=e^{-\lambda^* t}A^{i(t_0,t)}$ with the pulsatile constant $A=1-\frac{p}{q}\lambda^*<0$. This completes the proof of the theorem. □

THEOREM 2.2. *Let all the assumptions of Theorem 2.1 hold. Then $(E)$ admits an eventually positive solution in the form of (1.3) if and only if (2.1) has at least one real root $\lambda$ with $\lambda<\frac{q}{p}$ for $pq>0$ and $\lambda>\frac{q}{p}$ for $pq<0$.*

*Proof.* The proof of the theorem follows from the proof of Theorem 2.1 and hence the details are omitted. □

COROLLARY 2.3. *Let $p,q,r\in\mathbb{R}\setminus\{0\}$, and $\sigma,\ \tau\in\mathbb{R}_+$ such that $\sigma=\tau\neq 0$ or $\sigma=0\neq\tau$ hold. Then the conclusion of the Theorems 2.1 and 2.2 are hold true.*

COROLLARY 2.4. *In Theorem 2.1, let $p=q\neq 0$. Then $(E)$ admits an oscillatory solution in the exponential impulsive form (1.3) if and only if $\lambda>1$ and eventually positive solution if and only if $\lambda<1$.*

REMARK 2.5. Following to Corollary 2.4, we may note that $\lambda=1$ if and only if $A=0$, that is, $(E)$ has the trivial solution.

THEOREM 2.6. *Let $\tau>\sigma>0$ and $p=q=0$. Then*

*i) for $r\in(-\infty,0)$ and $n_2$ odd or $r\in(0,\infty)$ and $n_2$ even, $(E)$ admits an oscillatory solution if and only if $\lambda^*\in(1,\infty)$ is a root of the characteristic equation of $(E)$;*

*ii) for $r\in(0,\infty)$, $(E)$ admits an eventually positive solution if and only if $\lambda^*\in(-\infty,1)$ is a root of the characteristic equation of $(E)$.*

*Proof.* Proceeding as in the proof of Theorem 2.1 we have the impulsive system

$$\lambda^2 A^{i(t_0,t)} - r\lambda^2 e^{\lambda\tau} A^{i(t_0,t-\tau)} = 0,$$
$$-\lambda(A-1)A^{i(t_0,\tau_k)} + r\lambda(A-1)e^{\lambda\tau}A^{i(t_0,\tau_k-\tau)} = 0$$

which in turn implies that

$$\lambda^2 A^k - r\lambda^2 e^{\lambda\tau} A^{k-n_2} = 0,$$
$$-\lambda(A-1)A^k + r\lambda(A-1)e^{\lambda\tau}A^{k-n_2} = 0.$$

Consequently, the above system becomes

$$\lambda^2 A^{n_2} - r\lambda^2 e^{\lambda\tau} = 0,$$
$$-\lambda(A-1)A^{n_2} + r\lambda(A-1)e^{\lambda\tau} = 0,$$

which is equivalent to say that

$$A = 1-\lambda, \quad \lambda^2 A^{n_2} - r\lambda^2 e^{\lambda\tau} = 0$$

and hence

$$\lambda^2(1-\lambda)^{n_2} - r\lambda^2 e^{\lambda\tau} = 0 \tag{2.6}$$

is the resulting characteristic equation for $(E)$. Clearly, $\lambda \neq 1$ for $r \neq 0$ in (2.6). Hence to solve (2.6), it happens that either $\lambda \in (-\infty,1)$ or $\lambda \in (1,\infty)$. If the former holds, then $1-\lambda > 0$, that is, $A > 0$ and (2.6) holds true when $r \in (0,\infty)$. Therefore, $(E)$ admits an eventually positive solution in the form (1.3) if and only if $\lambda^* \in (-\infty,1)$ is a root of (2.6). Assume that the latter holds. Then $1-\lambda < 0$, that is, $A < 0$ and (2.6) holds true when $r \in (-\infty,0)$ with odd $n_2$ or $r \in (0,\infty)$ with even $n_2$. Therefore, $(E)$ admits an oscillatory solution in the form (1.3) if and only if $\lambda^* \in (1,\infty)$ is a root of (2.6). This completes the proof of the theorem. □

REMARK 2.7. Indeed, (2.6) doesn't hold if $r \in (-\infty,0)$ and $\lambda \in (-\infty,1)$.

THEOREM 2.8. *Let* $p, r \in \mathbb{R}\setminus\{0\}$, $\tau = \sigma \neq 0$, $q = 0$ *and* $i(t-\tau,t) = 1$. *Then for* $r \in (0,\infty)$, $(E)$ *admits an eventually positive solution, and for* $r \in (-\infty,0)$, $(E)$ *admits an oscillatory solution.*

*Proof.* Let $y(t)$ be a regular nontrivial solution of $(E)$ in the form of (1.3). Then proceeding as in Theorem 2.1, we have the system of equations

$$\lambda^2 A - r\lambda^2 e^{\lambda\tau} = 0,$$
$$-\lambda(A-1)A + [p + r\lambda(A-1)]e^{\lambda\tau} = 0.$$

In the above system of equations, if $A = re^{\lambda\tau}$, then $p = 0$ which is absurd. Hence $\lambda = 0$, and either $A - r > 0$ or $A - r < 0$. Thus for $r \in (0,\infty)$, we have $A > 0$, that is, $(E)$ admits a nonoscillatory solution and when $r \in (-\infty,0)$, $(E)$ admits an oscillatory solution. This completes the proof of the theorem. □

COROLLARY 2.9. *Let* $p,r\in\mathbb{R}\setminus\{0\}$, $\sigma=0=q$ *and* $i(t-\tau,t)=1$. *Then the conclusion of Theorem* 2.8 *holds true.*

REMARK 2.10. If we denote

$$F(\lambda)=\lambda^2\left(1-\frac{p}{q}\lambda\right)^{n_1}-r\lambda^2e^{\lambda\tau}\left(1-\frac{p}{q}\lambda\right)^{n_1-n_2}+qe^{\lambda\sigma},$$

then it is easy to verify that $F(0)=q>0$,

$$F\left(\frac{q}{p}\right)\longrightarrow+\infty,\ \text{for }\ r\neq0,\ \ p>0,\ q>0$$

and

$$F\left(-\frac{q}{p}\right)=\frac{q^2}{p^2}2^{n_1}\left[1-r2^{-n_2}e^{-\frac{q}{p}\tau}\right]+qe^{-\frac{q}{p}\sigma}>0,$$

for $r\in(0,1)$, $p>0$ and $q>0$. Keeping in view of Theorem 2.1, we have proved the following result:

THEOREM 2.11. *Let* $p,q>0$, $r\in(0,1)$ *and* $\tau>\sigma>0$. *Then every solution of* $(E)$ *which is of the form* (1.3) *oscillates if and only if* (2.1) *has no real roots* $\lambda^*\in\left[-\frac{q}{p},\frac{q}{p}\right]$.

EXAMPLE 2.12. Consider the system of equations

$$(E_1)\begin{cases}(y(t)-ry(t-2))''+qy(t-1)=0, t\neq\tau_k,\ \ t>2,\ \ k\in\mathbb{N}\\ \Delta(y(\tau_k)-ry(\tau_k-2))'+py(\tau_k-1)=0,\ \ k\in\mathbb{N},\end{cases}$$

where $r=0.401713262$, $q=2$, $p=1$ and $\tau_k=k+1$, $k\in\mathbb{N}$. If we choose $n_1=1$ and $n_2=2$, then from the characteristic equation of $(E_1)$ it follows that $A=0.5$, $\lambda=1$ and

$$y(t)=e^{-t}(0.5)^{i(2,t)}$$

is a solution of the impulsive system $(E_1)$. Hence, by Theorem 2.2, the system $(E_1)$ has an eventually positive solution.

EXAMPLE 2.13. Consider the system of equations

$$(E_2)\begin{cases}(y(t)-ry(t-\frac{1}{3}))''+qy(t-\frac{1}{6})=0,\ \ t\neq\tau_k,\ \ t>\frac{1}{3},\ \ k\in\mathbb{N}\\ \Delta(y(\tau_k)-ry(\tau_k-\frac{1}{3}))+py(\tau_k-\frac{1}{6})=0,\ \ k\in\mathbb{N},\end{cases}$$

where $r=0.01625323552$, $q=0.2$, $p=0.1$ and $\tau_k=k$, $k\in\mathbb{N}$. If we choose $n_1=3$ and $s=4$, then from the characteristic equation of $(E_2)$, it follows that $A=-0.5$, $\lambda=3$ and

$$y(t)=e^{-3t}(-0.5)^{i(\frac{1}{3},t)}$$

is an oscillatory solution of the impulsive system $(E_2)$. Hence by Theorem 2.1, the system $(E_2)$ admits an oscillatory solution.

## 3. Linearized oscillation criteria

Consider the nonlinear neutral impulsive delay differential equations of the form:

$$\begin{aligned}&(y(t)-r(t)g(y(t-\tau)))''+q(t)f(y(t-\sigma))=0,\ \ t\neq\tau_k,\ \ k\in\mathbb{N}\\&\Delta(y(\tau_k)-r(\tau_k)g(y(\tau_k-\tau)))'+p(\tau_k)f(y(\tau_k-\sigma))=0,\ \ k\in\mathbb{N},\end{aligned}\tag{3.1}$$

where $\tau,\ \sigma\in\mathbb{R}_+$, $g,f\in C(\mathbb{R},\mathbb{R})$ and $r,p,q\in C(\mathbb{R}_+,\mathbb{R}_+)$. We introduce the following assumptions for the system (3.1):

$(A_1)$ $\lim\limits_{t\to\infty}r(t)=r_0,\ r_0\in[0,1);\ \lim\limits_{t\to\infty}q(t)=q_0\in\mathbb{R}_+,\ \liminf\limits_{t\to\infty}p(t)=p_0\in\mathbb{R}_+,$

$(A_2)$ $ug(u)>0,\ vf(v)>0$ for $u,v\neq 0$ and $g(u)\leqslant u,\ f(v)\leqslant v$ for $u,v\geqslant 0$; $|f(u)|\geqslant h$ for $|u|\geqslant h>0$ and $\lim\limits_{u\to 0}\frac{g(u)}{u}=1=\lim\limits_{v\to 0}\frac{f(v)}{v}$.

With the system of equations (3.1), we associate the linear system of equations

$$\begin{aligned}&(x(t)-r_0x(t-\tau))''+q_0x(t-\sigma)=0,\ \ t\neq\tau_k,\ \ k\in\mathbb{N}\\&\Delta(x(\tau_k)-r_0x(\tau_k-\tau))'+p_0x(\tau_k-\sigma)=0,\ \ k\in\mathbb{N}.\end{aligned}\tag{3.2}$$

In this section, our aim is to establish conditions for the oscillation of solutions of the system (3.1) in terms of the oscillation of solutions of the limiting equations (3.2). We note that the associated characteristic equation for the system (3.2) is given by

$$\lambda^2\left(1-\frac{p_0}{q_0}\lambda\right)^{n_1}-r_0\lambda^2e^{\lambda\tau}\left(1-\frac{p_0}{q_0}\lambda\right)^{n_1-n_2}+q_0e^{\lambda\sigma}=0.\tag{3.3}$$

By Theorem 2.1, (3.2) admits an oscillatory solution in the form (1.3) if and only if (3.3) has at least one real root $\lambda$ with $\lambda>\frac{q_0}{p_0}$.

THEOREM 3.1. *Assume that* (3.3) *has no real roots in* $\left[-\frac{q_0}{p_0},\frac{q_0}{p_0}\right]$. *Furthermore, assume that* $(A_1)$ *and* $(A_2)$ *hold. If*

$(A_3)$ $\int_0^\infty q(t)dt+\sum\limits_{k=1}^{\infty}p(\tau_k)=\infty$

*and*

$(A_4)$ $\int_t^\infty\left[\int_T^t q(s)ds+\sum\limits_{k=1}^{\infty}p(\tau_k)\right]d\theta=\infty,\ T>0,$

*then the system* (3.1) *admits oscillatory solutions.*

*Proof.* Suppose that (3.1) doesn't admit an oscillatory solution and let $y(t)$ be a regular nonoscillatory solution of the system (3.1). Then there exists $t_0\geqslant\max\{\sigma,\tau\}$ such that $y(t)>0$, $y(t-\tau)>0$ and $y(t-\sigma)>0$ for $t\geqslant t_0$. If we set

$$z(t)=y(t)-r(t)g(y(t-\tau)),$$

then the system (3.1) becomes

$$\begin{aligned}&z''(t)=-q(t)f(y(t-\sigma))\leqslant 0,\ \ t\neq\tau_k,\ \ k\in\mathbb{N},\\&\Delta z'(\tau_k)=-p(\tau_k)f(y(\tau_k-\sigma))\leqslant 0,\ \ k\in\mathbb{N}\end{aligned}\tag{3.4}$$

for $t \geqslant t_1 > t_0$. As a result, $z'(t)$ is nonincreasing on $[t_2, \infty)$, $t_2 > t_1$. Consider $z'(t) > 0$ for $t \geqslant t_2$. Then integrating the system (3.4) from $t_2$ to $t(> t_2)$, we get

$$[z'(s)]_{t_2}^{t} + \int_{t_2}^{t} q(s)f(y(s-\sigma))ds - \sum_{t_2 \leqslant \tau_k < t} \Delta z'(\tau_k) = 0,$$

that is,

$$\int_{t_2}^{t} q(s)f(y(s-\sigma))ds + \sum_{t_2 \leqslant \tau_k < t} p(\tau_k)f(y(\tau_k - \sigma)) = -[z'(s)]_{t_2}^{t}$$

implies that

$$\begin{aligned} h\left[\int_{t_2}^{t} q(s)ds + \sum_{t_2 \leqslant \tau_k < t} p(\tau_k)\right] &\leqslant -[z'(s)]_{t_2}^{t} \\ &< \infty, \text{ as } t \to \infty \end{aligned}$$

due to $(A_2)$, a contradiction. Hence $z'(t) < 0$ for $t \geqslant t_2$. Consequently, $z(t)$ is nonincreasing on $[t_2, \infty)$, $t_2 > t_1$. We claim that $y(t)$ is bounded for $t \geqslant t_2$. If not, there exists $\{\eta_n\}$ with $\lim_{n \to \infty} \eta_n = \infty$ such that $\lim_{n \to \infty} y(\eta_n) = \infty$ and $y(\eta_n) = \max_{t_2 \leqslant s \leqslant \eta_n} y(s)$. Consequently,

$$\begin{aligned} z(\eta_n) &= y(\eta_n) - r(\eta_n)g(y(\eta_n - \tau)) \\ &\geqslant y(\eta_n) - r(\eta_n)y(\eta_n - \tau) \\ &\geqslant (1 - r(\eta_n))y(\eta_n) \\ &\to +\infty \text{ as } n \to \infty \end{aligned}$$

implies that $z(t)$ is nondecreasing, a contradiction. So, our claim holds and $z(t)$ is bounded ultimately. Integrating (3.1) from $t_2$ to $t(> t_2)$, we obtain

$$\int_{t_2}^{t} q(s)f(y(s-\sigma))ds + \sum_{t_2 \leqslant \tau_k < \infty} p(\tau_k)f(y(\tau_k - \sigma)) \leqslant -z'(t).$$

Again integrating the last inequality from $t$ to $+\infty$, we get

$$\int_{t}^{\infty} \left[\int_{t_2}^{\theta} q(s)f(y(s-\sigma))ds + \sum_{t_2 \leqslant \tau_k < \infty} p(\tau_k)f(y(\tau_k - \sigma))\right] d\theta < \infty.$$

We assert that $\liminf_{t \to \infty} y(t) = 0$. If not, let $\liminf_{t \to \infty} y(t) > 0$. Then there exists a constant $\beta > 0$ such that $y(t) \geqslant \beta > 0$ for $t \geqslant t_3$. Therefore,

$$\begin{aligned} &\int_{t}^{\infty} \left[\int_{t_2}^{\theta} q(s)f(y(s-\sigma))ds + \sum_{t_2 \leqslant \tau_k < \infty} p(\tau_k)f(y(\tau_k - \sigma))\right] d\theta \\ &\geqslant f(\beta) \int_{t}^{\infty} \left[\int_{t_2}^{\theta} q(s)ds + \sum_{t_2 \leqslant \tau_k < \infty} p(\tau_k)\right] d\theta \to \infty, \text{ as } t \to \infty \end{aligned}$$

due to $(A_4)$, a contradiction. So, $\liminf_{t\to\infty} y(t) = 0$. By Lemma 1.5.2 [8], it follows that $\lim_{t\to\infty} z(t) = 0$. Let $\varepsilon \in (0, 1 - r_0)$ be given. Then there exists $t_3 > t_2$ such that $r(t) \leqslant r_0 + \varepsilon < 1$, for $t \geqslant t_3$. Now,

$$\begin{aligned}
0 &= \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} z(t) \\
&\geqslant \limsup_{t\to\infty} (y(t) - (r_0+\varepsilon)\, y(t-\tau)) \\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} (-(r_0+\varepsilon)\, y(t-\tau)) \\
&= (1 - r_0 - \varepsilon) \limsup_{t\to\infty} y(t)
\end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$ for $t \neq \tau_k$, $k \in \mathbb{N}$. As $\{y(\tau_k - 0)\}_1^\infty$ and $\{y(\tau_k + 0)\}_1^\infty$ are the sequences of real values, and because of the continuity of $y$, it follows that $\lim_{k\to\infty} y(\tau_k - 0) = 0 = \lim_{k\to\infty} y(\tau_k + 0)$ due to $\liminf_{t\to\infty} y(t) = 0 = \limsup_{t\to\infty} y(t)$. Hence, for the system (3.1), $\lim_{t\to\infty} y(t) = 0 = \lim_{k\to\infty} y(\tau_k)$. Let's set

$$Q(t) = q(t)\frac{f(y(t-\sigma))}{y(t-\sigma)}, \quad P(t) = p(t)\frac{f(y(t-\sigma))}{y(t-\sigma)}, \quad R(t) = r(t)\frac{g(y(t-\tau))}{y(t-\tau)}$$

for $t > t_4 \geqslant t_3$. Then it is easy to see that $\lim_{t\to\infty} Q(t) = q_0$, $\lim_{t\to\infty} R(t) = r_0$ and $\liminf_{t\to\infty} P(t) = \liminf_{t\to\infty} p(t) \lim_{t\to\infty} \frac{f(y(t-\sigma))}{y(t-\sigma)} = p_0$. Ultimately, the system (3.1) becomes

$$\begin{aligned}
&(y(t) - R(t)y(t-\tau))'' + Q(t)y(t-\sigma) = 0, \;\; t \neq \tau_k, \;\; k \in \mathbb{N} \\
&\Delta(y(\tau_k) - R(\tau_k)y(\tau_k - \tau))' + P(\tau_k)y(\tau_k - \sigma) = 0, \;\; k \in \mathbb{N}
\end{aligned} \tag{3.5}$$

for $t > t_5 \geqslant t_4$. Let $0 < \varepsilon_1 < q_0$ be given such that $Q(t) \geqslant q_0 - \varepsilon_1$ for $t \geqslant t_5$. Choose $Z(t) = y(t) - R(t)y(t-\tau)$. Then integrating (3.5) from $t_5$ to $t (> t_5)$, we get

$$\int_{t_5}^{t} Q(s)y(s-\sigma)ds + \sum_{t_5 \leqslant \tau_k < \infty} P(\tau_k)y(\tau_k - \sigma) \leqslant -Z'(t).$$

Again integrating the last inequality from $t$ to $+\infty$, we obtain

$$\int_t^\infty \left[\int_{t_5}^{\theta} Q(s)y(s-\sigma)ds + \sum_{t_5 \leqslant \tau_k < \infty} P(\tau_k)y(\tau_k - \sigma)\right] d\theta \leqslant -\lim_{\theta\to\infty} Z(\theta) + Z(t) = Z(t)$$

due to $\lim_{t\to\infty} Z(t) = 0$. Consequently,

$$\begin{aligned}
y(t) &= R(t)y(t-\tau) + \int_t^\infty \left[\int_{t_5}^{\theta} Q(s)y(s-\sigma)ds + \sum_{t_5 \leqslant \tau_k < \infty} P(\tau_k)y(\tau_k - \sigma)\right] d\theta \\
&\geqslant (r_0 - \varepsilon)y(t-\tau) + \int_t^\infty \left[(q_0 - \varepsilon_1)\int_{t_5}^{\theta} y(s-\sigma)ds + p_0 \sum_{t_5 \leqslant \tau_k < \infty} y(\tau_k - \sigma)\right] d\theta
\end{aligned}$$

for any large $t$. Let $Y = BC([t^*,\infty),\mathbb{R})$ be the space of all real valued bounded continuous functions defined on $\mathbb{R}$ such that $Y$ is a Banach space with respect to the sup norm defined by

$$\|x\| = \sup_{t \geqslant t^*} |x(t)|.$$

Let

$$S = \{x \in Y : 0 \leqslant x(t) \leqslant 1,\ t \geqslant t^*\}.$$

Clearly, $S$ is a closed and convex subspace of $Y$. For $\rho = \max\{\sigma,\tau\}$ and $y \in S$, we define

$$\Phi x(t) = \begin{cases} \Phi x(t^* + \rho), & t \in [t^*, t^* + \rho] \\ \frac{\alpha}{y(t)}[(r_0 - \varepsilon)x(t-\tau)y(t-\tau) + \int_t^\infty [(q_0 - \varepsilon_1)\int_{t_5}^\theta x(s-\tau)y(s-\sigma)ds \\ +p_0 \sum\limits_{t_5 \leqslant \tau_k < \infty} x(\tau_k - \tau)y(\tau_k - \sigma)]d\theta], & t > t^* + \rho, \end{cases}$$

where $\alpha < 1$. Clearly, $\Phi x(t) \geqslant 0$ for $t \geqslant t^*$ and

$$\begin{aligned} \Phi x(t) &\leqslant \frac{\alpha}{y(t)}\Big\{(r_0 - \varepsilon)y(t-\tau) + \int_t^\infty \Big[(q_0 - \varepsilon_1)\int_{t_5}^\theta y(s-\sigma)ds + p_0 \sum_{t_5 \leqslant \tau_k < \infty} y(\tau_k - \sigma)\Big]d\theta\Big\}, \\ &\leqslant \alpha < 1 \end{aligned}$$

implies that $\Phi x \in S$ and $\Phi : S \longrightarrow S$. For $x_1, x_2 \in S$,

$$\begin{aligned} |\Phi x_1(t) - \Phi x_2(t)| \leqslant \frac{\alpha}{y(t)}\{&(r_0 - \varepsilon)y(t-\tau)|x_1(t-\tau) - x_2(t-\tau)| \\ &+ \int_t^\infty [(q_0 - \varepsilon_1)\int_{t_5}^\theta y(s-\sigma)|x_1(s-\sigma) - x_2(s-\sigma)|ds \\ &+ p_0 \sum_{t_5 \leqslant \tau_k < \infty} y(\tau_k - \sigma)|x_1(\tau_k - \sigma) - x_2(\tau_k - \sigma)|]d\theta\}, \end{aligned}$$

that is,

$$\begin{aligned} |\Phi x_1(t) - \Phi x_2(t)| &\leqslant \frac{\alpha\|x_1 - x_2\|}{y(t)}\{(r_0 - \varepsilon)y(t-\tau) + \int_t^\infty [(q_0 - \varepsilon_1)\int_{t_5}^\theta y(s-\sigma)ds \\ &\quad + p_0 \sum_{t \leqslant \tau_k < \infty} y(\tau_k - \sigma)]d\theta\} \\ &\leqslant \alpha\|x_1 - x_2\| \end{aligned}$$

implies that $\Phi$ is contraction. By the Banach's fixed point theorem, $\Phi$ has a unique fixed point in $[0,1]$. Hence,

$$x(t) = \begin{cases} \Phi x(t^* + \rho), & t \in [t^*, t^* + \rho] \\ \frac{\alpha}{y(t)}[(r_0 - \varepsilon)x(t-\tau)y(t-\tau) + \int_t^\infty [(q_0 - \varepsilon_1)\int_{t_5}^\theta x(s-\tau)y(s-\sigma)ds \\ +p_0 \sum\limits_{t_5 \leqslant \tau_k < \infty} x(\tau_k - \tau)y(\tau_k - \sigma)]d\theta], & t > t^* + \rho. \end{cases}$$

Setting $w(t) = x(t)y(t)$ for $t \geqslant t^* + \rho$, we obtain

$$\begin{aligned}&(w(t) - \alpha(r_0 - \varepsilon)w(t - \tau))'' + \alpha(q_0 - \varepsilon_1)w(t - \sigma) = 0, \ \ t \neq \tau_k, \ \ k \in \mathbb{N}\\ &\Delta(w(\tau_k) - \alpha(r_0 - \varepsilon)w(\tau_k - \tau))' + p_0 \ \ \alpha \, w(\tau_k - \sigma)) = 0, \ \ k \in \mathbb{N},\end{aligned} \tag{3.6}$$

that is, $w(t)$ is a positive solution of (3.6) whose characteristic equation is given by

$$\lambda^2 \left[1 - \frac{p_0\alpha}{\alpha(q_0-\varepsilon_1)}\lambda\right]^{n_1} - \alpha(r_0-\varepsilon)\lambda^2 e^{\lambda\tau}\left[1 - \frac{p_0\alpha}{\alpha(q_0-\varepsilon_1)}\lambda\right]^{n_1-n_2} + \alpha(q_0 - \varepsilon_1)e^{\lambda\sigma} = 0.$$

From Theorem 2.2, it follows that $w(t)$ is the positive solution of (3.6) if and only if

$$\lambda < \frac{\alpha(q_0 - \varepsilon_1)}{p_0\alpha} = \frac{q_0 - \varepsilon_1}{p_0} < \frac{q_0}{p_0}$$

which in turn implies that (3.3) has a real root in $\left[-\frac{q_0}{p_0}, \frac{q_0}{p_0}\right]$ due to Theorem 2.11, a contradiction. This completes the proof of the theorem. □

EXAMPLE 3.2. Consider the system of equations

$$\begin{aligned}&(y(t) - r(t)g(y(t-2)))'' + q(t)f(y(t-1)) = 0, \ \ t \neq \tau_k, \ \ t > 2, \ \ k \in \mathbb{N}\\ &\Delta(y(\tau_k) - r(\tau_k)g(y(\tau_k - 2)))' + p(\tau_k)f(y(\tau_k - 1)) = 0, \ \ k \in \mathbb{N},\end{aligned} \tag{3.7}$$

where $r(t) = 0.2325441579 + 2e^{-t}$, $q(t) = 1 + e^{-t}$, $p(t) = 2(2 + \cos t)$, $\tau_k = k + 2$, $k \in \mathbb{N}$, and $g(u) = (3 - 2e^{-|u|})u = f(u)$. The limiting equation for (3.7) is given by

$$\begin{aligned}&(x(t) - r_0 x(t-2))'' + q_0 x(t-1) = 0, \ \ t \neq \tau_k, \ \ k \in \mathbb{N}\\ &\Delta(x(\tau_k) - r_0 x(\tau_k - 2))' + p_0 x(\tau_k - 1) = 0, \ \ k \in \mathbb{N},\end{aligned} \tag{3.8}$$

where $r_0 = 0.2325441579$, $q_0 = 1$ and $p_0 = 2$. If we choose $n_1 = 1$ and $n_2 = 3$, then from the characteristic equation of (3.8), it follows that $A = -1$, $\lambda = 1$ and

$$x(t) = e^{-t}(-1)^{i(2,t)}$$

is an oscillatory solution of (3.7). Hence by Theorem 3.1, (3.7) admits an oscillatory solution.

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