Differential
Equations
& Applications
Volume 9, Number 1 (2017), 99–104
doi:10.7153/dea-09-08

MINIMIZATION PRINCIPLE IN ORDERED BANACH SPACES AND APPLICATION VIA EKELAND'S VARIATIONAL PRINCIPLE

AMINA BOUCENNA, MABROUK BRIKI, TOUFIK MOUSSAOUI
AND DONAL O'REGAN

(*Communicated by Claudianor O. Alves*)

Abstract. In this paper we establish a minimization principle in an ordered Banach space (in particular in a Riesz-Banach space). As an application we discuss the existence of a positive solution for a boundary value problem on the half-line even when the nonlinear term is sign-changing.

1. Introduction

In [4] the author established a mountain pass theorem in an ordered Banach space (in particular in a Riesz-Banach space). In this paper we present a version of a minimization principle in an ordered Banach space (in particular in a Riesz-Banach space) using a simple argument based on Ekeland's variational principle. As an application we establish the existence of a positive solution for a boundary value problem on the half-line even when the nonlinear term is sign-changing.

DEFINITION 1.1. Let $(E,\|.\|)$ be a real Banach space. Now E is called an ordered Banach space if the following conditions hold:

(1) $(E,\leqslant)$ is an ordered set.

(2) Given $u,v,w\in E$, if $u\leqslant v$, then $u+w\leqslant v+w$. If $u\leqslant v$, then $\lambda u\leqslant \lambda v$ for any $\lambda\in[0,+\infty)$.

(3) $E^{+}:=\{u\in E: 0\leqslant u\}$ is a closed subset of E.

DEFINITION 1.2. [6]

1) We say that a Banach space E is ordered by a cone K, that is $u\leqslant v$ if and only if $v-u\in K$.

2) An ordered Banach space E is called a Riesz-Banach space if $u\vee v:=\sup\{u,v\}$, $u\wedge v:=\inf\{u,v\}$ exist for any $u,v\in E$.

For a Riesz-Banach space E, we define $|u|:=u\vee(-u)$, $u^{+}:=u\vee 0$, $u^{-}:=(-u)\vee 0$.

Mathematics subject classification (2010): 35B38, 47L07.
Keywords and phrases: Critical point, Ekeland's variational principal, cone, Riesz-Banach space, minimization principle.

Paper DEA-09-08

REMARK 1.1. [4]
(1) The Lebesgue space $L^p(\Omega)$ and the Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are ordered Banach spaces, where we define the order $u \leqslant v$ if $u(x) \leqslant v(x)$ a.e. $x \in \Omega$. Note $L^p(\Omega)$ and the first order Sobolev spaces $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ are Riesz-Banach spaces.
(2) If $u \in W^{m,p}(\Omega)$, then $|u| \in W^{m,p}(\Omega)$. Moreover we have

$$\nabla|u(x)| = \begin{cases} \nabla u(x), & \text{if } \ u(x) > 0 \\ 0, & \text{if } \ u(x) = 0 \\ -\nabla u(x), & \text{if } \ u(x) < 0, \end{cases}$$

$$|\nabla|u||_p = \|u\|_p \quad \text{for} \quad u \in W^{1,p}(\Omega),$$

where $|u|_p$ denotes the $L^p(\Omega)-$ norm.

DEFINITION 1.3. Let X be a real Banach space, $J \in C^1(X,\mathbb{R})$. The functional J is said to satisfy the Palais-Smale condition ((PS) for short) if any sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that

$$(J(u_n))_{n\in\mathbb{N}} \text{ is bounded and } J'(u_n) \to 0, \quad \text{as} \quad n \to \infty \tag{1.1}$$

possesses a convergent subsequence.

LEMMA 1.1. (Ekeland's variational principle) [5] *Let (E,d) be a complete metric space, and let $J : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional, bounded from below, and not identically equal to $+\infty$ $(J \not\equiv +\infty)$. Let $\varepsilon > 0$ and $u_0 \in E$ such that*

$$J(u_0) \leqslant \inf_{u\in E} J(u) + \varepsilon.$$

Then, there exists $u_\varepsilon \in E$ such that

(1) $J(u_\varepsilon) \leqslant J(u_0)$,

(2) $d(u_\varepsilon, u_0) \leqslant 1$,

(3) $J(u_\varepsilon) < J(v) + \varepsilon d(v, u_\varepsilon)$ for all $v \in E$ such that $v \neq u_\varepsilon$.

COROLLARY 1.1. [5] *Let E be a Banach space and $J : E \longrightarrow \mathbb{R}$, a C^1- functional that is bounded from below and satisfies the (PS) condition. Then there exists a critical point $u \in E$ of J.*

2. Main result

Our goal in this section is to prove a version of Corollary 1.1 in Riesz-Banach spaces.

THEOREM 2.1. *Let E be a Riesz- Banach space ordered by a cone K and let the functional $J \in C^1(E,\mathbb{R})$ be bounded from below, and satisfy the (PS) condition. Suppose that*

$$J(|u|) \leqslant J(u), \ \ \forall u \in E.$$

Then J admits a critical point u in K.

Proof. For $\varepsilon = \frac{1}{n}$, let $u_0 \in E$ be such that $J(u_0) \leqslant \inf_E J(u) + \frac{1}{n}$. From Ekeland's variational principle, there exists $(u_n) \subset E$, such that

$$J(u_n) < J(v) + \frac{1}{n}\|u_n - v\| \text{ for all } v \in E \text{ such that } v \neq u_n. \tag{2.1}$$

Let $v = u_n + th$, $t > 0$, $h \neq 0$. Then by a standard technique, one has $\lim_{n \longrightarrow +\infty} J'(u_n) = 0$. Now

$$\inf_{u \in E} J(u) \leqslant J(u_n) \leqslant J(u_0) \leqslant \inf_E J(u) + \frac{1}{n},$$

so (u_n) is a Palais-Smale sequence, and since J satisfies the (PS) condition, then there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that $u_{n_k} \longrightarrow w$ and $J(w) = \inf_{u \in E} J(u)$, $J'(w) = 0$. Since $J(|w|) \leqslant J(w)$, we have $J(|w|) = \inf_{u \in E} J(u)$ and because $J \in C^1(E, \mathbb{R})$, then $|w| \in K$ is a critical point of J. $\square$

As an application of the above result, consider the problem

$$\begin{cases} -(p(t)u'(t))' = f(t, u(t)), & \text{a.e. } t \in [0, +\infty), \\ u(0) = u(+\infty) = 0, \end{cases} \tag{2.2}$$

where $f : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, and may change sign, $p : [0, +\infty) \longrightarrow (0, +\infty)$ satisfies $\frac{1}{p} \in L^1[0, +\infty)$, and

$$\int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

Examples of p are the exponential function or

$$p(t) = \begin{cases} \sqrt{t}, & \text{if } t \in [0, 1], \\ \frac{1}{2}t(1 + t^2), & \text{if } t \geqslant 1. \end{cases}$$

Define the space

$$H^1_{0,p}(0, +\infty) = \{u \in AC([0, +\infty), \mathbb{R}) \mid u(0) = u(+\infty) = 0, \sqrt{p}u' \in L^2[0, +\infty)\}$$

and the cone

$$K = \{u \in H^1_{0,p}(0, +\infty),\ 0 \leqslant u\}.$$

LEMMA 2.1. [3], [1] *$H^1_{0,p}(0, +\infty)$ is embedded in $L^2(0, +\infty)$.*

Now $H^1_{0,p}(0, +\infty)$ is a Hilbert space equipped with the norm

$$\|u\|_p^2 = \int_0^{+\infty} p(t)u'^2(t)dt + \int_0^{+\infty} u^2(t)dt,$$

associated with the scalar product

$$(u, v) = \int_0^{+\infty} p(t)u'(t)v'(t)dt + \int_0^{+\infty} u(t)v(t)dt.$$

LEMMA 2.2. [3], [1] *On $H^1_{0,p}(0, +\infty)$, the quantity $\|u\|^2 = \int_0^{+\infty} p(t)u'^2(t)dt$ is a norm which is equivalent to the $H^1_{0,p}(0, +\infty)-$norm.*

LEMMA 2.3. [3], [1] *$(H^1_{0,p}(0,+\infty), \|\cdot\|)$ is embedded in $(C_l[0,+\infty), \|u\|_\infty)$, where $C_l[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} u(t)$ exists$\}$ and $\|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|$ with $d = \sqrt{\|1/p\|_{L^1}}$ the constant of the embedding.*

COROLLARY 2.1. [3], [1] *$H^1_{0,p}(0,+\infty)$ is embedded continuously in $C_l[0,+\infty)$ and in $L^2(0,+\infty)$.*

LEMMA 2.4. [3], [1] *The embedding*

$$H^1_{0,p}(0,+\infty) \hookrightarrow C_l[0,+\infty)$$

is compact.

2.1. Weak solutions

Take $v \in H^1_{0,p}(0,+\infty)$, and multiply the equation in (2.2) by v and integrate between 0 and $+\infty$, to obtain

$$-\int_0^{+\infty} (p(t)u'(t))'v(t)dt = \int_0^{+\infty} f(t,u(t))v(t)dt.$$

Hence

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt = \int_0^{+\infty} f(t,u(t))v(t)dt.$$

This leads to the natural concept of a weak solution for (2.2).

DEFINITION 2.1. We say that a function $u \in H^1_{0,p}(0,+\infty)$ is a weak solution of (2.2) if

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt - \int_0^{+\infty} f(t,u(t))v(t)dt = 0,$$

for all $v \in H^1_{0,p}(0,+\infty)$.

To study (2.2), consider the functional $J : H^1_{0,p}(0,+\infty) \longrightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt,$$

where

$$F(t,u) = \int_0^u f(t,s)ds.$$

Let the operator $A : H^1_{0,p} \longrightarrow H^1_{0,p}$ be defined by

$$Au(t) = \int_0^{+\infty} G(t,s)f(s,u(s))ds$$

with the Green's function

$$G(t,s) = \frac{1}{\|\frac{1}{p}\|_{L^1}} \begin{cases} \varphi_1(t)\varphi_2(s), & t \leqslant s, \\ \varphi_1(s)\varphi_2(t), & s \leqslant t, \end{cases}$$

and the fundamental system of solutions $\varphi_1(t) = \int_0^t \frac{ds}{p(s)}$ and $\varphi_2(t) = \int_t^{+\infty} \frac{ds}{p(s)}$.

THEOREM 2.2. *Suppose the following condition holds:*

(f_1) f is an odd function in u and there exist a constant $\mu \in [0,1)$, and positive functions $a_1, b_1 \in L^1[0,+\infty)$ such that

$$|f(t,u)| \leqslant a_1(t)|u|^\mu + b_1(t), \quad \text{for a.e.} \quad t \in [0,+\infty) \quad \text{and all} \quad u \in \mathbb{R}.$$

Then (2.2) has at least one weak solution in K.

LEMMA 2.5. [1] *Under assumption (f_1), we have*

(1) A is well defined,

(2) A is compact.

Proof of Theorem 2.2. We will apply Theorem 2.1. First we note that J is well defined. In fact, given $u \in H^1_{0,p}(0,+\infty)$, then (f_1) guarantees that

$$|F(t,u(t))| \leqslant \frac{a_1(t)}{\mu+1}|u(t)|^{\mu+1} + b_1(t)|u(t)|.$$

Hence using Lemma 2.3 we have

$$\begin{aligned} \left|\int_0^{+\infty} F(t,u(t))dt\right| &\leqslant \tfrac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}\int_0^{+\infty} a_1(t)dt + d\|u\|\int_0^{+\infty} b_1(t)dt \\ &\leqslant \tfrac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}|a_1|_1 + d\|u\||b_1|_1, \end{aligned}$$

so

$$|J(u)| \leqslant \tfrac{1}{2}\|u\|^2 + \tfrac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}|a_1|_1 + d\|u\||b_1|_1.$$

Now we show J is bounded from below. To see this note (f_1) and Lemma 2.3 guarantee that

$$J(u) \geqslant \tfrac{1}{2}\|u\|^2 - \tfrac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}|a_1|_1 - d\|u\||b_1|_1. \tag{2.3}$$

Since $\mu < 1$, (2.3) implies

$$\lim_{\|u\| \longrightarrow +\infty} J(u) = +\infty.$$

Next from (f_1), J is continuously differentiable and satisfies

$$(J'(u),v) = \int_0^{+\infty} p(t)u'(t)v'(t)dt - \int_0^{+\infty} f(t,u(t))v(t)dt$$

for all $u,v \in H^1_{0,p}$ and

$$J' = I - A.$$

Finally J satisfies the (PS) condition. Indeed, suppose that $(u_n) \subset H^1_{0,p}(0,+\infty)$ and there exists $M > 0$ such that $|J(u_n)| \leqslant M$ and $J'(u_n) = u_n - Au_n \longrightarrow 0$ on $H^1_{0,p}(0,+\infty)$ when $n \longrightarrow +\infty$. From the above ($J$ is bounded from below) we see that (u_n) is bounded in $H^1_{0,p}(0,+\infty)$. From the compactness of A there is a subsequence (Au_{n_k}) such that $Au_{n_k} \longrightarrow w$. Then

$$\|u_{n_k} - w\| \leqslant \|u_{n_k} - Au_{n_k}\| + \|Au_{n_k} - w\|,$$

and since $u_{n_k} - Au_{n_k} \longrightarrow 0$ in $H^1_{0,p}(0,+\infty)$, when $n \longrightarrow +\infty$, we have that (u_n) has a convergent subsequence (u_{n_k}) with $u_{n_k} \longrightarrow w$. Now $J(|u|) = J(u)$, $\forall u \in H^1_{0,p}(0,+\infty)$ since f is odd and now apply Theorem 2.1. □

REMARK 2.1. An example of f is the odd function

$$f(t,u) = a(t)u^{\frac{1}{3}} - b(t)u^{\frac{1}{5}},$$

with $a,b \in L^1(0,+\infty)$.

REFERENCES

[1] K. AIT-MAHIOUT, S. DJEBALI AND T. MOUSSAOUI, *Multiple solutions for an impulsive boundary value problem on the half line via Morse theory*, to appear in Topological Methods in Nonlinear Analysis, DOI: 10.12775/TMNA.2016.003.

[2] C. D. ALIPRANTIS AND O. BURKINSHAW, *Positive operators*, Academic Press, New York, 1985.

[3] M. BRIKI, S. DJEBALI AND T. MOUSSAOUI, *Solvability of an impulsive boundary value problems on the half-line via critical point theory*, accepted in Bullet. Iranian Math. Soc.

[4] R. KAJIKIYA, *Mountain pass theorem in ordered Banach spaces and its applications to semilinear elliptic equations*, Nonlinear. Differ. Appl. **19** (2012), 159–175.

[5] O. KAVIAN, *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, Springer-Verlag, 1993.

[6] W. A. J. LUXEMBURG AND A. C. ZAANEN, *Riesz Spaces*, vol. 1. North-Holland, London, 1971.

(Received June 13, 2016)

Amina Boucenna
Laboratory of Fixed Point Theory and Applications
Department of Mathematics, E. N. S. Kouba
Algiers, Algeria
e-mail: boucenna-amina@hotmail.fr

Mabrouk Briki
Laboratory of Fixed Point Theory and Applications
Department of Mathematics, E. N. S. Kouba
Algiers, Algeria
e-mail: mabroukbriki@yahoo.fr

Toufik Moussaoui
Laboratory of Fixed Point Theory and Applications
Department of Mathematics, E. N. S. Kouba
Algiers, Algeria
e-mail: moussaoui@ens-kouba.dz

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway, Ireland
e-mail: donal.oregan@nuigalway.ie