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ON NONLINEAR FRACTIONAL–ORDER BOUNDARY VALUE PROBLEMS WITH NONLOCAL MULTI–POINT CONDITIONS INVOLVING LIOUVILLE–CAPUTO DERIVATIVE

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Abstract. In this paper, we study some new nonlinear boundary value problems of Liouville-Caputo type fractional differential equations supplemented with nonlocal multi-point conditions involving lower order fractional derivative. We make use of some well known tools of the fixed point theory to establish the existence of solutions for problems at hand. For illustration of the obtained results, several examples are discussed.

1. Introduction

We introduce a new class of boundary value problems of Liouville-Caputo type fractional differential equations supplemented with nonlocal multi-point boundary conditions involving lower-order fractional derivative. Instead of writing the so-called "Caputo" derivative, we will call it "Liouville-Caputo" derivative as it was introduced by Liouville many decades ago. As a first problem, for $1 < q \leqslant 2$ and $0 < \sigma < \zeta_1 < \beta_1 < \beta_2 < \ldots < \beta_{m-2} < \zeta_2 < 1$, we consider

$$^cD^q x(t) = f(t,x), \quad 1 < q \leqslant 2, \quad t \in [0,1], \tag{1.1}$$

$$x(0) = \delta x(\sigma), \quad a\, {}^cD^p x(\zeta_1) + b\ {}^cD^p x(\zeta_2) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i), \quad 0 < p < 1, \tag{1.2}$$

where $^cD^q$ denote Caputo derivative of order q and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and δ, a, b, $\alpha_i \in \mathbb{R}$. The multi-point boundary conditions in (1.2) implies that the linear combination of the values of the fractional derivative of the unknown function at nonlocal positions ζ_1 and ζ_2 is equal to the linear combination of the values of the unknown function at β_i, $i = 1,2,\ldots m-2$, while the value of the unknown function at the left end point $(t = 0)$ of the interval $[0,1]$ is proportional to its value at the nonlocal position σ.

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In the second problem, we discuss the existence of solutions of (1.1) subject to the boundary conditions:

$$x(0)=\delta_1 \int_0^{\sigma} x(s)ds, \;\; a\,{}^cD^p x(\zeta_1)+b\;{}^cD^p x(\zeta_2)=\sum_{i=1}^{m-2}\alpha_i x(\beta_i), \;\; 0<p<1. \tag{1.3}$$

In (1.3), the first condition can be interpreted as the value of the unknown function at $t=0$ is proportional to the continuous distribution of the unknown function over a strip of an arbitrary length σ.

Fractional differential equations are found to be of great value and interest in view of their extensive applications. As a matter of fact, fractional-order operators describe some interesting characteristics of the real world phenomena such as hereditary properties of the processes and materials involved in the phenomena, which could not be explored with the modeling techniques of traditional calculus. For examples and details, see [16, 17, 18, 19, 23, 24].

The boundary value problems of fractional differential equations supplemented with a variety of initial, boundary, nonlocal and integral conditions have been investigated by many researchers and the literature on the topic is now much enriched. Examples and details can be found in a series of articles [1, 2, 4, 5, 6, 8, 9, 12, 14, 21, 22, 25, 26, 27] and the references cited therein.

We emphasize that nonlocal conditions are important as they can describe some peculiarities of physical, chemical or other processes happening inside the domain [10], while the integral boundary conditions find useful applications in blood flow problem [3] and regularization of ill-posed parabolic backward problems in time partial differential equations, for example, mathematical models for bacterial self-regularization [11].

The paper is organized as follows. Section 2 contains some preliminary concepts of fractional calculus and an auxiliary lemma related to the linear variant of problem (1.1)–(1.2). In Section 3, we derive the existence and uniqueness results for the given problem via some standard tools of the fixed point theory. Examples are also included for illustration of the main results. The paper concludes with some interesting remarks.

2. Preliminaries

This section is devoted to some preliminary concepts of fractional calculus that we need in the forthcoming analysis [15, 28].

DEFINITION 1. The fractional integral of order r with the lower limit zero for a function $f:[0,\infty)\to R$ is defined as

$$I^r f(t)=\frac{1}{\Gamma(r)}\int_0^t \frac{f(s)}{(t-s)^{1-r}}ds, \quad t>0, \quad r>0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r)=\int_0^\infty t^{r-1}e^{-t}dt$.

DEFINITION 2. The Riemann-Liouville fractional derivative of order $r>0$, $n-1<r<n$, $n\in N$ for a function $f:[0,\infty)\to R$ is defined as

$$D^r_{0+}f(t)=\frac{1}{\Gamma(n-r)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-r-1}f(s)ds,$$

where the function $f:[0,\infty)\to R$ has absolutely continuous derivative up to order $(n-1)$.

DEFINITION 3. The Caputo derivative of order r for a function $f:[0,\infty)\to R$ can be written as

$$^cD^rf(t)=D^r_{0+}\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right),\quad t>0,\quad n-1<r<n.$$

REMARK 1. If $f(t)\in C^n[0,\infty)$, then

$$^cD^rf(t)=\frac{1}{\Gamma(n-r)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{r+1-n}}ds=I^{n-r}f^{(n)}(t),\ \ t>0,\ \ n-1<q<n.$$

Now we present an auxiliary lemma to define the solution for the problem (1.1)–(1.2).

LEMMA 1. *Let $y\in C[0,1]$. Then the problem consisting of linear fractional differential equation*

$$^cD^qx(t)=y(t),\ \ n-1<q\leqslant n,\ \ n\geqslant 2,\ \ t\in[0,1],\tag{2.1}$$

supplemented with boundary conditions (1.2) is equivalent to the fractional integral equation

$$\begin{aligned}x(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}y(s)ds+\frac{\delta}{1-\delta}\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}y(s)ds\\&+\Big[\frac{\delta\sigma}{A(1-\delta)}+\frac{t}{A}\Big]\Big[\Big(1-\delta\Big)\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}y(s)ds\\&-a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}y(s)ds-b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}y(s)ds\Big)\\&+\delta\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}y(s)ds\sum_{i=1}^{m-2}\alpha_i\Big],\end{aligned}\tag{2.2}$$

where

$$A=\Big[\Big(\frac{a\zeta_1^{1-p}+b\zeta_2^{1-p}}{\Gamma(2-p)}-\sum_{i=1}^{m-2}\alpha_i\beta_i\Big)\Big(1-\delta\Big)-\delta\sigma\sum_{i=1}^{m-2}\alpha_i\Big]\neq 0.\tag{2.3}$$

Proof. As argued in [15], the solution of fractional differential equation (2.1) can be written as

$$x(t)=\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)ds+c_0+c_1t, \tag{2.4}$$

where $c_0,\ c_1\in\mathbb{R}$ are arbitrary constants. Using the boundary conditions (1.2) in (2.4), we find that

$$\begin{aligned}c_0=&\frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}y(s)ds\\&+\Big[\frac{\delta\sigma}{A(1-\delta)}\Big]\Big[\Big(1-\delta\Big)\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}y(s)ds\\&-a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}y(s)ds-b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}y(s)ds\Big)\\&+\delta\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}y(s)ds\sum_{i=1}^{m-2}\alpha_i\Big]\end{aligned}$$

and

$$\begin{aligned}c_1=&\frac{1}{A}\Big[\Big(1-\delta\Big)\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}y(s)ds\\&-a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}y(s)ds-b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}y(s)ds\Big)\\&+\delta\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}y(s)ds\sum_{i=1}^{m-2}\alpha_i\Big].\end{aligned}$$

Substituting the values of c_0 and c_1 in (2.4) completes the solution (2.2). □

3. Existence results

In Lemma 1, we replace $y(t)$ by $f(t,x(t))$ and define an operator $\mathscr{S}:\mathscr{K}\longrightarrow\mathscr{K}$ associated with problem (1.1)–(1.2) as follows:

$$\begin{aligned}(\mathscr{S}x)(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\frac{\delta}{1-\delta}\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\\&+\Big[\frac{\delta\sigma}{A(1-\delta)}+\frac{t}{A}\Big]\Big[\Big(1-\delta\Big)\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\\&-a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s))ds-b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s))ds\Big)\\&+\delta\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\sum_{i=1}^{m-2}\alpha_i\Big],\ \ t\in[0,1]\end{aligned}\tag{3.1}$$

where $\mathscr{K} = C([0,1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm: $\|x\| = \sup\{|x(t)|,\ t \in [0,1]\}$. Observe that the problem (1.1)–(1.2) has solutions if and only if the operator $\mathscr{S}$ has fixed points.

In the sequel, we set

$$\begin{aligned}\vartheta = &\frac{1}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \Big[\frac{|\delta|\sigma}{|A(1-\delta)|} + \frac{1}{|A|}\Big] \\ &\times\Big[|1-\delta|\Big(\sum_{i=1}^{m-2}\frac{|\alpha_i|\beta_i^q}{\Gamma(q+1)} + \frac{|a|\zeta_1^{q-p}+|b|\zeta_2^{q-p}}{\Gamma(q-p+1)}\Big) + \frac{|\delta|\sigma^q}{\Gamma(q+1)}\sum_{i=1}^{m-2}|\alpha_i|\Big].\end{aligned} \tag{3.2}$$

Now we are in a position to present the main results of our paper. The first one dealing with the existence and uniqueness of solutions for problem (1.1)–(1.2) is based on Banach's contraction mapping principle.

THEOREM 1. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying the Lipschitz condition:*

(A_1) $|f(t,x) - f(t,y)| \leqslant \ell|x-y|,\ \ell > 0,\ \forall t \in [0,1],\ x,y \in \mathbb{R}.$

Then the problem (1.1)–(1.2) has a unique solution if $\vartheta\ell < 1$, where ϑ is given by (3.2).

Proof. In the first step, we show that the operator $\mathscr{S}$ defined by (3.1) satisfies the relation: $\mathscr{S}B_r \subset B_r$, where $B_r = \{x \in \mathscr{K} : \|x\| \leqslant r\}$, $r \geqslant \vartheta\varphi/(1-\vartheta\ell)$, $\sup_{t\in[0,1]}|f(t,0)| = \varphi$. For $x \in B_r$, $t \in [0,1]$, using the assumption (A_1), we get

$$\begin{aligned}|f(t,x(t))| &= |f(t,x(t)) - f(t,0) + f(t,0)| \\ &\leqslant |f(t,x(t)) - f(t,0)| + |f(t,0)| \leqslant \ell\|x\| + \varphi \leqslant \ell r + \varphi.\end{aligned} \tag{3.3}$$

In view of (3.2) and (3.3), we obtain

$$\begin{aligned}\|(\mathscr{S}x)\| \leqslant &\sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\ &+\Big[\frac{|\delta|\sigma}{|A(1-\delta)|} + \frac{t}{|A|}\Big]\Big[|1-\delta|\Big(\sum_{i=1}^{m-2}|\alpha_i|\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\ &+|a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}|f(s,x(s))|ds + |b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}|f(s,x(s))|ds\Big) \\ &+|\delta|\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\sum_{i=1}^{m-2}|\alpha_i|\Big]\Big\} \\ \leqslant &(\ell r+\varphi)\sup_{t\in[0,1]}\Big\{\frac{t^q}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \Big[\frac{|\delta|\sigma}{|A(1-\delta)|} + \frac{t}{|A|}\Big] \\ &\times\Big[|1-\delta|\Big(\sum_{i=1}^{m-2}\frac{|\alpha_i|\beta_i^q}{\Gamma(q+1)} + \frac{|a|\zeta_1^{q-p}+|b|\zeta_2^{q-p}}{\Gamma(q-p+1)}\Big) + \frac{|\delta|\sigma^q}{\Gamma(q+1)}\sum_{i=1}^{m-2}|\alpha_i|\Big]\Big\} \\ \leqslant &(\ell r+\varphi)\vartheta \leqslant r.\end{aligned}$$

This shows that $\mathscr{S}B_r \subset B_r$.

Again making use of the condition (A_1) and (3.2), we obtain

$$\begin{aligned}
\|(\mathscr{S}x)-(\mathscr{S}y)\| \leqslant \sup_{t\in[0,1]} \Big\{ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}ds \\
& + \Big[\frac{|\delta|\sigma}{|A(1-\delta)|}+\frac{t}{|A|}\Big]\Big[|1-\delta|\Big(\sum_{i=1}^{m-2}|\alpha_i|\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}ds \\
& + |a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}ds + |b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}ds\Big) \\
& + |\delta|\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}ds\sum_{i=1}^{m-2}|\alpha_i|\Big]\Big\}\ell\|x-y\| \\
\leqslant {} & \vartheta\ell\|x-y\|,
\end{aligned}$$

which shows that the operator $\mathscr{S}$ is a contraction according to the given condition $\vartheta\ell < 1$. Thus, by Banach's contraction mapping principle, there exists a unique fixed point for the operator $\mathscr{S}$ which corresponds to the unique solution for the problem (1.1)–(1.2). This completes the proof. $\square$

Our next existence result is based on Krasnoselskii's fixed point theorem [20].

THEOREM 2. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (A_1). In addition it is assumed that $|f(t,x)|\leqslant\mu(t)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$, and $\mu\in C([0,1],\mathbb{R}^+)$. Then the problem (1.1)–(1.2) has at least one solution on $[0,1]$ if $\ell(\vartheta-1/\Gamma(q+1))<1$, where ϑ is given by (3.2).*

Proof. Let us consider a set $\mathscr{B}_\nu=\{x\in\mathscr{K}:\|x\|\leqslant\nu\}$ with $\nu\geqslant\vartheta\|\mu\|$ ($\sup_{t\in[0,1]}|\mu(t)|=\|\mu\|$). In order to satisfy the hypothesis of Krasnoselskii's fixed point theorem, we define two operators $\mathscr{S}_1$ and $\mathscr{S}_2$ on $\mathscr{B}_\nu$ as

$$\begin{aligned}
(\mathscr{S}_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds, \\
(\mathscr{S}_2x)(t) &= \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\
&\quad + \Big[\frac{\delta\sigma}{A(1-\delta)}+\frac{t}{A}\Big]\Big[\Big(1-\delta\Big)\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\
&\quad - a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s))ds - b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s))ds\Big) \\
&\quad + \delta\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\sum_{i=1}^{m-2}\alpha_i\Big].
\end{aligned}$$

For $x,y\in\mathscr{B}_\nu$, it is easy to show that $\|(\mathscr{S}_1x)+(\mathscr{S}_2y)\|\leqslant\|\mu\|\vartheta\leqslant\nu$ (ϑ is given by (3.2)), which means that $\mathscr{S}_1x+\mathscr{S}_2y\in\mathscr{B}_\nu$.

Using (A_1) and (3.2), for $x, y \in \mathbb{R}$, $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\|(\mathscr{S}_2 x) - (\mathscr{S}_2 y)\| \\
\leqslant &\sup_{t\in[0,1]} \Big\{ \frac{|\delta|}{|1-\delta|} \int_0^{\sigma} \frac{(\sigma-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
&+ \Big[\frac{|\delta|\sigma}{|A(1-\delta)|} + \frac{t}{|A|} \Big] \Big[|1-\delta| \Big(\sum_{i=1}^{m-2} |\alpha_i| \int_0^{\beta_i} \frac{(\beta_i - s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
&+ |a| \int_0^{\zeta_1} \frac{(\zeta_1 - s)^{q-p-1}}{\Gamma(q-p)} |f(s,x(s)) - f(s,y(s))| ds \\
&+ |b| \int_0^{\zeta_2} \frac{(\zeta_2 - s)^{q-p-1}}{\Gamma(q-p)} |f(s,x(s)) - f(s,y(s))| ds \Big) \\
&+ |\delta| \int_0^{\sigma} \frac{(\sigma-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \sum_{i=1}^{m-2} |\alpha_i| \Big] \Big\} \\
&\leqslant \ell(\vartheta - 1/\Gamma(q+1)) \|x-y\|.
\end{aligned}
$$

This shows that $\mathscr{S}_2$ a contraction in view of the condition $\ell(\vartheta - 1/\Gamma(q+1)) < 1$.

Continuity of f implies that the operator $\mathscr{S}_1$ is continuous. Also, $\mathscr{S}_1$ is uniformly bounded on $\mathscr{B}_\nu$ as

$$
\|(\mathscr{S}_1 x)\| \leqslant \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \Big\} \leqslant \frac{\|\mu\|}{\Gamma(q+1)}.
$$

Moreover, with $\sup_{(t,x)\in[0,1]\times\mathscr{B}_\nu} |f(t,x)| = \overline{f} < \infty$ and $0 < t_1 < t_2 < 1$, we have

$$
\Big| (\mathscr{S}_1 x)(t_2) - (\mathscr{S}_1 x)(t_1) \Big| \leqslant \frac{\overline{f}}{\Gamma(q+1)} \Big(2|t_2 - t_1|^q + |t_2^q - t_1^q| \Big),
$$

which tends to zero independent of x as $(t_2 - t_1) \to 0$. This implies that $\mathscr{S}_1$ is relatively compact on $\mathscr{B}_\nu$. Hence by the $Arzel\acute{a} - Ascoli$ theorem, $\mathscr{S}_1$ is compact on $\mathscr{B}_\nu$. Thus the hypothesis of Krasonselskii's fixed theorem is satisfied and consequently the problem (1.1)–(1.2) has at least one solution on $[0,1]$. This completes the proof. □

Our next result relies on the following fixed point theorem [20].

THEOREM 3. *Let X be a Banach space. Assume that $T : X \longrightarrow X$ is a completely continuous operator and the set $V = \{u \in X | u = \varepsilon T u,\ 0 < \varepsilon < 1\}$ is bounded. Then T has a fixed point in X.*

THEOREM 4. *Assume that exists a positive constant L_1 such that $|f(t,x)| \leqslant L_1$ for all $t \in [0,1]$, $x \in \mathbb{R}$. Then there exists at least one solution for the problem (1.1)–(1.2) on $[0,1]$.*

Proof. In the first step, we show that the operator $\mathscr{S}$ is completely continuous. Clearly continuity of $\mathscr{S}$ follows from the continuity of f and it is easy to establish by the given assumption that $|(\mathscr{S}x)(t)| \leqslant L_1\vartheta = L_2$, where ϑ is given by (3.2). Let $0 < t_1 < t_2 < 1$, we get

$$\begin{aligned} &|(\mathscr{S}x)(t_2) - (\mathscr{S}x)(t_1)| \\ &\leqslant L_1\Big\{\frac{2|t_2-t_1|^q + |t_2^q - t_1^q|}{\Gamma(q+1)} + \frac{|t_2-t_1|}{|A|}\Big[|1-\delta|\Big(\sum_{i=1}^{m-2}\frac{|\alpha_i|\beta_i^q}{\Gamma(q+1)} \\ &+\frac{|a|\zeta_1^{q-p}+|b|\zeta_2^{q-p}}{\Gamma(q-p+1)}\Big) + \frac{|\delta|\sigma^q}{\Gamma(q+1)}\sum_{i=1}^{m-2}|\alpha_i|\Big]\Big\}. \end{aligned}$$

Clearly, the right-hand side tends to zero independently of $x \in B_\rho$ as $t_2 \longrightarrow t_1$. Thus, by the Arzelá theorem, the operator $\mathscr{S}$ is completely continuous.

Next, we consider the set $V = \{x \in \mathscr{K} : x = \varepsilon\mathscr{S}x,\ 0 < \varepsilon < 1\}$. To show that V is bounded, let $t \in [0,1]$. Then

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds + \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\ &+\Big[\frac{\delta\sigma}{A(1-\delta)} + \frac{t}{A}\Big]\Big[\Big(1-\delta\Big)\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\ &-a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s))ds - b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s))ds\Big) \\ &+\delta\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\sum_{i=1}^{m-2}\alpha_i\Big]. \end{aligned}$$

Then it is easy to show that $|x(t)| = \varepsilon|(\mathscr{S}x)(t)| \leqslant L_1\vartheta = L_2$. Hence, $\|x\| \leqslant L_2$, $\forall x \in V$, $t \in [0,1]$. So V is bounded. Thus, the conclusion of Theorem 3 applies and the problem (1.1)–(1.2) has at least one solution on $[0,1]$. This completes the proof. □

LEMMA 2. (Nonlinear alternative for single valued maps [13]) *Let E be a Banach space E_1 a closed, convex subset of E, V an open subset of E_1, and $0 \in V$. Suppose that $\mathscr{U} : \overline{V} \longrightarrow E_1$ is a continuous, compact (that is, $\mathscr{U}(\overline{V})$ is a relatively compact subset ofE_1) map. Then either*

(i) *$\mathscr{U}$ has a fixed point in $\overline{V}$, or*

(ii) *there is a $x \in \partial V$ (the boundary of V in E_1) and $\kappa \in (0,1)$ with $x = \kappa\ \mathscr{U}(x)$.*

THEOREM 5. *Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Further, it is assumed that*

(A_2) *there exist a function $p \in C([0,1],\mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $|f(t,x)| \leqslant p(t)\psi(\|x\|)$, $\forall (t,x) \in [0,1]\times\mathbb{R}$;*

(A_3) *there exists a constant $M > 0$ such that $M/\psi(M)\|p\|\vartheta > 1$, where ϑ is given by (3.2).*

Then the problem (1.1)–(1.2) *has at least one solution on* $[0,1]$.

Proof. Let us consider the operator $\mathscr{S}:\mathscr{K}\longrightarrow\mathscr{K}$ defined by (3.1) and show that $\mathscr{S}$ maps bounded sets into bounded sets in $\mathscr{K}$. For a given positive number ρ, let $B_\rho=\{x\in\mathscr{K}:\|x\|\leqslant\rho\}$ be a bounded set in $\mathscr{K}$. Then, for $x\in B_\rho$ together with (A_2), we obtain

$$\begin{aligned}
|(\mathscr{S}x)(t)| \leqslant{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)ds+\frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)ds\\
&+\Big[\frac{|\delta|\sigma}{|A(1-\delta)|}+\frac{t}{|A|}\Big]\Big[|1-\delta|\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)ds\\
&+|a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)}p(s)\psi(\|x\|)ds+|b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)}p(s)\psi(\|x\|)ds\Big)\\
&+|\delta|\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)ds\sum_{i=1}^{m-2}|\alpha_i|\Big]\\
\leqslant{}& \psi(\rho)\|p\|\vartheta,
\end{aligned}$$

where A is given by (2.3). As in the proof of the previous result, for $0<t_1<t_2<1$ and $x\in B_\rho$. we have that the operator $\mathscr{S}$ is completely continuous. Thus, it follows that $\mathscr{S}$ maps bounded sets into equicontinuous sets of $\mathscr{K}$.

Let x be a solution for the given problem. Then, for $\lambda\in(0,1)$, as before, we obtain

$$|x(t)|=\|\lambda(\mathscr{S}x)(t)\|\leqslant\psi(\|x\|)\|p\|\vartheta,$$

which, on taking the norm for $t\in[0,1]$, yields

$$\|x\|/\psi(\|x\|)\|p\|\vartheta\leqslant 1.$$

In view of (A_3), there exists M such that $\|x\|\neq M$. Let us choose $M_1=\{x\in\mathscr{K}:\|x\|<M+1\}$. Since the operator $\mathscr{S}:\overline{M}_1\to\mathscr{K}$ is continuous and completely continuous. From the choice of M_1, there is no $x\in\partial M_1$ such that $x=\lambda\mathscr{S}(x)$ for some $\lambda\in(0,1)$. Consequently, by Lemma 2, we deduce that the operator $\mathscr{S}$ has a fixed point $x\in\overline{M}_1$ which is a solution of the problem (1.1)–(1.2). This completes the proof. □

EXAMPLE 1. Consider a fractional boundary value problem given by

$$\begin{cases}
{}^cD^qx(t)=f(t,x),\;\;1<q\leqslant 2,\;\;t\in[0,1],\\
x(0)=\delta x(\sigma),\\
a\;{}^cD^px(\zeta_1)+b\;{}^cD^px(\zeta_2)=\sum_{i=1}^{m-2}\alpha_ix(\beta_i),\\
0<\sigma<\zeta_1<\beta_1<\beta_2<\ldots<\beta_{m-2}<\zeta_2<1,\;\;0<p<1,\;\;\alpha_i\in\mathbb{R}.
\end{cases}\tag{3.4}$$

Here, $\delta = 1/2,\ q = 7/4,\ a = 1,\ b = 2,\ m = 5,\ \sigma = 1/6,\ \zeta_1 = 1/3,\ \beta_1 = 1/2,$ $\beta_2 = 2/3,\ \beta_3 = 3/4,\ \zeta_2 = 4/5,\ \alpha_1 = \alpha_2 = \alpha_3 = 1,\ p = 1/2$, and $f(t,x) = \frac{\sin x}{\sqrt{t^2+225}} + (t+5)^{\frac{1}{2}}$. With the given data, $\ell = 1/15$, $|A| \simeq 0.126655$ and $\vartheta \simeq 12.193711$, where ϑ is given by(3.2). Obviously all the conditions of Theorem 1 are satisfied with $\ell\vartheta < 1$. Therefore, by the conclusion of Theorem 1, there exists a unique solution for the problem (3.4) on $[0,1]$.

EXAMPLE 2. Consider the problem (3.4) with

$$f(t,x) = \frac{1}{5t+6}\Big[\frac{1}{1+|x|} + \frac{\sin x}{3}\Big]. \tag{3.5}$$

Clearly $|f(t,x)| \leqslant p(t)\psi(\|x\|)$ with $p(t) = \frac{1}{5t+6}$, $\psi(\|x\|) = 1 + \frac{\|x\|}{3}$. By the assumption (A_3) of Theorem 5, we find that $M > 6.30026$. Thus, by Theorem 5, there exists at least one solution for the problem (3.4) with $f(t,x)$ given by (3.5).

REMARK 2. (Concerning problem (1.1)–(1.3)) As done for problem (1.1)–(1.2), we can find the operator $\mathscr{S}_1 : \mathscr{K} \to \mathscr{K}$ associated with problem (1.1)–(1.3), which is given by

$$\begin{aligned}
(\mathscr{S}_1 x)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{\delta_1}{1-\delta_1\sigma}\int_0^\sigma \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))duds \\
& + \Big[\frac{\delta_1\sigma^2}{2A_1(1-\delta_1\sigma)} + \frac{t}{A_1}\Big]\Big[\Big(1-\delta_1\sigma\Big)\Big(\sum_{i=1}^{m-2}\alpha_i \int_0^{\beta_i}\frac{(\beta_i-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
& - a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds - b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds\Big) \\
& + \delta_1 \int_0^\sigma \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))duds \sum_{i=1}^{m-2}\alpha_i\Big],
\end{aligned} \tag{3.6}$$

where

$$A_1 = \Big[\Big(\frac{a\zeta_1^{1-p} + b\zeta_2^{1-p}}{\Gamma(2-p)} - \sum_{i=1}^{m-2}\alpha_i\beta_i\Big)\Big(1-\delta_1\sigma\Big) - \frac{\delta_1\sigma^2}{2}\sum_{i=1}^{m-2}\alpha_i\Big] \neq 0. \tag{3.7}$$

Using the operator $\mathscr{S}_1$, we can obtain the existence results for problem (1.1)–(1.3) similar to the ones obtained for problem (1.1)–(1.2) in Section 3.

REMARK 3. (Special cases) We can record some special cases (of course new) of the results obtained in this paper. For instance, by taking $\delta = 0$ in (1.2), ours results correspond the ones for fractional differential equation (1.1) equipped with the boundary conditions of the form:

$$x(0) = 0,\ \ a\,{}^cD^p x(\zeta_1) + b\ {}^cD^p x(\zeta_2) = \sum_{i=1}^{m-2}\alpha_i x(\beta_i).$$

If we choose $a = 0$, $b = 1$ and $\zeta_2 \to 1$ in (1.2), then we obtain the results for fractional differential equation (1.1) subject to the boundary conditions:

$$x(0) = \delta x(\sigma), \ {}^cD^p x(1) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i).$$

In a similar manner, we can get the results for fractional differential equation (1.1) with the boundary data:

$$x(0) = \delta_1 \int_0^{\sigma} x(s)ds, \ {}^cD^p x(1) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i).$$

REMARK 4. (Some more problems) In (1.2), we replace the multi-point boundary condition by the following one:

$${}^cD^p x(1) = \sum_{i=1}^{m-2} \alpha_i \ {}^cD^p x(\beta_i). \tag{3.8}$$

In this case, the associated fixed point problem is

$$\mathscr{S}_2 x = x,$$

where

$$\begin{aligned}(\mathscr{S}_2 x)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{\delta}{1-\delta}\int_0^{\sigma} \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ & + \Big[\frac{\delta\sigma}{A_2(1-\delta)} + \frac{t}{A_2}\Big]\Big(\sum_{i=1}^{m-2}\alpha_i \int_0^{\beta_i} \frac{(\beta_i-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds \\ & - \int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds\Big),\end{aligned} \tag{3.9}$$

$$A_2 = \Big[\frac{1-\sum_{i=1}^{m-2}\alpha_i\beta_i^{1-p}}{\Gamma(2-p)}\Big] \neq 0. \tag{3.10}$$

Instead of (1.2), if we take the boundary conditions:

$$x(0) = 0, \ {}^cD^p x(\zeta_1) = \sum_{i=1}^{m-2} \alpha_i \ {}^cD^p x(\beta_i), \tag{3.11}$$

then the related fixed point problem is

$$\mathscr{S}_3 x = x,$$

where

$$\begin{aligned}(\mathscr{S}_3 x)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{t}{A_3}\Big(\sum_{i=1}^{m-2}\alpha_i \int_0^{\beta_i} \frac{(\beta_i-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds \\ & - \int_0^{\zeta_1} \frac{(\zeta_1-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds\Big),\end{aligned} \tag{3.12}$$

$$A_3 = \Big[\frac{\zeta_1^{1-p} - \sum_{i=1}^{m-2} \alpha_i \beta_i^{1-p}}{\Gamma(2-p)}\Big] \neq 0. \tag{3.13}$$

Next, replacing (1.2) with the following boundary conditions

$$x(0) = 0, \ \ {}^cD^p x(0) = \sum_{i=1}^{m-2} \alpha_i \ \ {}^cD^p x(\beta_i), \tag{3.14}$$

the associated fixed point problem is

$$\mathscr{S}_4 x = x,$$

where

$$(\mathscr{S}_4 x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \frac{t}{A_4}\Big(\sum_{i=1}^{m-2} \alpha_i \int_0^{\beta_i} \frac{(\beta_i - s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds\Big),$$

$$A_4 = \sum_{i=1}^{m-2} \frac{\alpha_i \beta_i^{1-p}}{\Gamma(2-p)} \neq 0. \tag{3.15}$$

We can obtain the existence and uniqueness results for all the problems introduced in this remark by following the methodology employed in Section 3.

4. Conclusions

We have studied the existence and uniqueness of solutions for nonlinear Liouville-Caputo type fractional differential equations equipped with nonlocal multi-point conditions involving lower order fractional derivatives. The uniqueness result is obtained by applying Banach's contraction mapping principle, while the existence results are established by means of Krasnoselskii's fixed point, Schaefer like fixed point theorem and Leray-Schauder type nonlinear alternative. Several variants of the given problem are also discussed. It is worth-mentioning that some existence results for Caputo type fractional differential equations equipped with nonlocal strip conditions were obtained in [7]. On the other hand, the present work deals with a variety of problems of Liouville-Caputo type fractional differential equations supplemented with nonlocal multi-point and integral boundary conditions.

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