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NONLINEAR BOUNDARY VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH CAUSAL OPERATORS

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Abstract. In this work, we investigate nonlinear boundary value problems for impulsive differential equations with causal operators. Our boundary condition is given by a nonlinear function, and more general than ones given before. To begin with, we prove a comparison theorem. Then by using this theorem, we show the existence of solutions for linear problems. Finally, by using the monotone iterative technique, we obtain the existence of extremal solutions for nonlinear boundary value problems with causal operators. An example satisfying the assumptions is presented.

1. Introduction

Impulsive differential equations are recognized as important models, which describe many evolution processes that abruptly change their state at a certain moment. This type of equations has been studied in depth by some authors in recent years [1, 2, 6, 10, 12, 15]. As an important branch, boundary value problems, especially, problems with nonlinear boundary conditions have drawn much attention. There are many ways to investigate this kind of problem. Among them, monotone iterative technique coupled with the method of upper and lower solutions is an effective method, readers can refer [8, 11, 13, 14, 19] for details. Recently, this method has been extended to boundary value problems with causal operators, see [3, 4, 5, 7, 9, 16, 17, 18, 20] and the references therein. For the case of differential equations, Jankowski [7] investigated nonlinear boundary value problems for first-order differential equations with causal operators by using the monotone iterative method. After it, Wang and Tian [16, 18] developed monotone iterative method, considered the generalized monotone iterative method for nonlinear boundary value problems and a class of integral boundary value problems, respectively, obtained the existence of extremal solutions for causal differential equations where the right-hand side is the sum of two monotone functions, one of which is monotone non-decreasing and the other is non-increasing. Moreover, for

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impulsive differential equations, Zhao et al. [20] studied integral boundary value problems for impulsive differential equations with causal operators. Motivated by the above excellent work, in this paper, we discuss the following impulsive differential equations with a causal operator:

$$\begin{cases} y'(t) = (Qy)(t), & t \neq t_k, t \in J, \\ \Delta y(t_k) = I_k(y(t_k)), & k = 1,2,\cdots,p, \\ B(y(0),y) = 0, \end{cases} \tag{1.1}$$

where $J=[0,T]$, $T>0$, $E=C(J,\mathbb{R})$, $Q \in C(E,E)$ is a causal operator, $0=t_0<t_1<\cdots<t_p<t_{p+1}=T$, $I_k \in C(\mathbb{R},\mathbb{R})$, $B \in C[\mathbb{R}\times\mathbb{R}^{T+1},\mathbb{R}]$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, $k=1,2,\cdots,p$.

The main interest of the paper lies in the fact that we consider nonlinear boundary conditions which, of course, includes the usual linear boundary conditions (such as initial and periodic) and other general conditions such as $B(y(0),y(T))=0$ and $y(0)=\int_0^T \mu(t,y(t))dt$, $(\mu \in C(J\times\mathbb{R},\mathbb{R}))$. Note that, the nonlinear boundary value problem (1.1) reduce to integral boundary value problems for $B(y(0),y) = y(0) - \lambda_1 y(\tau) - \lambda_2 \int_0^T \varphi(t,y(t))dt - c$ which has been studied in [20], and other general conditions such as $B(y(0),y(T))=0, I_k \equiv 0$ which has studied in [7]. Thus our boundary condition has a very general form.

The rest of the paper is organized as follows. In Section 2, a comparison principle is established. In Sections 3, after introducing the definition of upper and lower solutions, we obtain the existence of solutions for linear problem of (1.1) by applying Schauder′s fixed point theorem. Moreover, the existence of extremal solutions for (1.1) is established by utilizing the monotone iterative technique. An example is given to illustrate our results.

2. Preliminaries

In this section, we present a definition and a lemma which help to prove our main results.

Taking $J' = J\setminus\{t_1,t_2,\cdots,t_p\}$, let us introduce the space:

$$PC(J,\mathbb{R}) = \left\{ \begin{array}{l} y: J\to\mathbb{R}; y(t) \text{ is continuous everywhere except for some } t_k \\ \text{at which } y(t_k^-) \text{ and } y(t_k^+) \text{ exist, and } y(t_k^-) = y(t_k), k=1,2,\cdots,p \end{array} \right\},$$

$$PC'(J,\mathbb{R}) = \left\{ \begin{array}{l} y \in PC(J,\mathbb{R}); y' \text{ is continuous on } J', \text{ where } y'(0^+), \\ y'(T^-),\ y'(t_k^+) \text{ and } y'(t_k^-) \text{ exist}, k=1,2,\cdots,p \end{array} \right\}.$$

Put $E_0 = PC(J,\mathbb{R})$, $\Omega = PC'(J,\mathbb{R})$. E_0 and Ω are Banach spaces with the respective norms:

$$\|y\|_{E_0} = \sup_{t\in J}|y(t)|, \qquad \|y\|_\Omega = \|y\|_{E_0} + \|y'\|_{E_0}.$$

A function $y\in\Omega$ is called a solution of (1.1) if it satisfies (1.1).

DEFINITION 1. Let $Q \in C(E,E)$. We said that Q is a *causal map* if $u(s) = v(s)$, $t_0 \leqslant s \leqslant t \leqslant T$, where $u, v \in E$, then

$$(Qu)(s) = (Qv)(s), \quad t_0 \leqslant s \leqslant t.$$

LEMMA 1. *Let $m \in \Omega$ and*

$$\begin{cases} m'(t) \leqslant -Mm(t) - (\mathscr{L}m)(t), & t \in J', \\ \Delta m(t_k) \leqslant -L_k m(t_k), & k = 1,2,\cdots,p, \\ m(0) \leqslant 0, \end{cases}$$

where $0 \leqslant L_k < 1$, $k = 1,2,\cdots,p$, $M \geqslant 0$ and $\mathscr{L} \in C(E,E)$ is a positive linear operator.

In addition, we assume that

$$\int_0^T e^{MT}(\mathscr{L}e^{-M})(t)dt + \prod_{k=1}^{p} L_k \leqslant 1. \tag{2.1}$$

Then $m(t) \leqslant 0$ for $t \in J$.

Proof. Let $v(t) = e^{Mt}m(t)$, $t \in J$. We have

$$\begin{cases} v'(t) \leqslant -e^{Mt}(\mathscr{L}e^{-M}v)(t), & t \in J', \\ \Delta v(t_k) \leqslant -L_k v(t_k), & k = 1,2,\cdots,p, \\ v(0) \leqslant 0. \end{cases} \tag{2.2}$$

From the definition of $v(t)$, obviously, $v(t) \leqslant 0$ implies $m(t) \leqslant 0$, $t \in J$. So it suffices to show $v(t) \leqslant 0$ for any $t \in J$. Suppose on the contrary, there exists $t^* \in (0,T]$ such that $v(t^*) > 0$. Let $\inf\limits_{0 \leqslant t \leqslant t^*} v(t) = -\lambda$, then $\lambda \geqslant 0$.

Case 1: if $\lambda = 0$, then $v(t) \geqslant 0$, for all $t \in [0,t^*]$. Thus, by (2.2), we get $v'(t) \leqslant 0$ on $t \in [0,t^*]$ and $\Delta v(t_k) \leqslant -L_k v(t_k) \leqslant 0$ on $t_k \in (0,t^*)$, hence $v(t)$ is nonincreasing in $[0,t^*]$. So we have $v(t^*) \leqslant v(0) \leqslant 0$, which is a contradiction.

Case 2: if $\lambda > 0$, then there exists a $t_* \in [0,t^*)$ such that $v(t_*) = -\lambda < 0$ or $v(t_*^+) = -\lambda$. We only consider $v(t_*) = -\lambda$ because when $v(t_*^+) = -\lambda$, the proof is similar. From (2.2) we get

$$\begin{aligned} 0 < v(t^*) &= v(t_*) + \int_{t_*}^{t^*} v'(t)dt + \sum_{t_* \leqslant t_k \leqslant t^*} \Delta v(t_k) \\ &\leqslant -\lambda + \lambda \int_0^T e^{MT}(\mathscr{L}e^{-M})(t)dt + \lambda \prod_{k=1}^{p} L_k \\ &= \lambda \left\{ \int_0^T e^{MT}(\mathscr{L}e^{-M})(t)dt + \prod_{k=1}^{p} L_k - 1 \right\}, \end{aligned}$$

which yields

$$\int_0^T e^{MT}(\mathscr{L}e^{-M})(t)dt + \prod_{k=1}^{p} L_k > 1.$$

A contradiction is then elicited due to (2.1). Hence $v(t) \leqslant 0$, and this completes the proof. $\square$

3. Main results

In this section, we shall establish the existence of extremal solutions of problem (1.1).

DEFINITION 2. A function $\alpha \in \Omega$ is called a *lower solution* of (1.1) if

$$\begin{cases} \alpha'(t) \leqslant (Q\alpha)(t), & t \in J', \\ \Delta\alpha(t_k) \leqslant I_k(\alpha(t_k)), & k = 1,2,\cdots,p, \\ B(\alpha(0),\alpha) \leqslant 0. \end{cases}$$

DEFINITION 3. A function $\beta \in \Omega$ is called an *upper solution* of (1.1) if

$$\begin{cases} \beta'(t) \geqslant (Q\beta)(t), & t \in J', \\ \Delta\beta(t_k) \geqslant I_k(\beta(t_k)), & k = 1,2,\cdots,p, \\ B(\beta(0),\beta) \geqslant 0. \end{cases}$$

DEFINITION 4. (see [7, Section 2]) *A solution $y \in \Omega$ of problem* (1.1) *is called maximal if $x(t) \leqslant y(t), t \in J$, for each solution x of* (1.1)*, and minimal if the reverse inequality holds. If both minimal and maximal solutions exist, we call them extremal solutions of* (1.1).

For $\alpha,\beta \in \Omega$, we write $\alpha \leqslant \beta$ if $\alpha(t) \leqslant \beta(t)$ for all $t \in J$. Also, we denote $[\alpha,\beta] = \{y \in \Omega, \alpha(t) \leqslant y(t) \leqslant \beta(t), t \in J\}$.

In the sequel, we state our theorems. First we discuss the existence of solutions for the following linear problem

$$\begin{cases} y'(t) + My(t) + (\mathscr{L}y)(t) = \sigma_\eta(t), & t \in J', \\ \Delta y(t_k) = -L_k y(t_k) + \gamma_k, & k = 1,2,\cdots,p, \\ B(y(0),y) = 0, \end{cases} \tag{3.1}$$

where $\eta \in [\alpha,\beta]$, $\sigma_\eta(t) = (Q\eta)(t) + M\eta(t) + (\mathscr{L}\eta)(t)$, $\gamma_k = I_k(\eta(t_k)) + L_k\eta(t_k)$.

Throughout this paper, we shall assume the following hypotheses hold:

(H_1) Let α and $\beta \in \Omega$ be lower and upper solutions of problem (1.1), respectively, such that $\alpha \leqslant \beta$.

(H_2) There exists $M \geqslant 0$ such that

$$(Qu)(t) - (Qv)(t) \geqslant -M(u-v) - (\mathscr{L}(u-v))(t), \text{ for } \alpha \leqslant v \leqslant u \leqslant \beta.$$

(H_3) There exist $0 \leqslant L_k < 1$, $k = 1, 2, \cdots, p$ such that

$$I_k(u) - I_k(v) \geqslant -L_k(u - v), \text{ for } \alpha \leqslant v \leqslant u \leqslant \beta.$$

(H_4) $B(u, \cdot)$ is a nonincreasing function for each $u \in [\alpha(0), \beta(0)]$.

THEOREM 1. *Suppose that all conditions of Lemma 1 are satisfied. Then linear problem (3.1) has at least one solution $y \in [\alpha, \beta]$.*

Proof. For $\eta \in [\alpha, \beta]$, we consider the following problem:

$$\begin{cases} y'(t) + My(t) + (\mathscr{L}y)(t) = \sigma_\eta(t), & t \in J', \\ \Delta y(t_k) = -L_k p(t_k, y(t_k)) + \gamma_k, & k = 1, 2, \cdots, p, \\ y(0) = p(0, y(0) - B(y(0), y)), \end{cases} \tag{3.2}$$

where $\sigma_\eta(t) = (Q\eta)(t) + M\eta(t) + (\mathscr{L}\eta)(t)$, $p(t, y) = \max[\alpha(t), \min(y, \beta(t))]$, $\gamma_k = I_k(\eta(t_k)) + L_k \eta(t_k)$.

The idea of the proof is to transform (3.2) into a fixed point problem in order to apply Schauder Fixed Point Theorem (see [17]).

Consider the operator

$$\mathscr{A} : E_0 \to E_0$$

given by

$$\mathscr{A}y(t) = y(0) + \int_0^t [\sigma_\eta(s) - My(s) - (\mathscr{L}y)(s)]ds + \sum_{0 \leqslant t_k \leqslant t} [\gamma_k - L_k p(t_k, y(t_k))]. \tag{3.3}$$

It is easy to see that $y \in \Omega$ is a solution of (3.2), if and only if $y \in E_0$ is a fixed point of $\mathscr{A}$. Also every solution y of (3.2) is a solution of (3.1).

In order to apply Schauder$'$s fixed point theorem, we shall prove that the operator $\mathscr{A}$ is continuous and compact. Note that E_0 is a Banach space with the norm $\|y\| = \sup_{t \in J} |y(t)|$. Let $y \in E_0$, the continuity of $\mathscr{L}$ and σ_η imply that $\sigma_\eta(s) - My(s) - (\mathscr{L}y)(s)$ is bounded. The definition of $p(t, y)$ implies that $\gamma_k - L_k p(t_k, y(t_k))$ is bounded, so $\mathscr{A}$ is continuous and bounded.

Let $t_1, t_2 \in J$, $t_1 < t_2$, then

$$|(\mathscr{A}y)(t_1) - (\mathscr{A}y)(t_2)| \leqslant \left| \int_{t_1}^{t_2} [\sigma_\eta(s) - My(s) - (\mathscr{L}y)(s)] \right| + \sum_{t_1 \leqslant t_k \leqslant t_2} |\gamma_k - L_k p(t_k, y(t_k))|,$$

as $t_2 \to t_1$. The right side of the above inequality tends to zero. This proves that the operator $\mathscr{A}$ is equicontinuous on J. Then Arzela-Ascoli theorem shows that $\mathscr{A}$ is compact. From Schauder$'$s fixed point theorem, $\mathscr{A}$ has a fixed point in E_0. Next, we shall prove $y \in [\alpha, \beta]$.

Firstly, we prove $\alpha \leqslant y$, set $m(t) = \alpha(t) - y(t)$, $t \in J$. Owing to (H_2) and (H_3), we acquire

$$\begin{aligned} m'(t) &= \alpha'(t) - y'(t) \\ &\leqslant (Q\alpha)(t) - (Q\eta)(t) - M\big(\eta(t) - y(t)\big) - \big(\mathscr{L}(\eta - y)\big)(t) \\ &\leqslant -Mm(t) - (\mathscr{L}m)(t), \end{aligned}$$

and

$$\begin{aligned} \Delta m(t_k) &= \Delta\alpha(t_k) - \Delta y(t_k) \\ &\leqslant I_k(\alpha(t_k)) - I_k(\eta(t_k)) - L_k\eta(t_k) + L_k m(t_k, y(t_k)) \\ &\leqslant -L_k(\alpha(t_k) - m(t_k, y(t_k))) \\ &\leqslant -L_k \min\{m(t_k), 0\}, \end{aligned}$$

note that $m(0) \leqslant 0$, by Lemma 1, we have $\alpha \leqslant y$. Similarly, we can show that $y \leqslant \beta$ and then $y \in [\alpha, \beta]$.

Finally, we shall prove $\alpha(0) \leqslant y(0) - B(y(0), y) \leqslant \beta(0)$.

If $y(0) - B(y(0), y) < \alpha(0)$, due to (3.2) and the definition of $p(t, y)$, we have $y(0) = \alpha(0)$, in consequence $\alpha(0) - B(\alpha(0), y) < \alpha(0)$. Since $B(\alpha(0), \cdot)$ is nonincreasing in $[\alpha, \beta]$ and we know $y \in [\alpha, \beta]$, we see that $B(\alpha(0), \alpha) > 0$, which contradicts with the definition of the lower solution. Similarly arguments may show that $y(0) - B(y(0), y) \leqslant \beta(0)$.

Thus every solution y of (3.2) is a solution of (3.1), and it belongs to $[\alpha, \beta]$.

This proves that problem (3.1) has a solution $y \in [\alpha, \beta]$. □

THEOREM 2. *Assume that the conditions of Theorem 1 hold. Then there exist monotone sequence* $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *in* E_0 *such that* $\alpha_0 = \alpha$, $\beta_0 = \beta$ *which converge uniformly to the extremal solutions of* (1.1) *in* $[\alpha, \beta]$.

Proof. Let $\eta \in [\alpha, \beta]$, we consider the following equations:

$$\begin{cases} y'(t) + My(t) + (\mathscr{L}y)(t) = (Q\eta)(t) + M\eta(t) + (\mathscr{L}\eta)(t), \ t \in J', \\ \Delta y(t_k) = -L_k y(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k), \qquad k = 1, 2, \cdots, p, \\ y(0) = \varsigma_\eta, \end{cases} \tag{3.4}$$

where ς_η is the minimal solution in $[\alpha(0), \beta(0)]$ such that $B(\varsigma_\eta, \eta) = 0$. Since B is continuous and $B(\alpha_0, \eta) \leqslant B(\alpha(0), \alpha) \leqslant 0$ and $0 \leqslant B(\beta(0), \beta) \leqslant B(\beta(0), \eta)$, ς_η is well defined.

By Theorem 1, (3.4) has at least one solution (defining $B(u, v) = u - \varsigma_\eta$). Next we prove the uniqueness of solution to this problem. If not, let $y_1(t)$, $y_2(t)$ be two solutions of (3.4). Set $v_1(t) = y_1(t) - y_2(t)$ and $v_2(t) = y_2(t) - y_1(t)$, then

$$v_1(0) = 0, \quad v_1'(t) + Mv_1(t) + (\mathscr{L}v_1)(t) = 0, \quad \Delta v_1(t_k) = -L_k v_1(t_k),$$

and

$$v_2(0)=0, \quad v_2'(t)+Mv_2(t)+(\mathscr{L}v_2)(t)=0, \quad \Delta v_2(t_k)=-L_kv_2(t_k),$$

from Lemma 1, we have $v_1=y_1-y_2\leqslant 0$ and $v_2=y_2-y_1\leqslant 0$, so $y_1=y_2$. Then problem (3.4) has exactly one solution.

Define a mapping A by $A\eta=y$, then A has the following properties:

(a) $\alpha\leqslant A\alpha$, $\beta\geqslant A\beta$;

(b) A is monotonically nondecreasing in $[\alpha,\beta]$, i.e., for any $\eta_1,\eta_2\in[\alpha,\beta]$, $\eta_1\leqslant\eta_2$ implies $A\eta_1\leqslant A\eta_2$.

To prove (a), set $m(t)=\alpha(t)-\alpha_1(t)$, $t\in J$, where $\alpha_1=A\alpha$. Employing (H_1), we have

$$\begin{aligned}m'(t)&=\alpha'(t)-\alpha_1'(t)\\&\leqslant(Q\alpha)(t)-[(Q\alpha)(t)+M\alpha(t)+(\mathscr{L}\alpha)(t)-M\alpha_1(t)-(\mathscr{L}\alpha_1)(t)]\\&=-Mm(t)-(\mathscr{L}m)(t),\end{aligned}$$

and

$$\begin{aligned}\Delta m(t_k)&=\Delta\alpha(t_k)-\Delta\alpha_1(t_k)\\&\leqslant I_k(\alpha(t_k))-[-L_k\alpha_1(t_k)+I_k(\alpha(t_k))+L_k\alpha(t_k)]\\&=-L_km(t_k),\end{aligned}$$

note that $m(0)\leqslant 0$, then based on Lemma 1, we get $\alpha\leqslant\alpha_1$. Analogously, we have $\beta\geqslant A\beta$.

To prove (b), let η_1, $\eta_2\in[\alpha,\beta]$, $\eta_1\leqslant\eta_2$. Suppose that $v_1=A\eta_1$, $v_2=A\eta_2$ and $m(t)=v_1(t)-v_2(t)$. Applying (H_2) and (H_3), we get

$$\begin{aligned}m'(t)+Mm(t)+(\mathscr{L}m)(t)&=(Q\eta_1)(t)+M\eta_1(t)+(\mathscr{L}\eta_1)(t)\\&\quad-(Q\eta_2)(t)-M\eta_2(t)-(\mathscr{L}\eta_2)(t)\leqslant 0,\end{aligned}$$

$$\begin{aligned}\Delta m(t_k)&=\Delta v_1(t_k)-\Delta v_2(t_k)\\&=[-L_kv_1(t_k)+I_k(\eta_1(t_k)+L_k\eta_1(t_k)]-[-L_kv_2(t_k)+I_k(\eta_2(t_k)+L_k\eta_2(t_k)]\\&\leqslant -L_k(v_1(t_k)-v_2(t_k))\\&=-L_km(t_k),\end{aligned}$$

and

$$B(x,\eta_1)\geqslant B(x,\eta_2), \text{ for all } x\in[\alpha_0,\beta_0],$$

then $A\eta_1(0)=\varsigma_{\eta_1}\leqslant\varsigma_{\eta_2}=A\eta_2(0)$, thus $m(0)\leqslant 0$. Based on Lemma 1 we have $m(t)\leqslant 0$, $t\in J$, which implies $A\eta_1\leqslant A\eta_2$.

Now, define the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0=\alpha$, $\beta_0=\beta$ such that $\alpha_{n+1}=A\alpha_n$, $\beta_{n+1}=\beta_n$. Following (a) and (b), we have

$$\alpha_0\leqslant\alpha_1\leqslant\alpha_2\leqslant\ldots\leqslant\alpha_n\leqslant\ldots\leqslant\beta_n\leqslant\ldots\leqslant\beta_2\leqslant\beta_1\leqslant\beta_0.$$

Consequently, there exist ρ and r such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$ and $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically. Clearly, $\rho(t)$ and $r(t)$ are the solutions of problem (1.1).

To prove that $\rho(t)$ and $r(t)$ are extremal solutions of problem (1.1), let $y(t)$ be any solution of (1.1) such that $\alpha(t) \leqslant y(t) \leqslant \beta(t)$. Suppose that there exists a positive integer n such that $\alpha_n(t) \leqslant y(t) \leqslant \beta_n(t)$ on J. Based on the monotonically nondecreasing property of A, we can easily see that $\alpha_{n+1} = A\alpha_n \leqslant Ay = y$, *i.e.*, $\alpha_{n+1}(t) \leqslant y(t)$ on J. Similarly, we can get $y(t) \leqslant \beta_{n+1}(t)$ on J. Since $\alpha_0(t) \leqslant y(t) \leqslant \beta_0(t)$ on J, by induction we derives $\alpha_n(t) \leqslant y(t) \leqslant \beta_n(t)$ on J for every n. Therefore $\rho(t) \leqslant y(t) \leqslant r(t)$ as $n \to \infty$. We complete the proof. □

4. Example

In this section, we give an example that proves the validity of Theorem 2.

EXAMPLE 1. Consider the following problem:

$$\begin{cases} y'(t) = -\dfrac{1}{2}y(t) + \dfrac{1}{2}\cos(y(\tfrac{1}{2}t)) - t^2\displaystyle\int_0^t e^{\frac{1}{2}s}y(s)ds = (Qy)(t), & t \neq \dfrac{1}{3}, t \in J, \\ \Delta y(t_k) = -\dfrac{1}{15}y^2(t_k), & t_k = \dfrac{1}{3}, \\ B(y(0), y) = y(0) + \displaystyle\int_0^1 (s - y(s))ds - \dfrac{1}{4}, \end{cases} \tag{4.1}$$

where $J = [0,1]$. Set

$$\alpha_0(t) = 0, \qquad \beta_0(t) = \begin{cases} \dfrac{3}{4}t + 1, & t \in [0, \dfrac{1}{3}], \\ \dfrac{3}{4}t + \dfrac{3}{4}, & t \in (\dfrac{1}{3}, 1], \end{cases}$$

we can easily verify that $\alpha_0(t)$ is a lower solution and $\beta_0(t)$ is an upper solution with $\alpha_0(t) \leqslant \beta_0(t)$.

By computing, we have

$$\begin{aligned} I_k(u(t_k)) - I_k(v(t_k)) &= -\frac{1}{15}\big(u^2(t_k) - v^2(t_k)\big) \\ &\geqslant -\frac{1}{6}(u(t_k) - v(t_k)) \\ &= -L_1(u(t_k) - v(t_k)), \end{aligned}$$

where $\alpha_0(t_k) \leqslant v(t_k) \leqslant u(t_k) \leqslant \beta_0(t_k)$, $L_1 = \dfrac{1}{6}$.

$$(Qu)(t) - (Qv)(t) \geqslant -\frac{1}{2}(u - v) - (\mathscr{L}(u - v))(t)$$

where $\alpha_0(t) \leqslant v(t) \leqslant u(t) \leqslant \beta_0(t)$, $M = \dfrac{1}{2}$.

It is easy to prove that $(\mathscr{L}y)(t) = t^2 \int_0^t e^{\frac{1}{2}s} y(s)ds$ and

$$\int_0^T e^{MT} (\mathscr{L}e^{-M})(t)dt + \prod_{k=1}^{p} L_k = \int_0^1 e^{\frac{1}{2}} \big(t^2 \int_0^t e^{\frac{1}{2}s} e^{-\frac{1}{2}s} ds\big) dt + \frac{1}{6} \leqslant 1.$$

Apparently, for any $y_1 < y_2$, $B(y(0), y_1) - B(y(0), y_2) = \int_0^1 (y_2(s) - y_1(s))ds > 0$. So $B(y(0), \cdot)$ is nonincreasing. Then all conditions of Theorem 2 are satisfied. Therefore, via Theorem 2, there exist monotone iterative sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ which converge uniformly on J to the extremal solutions of (4.1).

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