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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF A LANCHESTER–TYPE MODEL

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Abstract. An ordinary differential system, referred to as Lanchester-type model, is treated. We examine how asymptotic behavior of every solution of the system varies according to the initial data. We can show the existence of critical values for initial data.

1. Introduction and statement of the main results

In the paper we consider the following binary system

$$\begin{cases} x' = -a(t)xy, \\ y' = -b(t)xy \end{cases} \tag{S}$$

under the following assumptions:

(A_1) $a(t)$ and $b(t)$ are positive continuous functions on $[0,\infty)$;

(A_2) For some constants λ_1, λ_2, $\mu_1 > -1$ and $\mu_2 > -1$, $a(t)$ and $b(t)$ satisfy the following growth conditions:

$$0 < \liminf_{t\to\infty} \frac{a(t)}{t^{\lambda_1}} \leqslant \limsup_{t\to\infty} \frac{a(t)}{t^{\lambda_2}} < \infty;$$

and

$$0 < \liminf_{t\to\infty} \frac{b(t)}{t^{\mu_1}} \leqslant \limsup_{t\to\infty} \frac{b(t)}{t^{\mu_2}} < \infty.$$

System (S) is a kind of Lanchester model, which describes many phenomena appearing in economics, logistics, biology, and so on. Originally, (S) was proposed by [7] to describe combat situations. It is said [1, 3, 4] that system (S) is a model of guerrilla engagements.

It seems that several scientists and technicians engaged in operational research treat such models via numerical methods; see, for example, [1, 3, 10]. However, as far as we know, there are few results treating mathematical models like system (S)

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rigorously. In [4, 9] differential systems similar to (S) were considered mathematically. In [2, 5, 6, 8] related results are obtained for Lanchester-type models.

In this paper we will study asymptotic behavior of positive solutions of (S). Let $x(0) > 0$ and $y(0) > 0$. Then we can show that the (local) solution $(x(t), y(t))$ of (S) exists globally on $[0, \infty)$, and $x(t) > 0$ and $y(t) > 0$ there, because for example, the formula

$$x(t) = x(0)\exp\left(-\int_0^t a(s)y(s)ds\right) \quad \text{and} \quad y(t) = y(0)\exp\left(-\int_0^t b(s)x(s)ds\right)$$

holds as long as $(x(t), y(t))$ exists. Therefore $x(t)$ and $y(t)$ both decrease, and $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$ exist as nonnegative numbers. We focus on the values of $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

To explain our motivation in detail, denote the global solution of (S) with the initial condition

$$x(0) = \alpha > 0 \quad \text{and} \quad y(0) = \beta > 0 \tag{I}$$

by $(x(t;\alpha,\beta), y(t;\alpha,\beta))$. Let $\alpha > 0$ be fixed arbitrarily, and let us move $\beta \in (0, \infty)$. We intend to examine how $\lim_{t\to\infty}(x(t;\alpha,\beta), y(t;\alpha,\beta))$ varies according to β.

As a typical example of system (S), consider the case where $a(t) \equiv a_0$ and $b(t) \equiv b_0$ for some positive constants a_0 and b_0:

$$\begin{cases} x' = -a_0xy, \\ y' = -b_0xy. \end{cases} \tag{S_0}$$

This system can be solved explicitly as seen below. In fact, the solution $(x(t), y(t)) \equiv (x(t;\alpha,\beta), y(t;\alpha,\beta))$ satisfies

$$(b_0x(t) - a_0y(t))' = -a_0b_0x(t)y(t) + a_0b_0x(t)y(t) \equiv 0,$$

and so the quantity $b_0x(t) - a_0y(t)$ is constant, that is, $b_0x(t) - a_0y(t) \equiv b_0\alpha - a_0\beta$ for $t \geqslant 0$. Let us put $m = b_0\alpha - a_0\beta$. Since $y(t) = (b_0x(t) - m)/a_0$, the first equation of (S_0) is rewritten as $x' = -x(b_0x - m)$. Therefore we get

$$x(t) = \frac{m\alpha e^{mt}}{b_0\alpha(e^{mt} - 1) + m} \quad \text{and} \quad y(t) = \frac{m\beta}{me^{mt} + a_0\beta(e^{mt} - 1)} \quad \text{if} \quad m \neq 0;$$

and

$$x(t) = \frac{\alpha}{b_0\alpha t + 1} \quad \text{and} \quad y(t) = \frac{\beta}{a_0\beta t + 1} \qquad \text{if} \quad m = 0.$$

Put $\beta_0 = \beta_0(\alpha) = b_0\alpha/a_0$. By these formulas we can derive the following fact concerning $\lim_{t\to\infty}(x(t;\alpha,\beta), y(t;\alpha,\beta))$:

(i) if $\beta < \beta_0$, then $\lim_{t\to\infty} x(t;\alpha,\beta) > 0$ and $\lim_{t\to\infty} y(t;\alpha,\beta) = 0$;
(ii) if $\beta = \beta_0$, then $\lim_{t\to\infty} x(t;\alpha,\beta) = \lim_{t\to\infty} y(t;\alpha,\beta) = 0$;
(iii) if $\beta > \beta_0$, then $\lim_{t\to\infty} x(t;\alpha,\beta) = 0$ and $\lim_{t\to\infty} y(t;\alpha,\beta) > 0$.

Accordingly, a simple problem naturally comes from this fact:

PROBLEM. Like the typical example (S_0), is there a critical value for system (S) with general coefficients $a(t)$ and $b(t)$?

The main objective of the paper is to answer this problem. In fact, we can settle this problem affirmatively in some sense.

The main result of the paper is as follows:

THEOREM 1. *Let $\alpha > 0$ be fixed. Then for system (S) there are two constants $\beta_1 = \beta_1(\alpha)$ and $\beta_2 = \beta_2(\alpha)$ $(\beta_1 \leqslant \beta_2)$ such that:*

(i) if $\beta < \beta_1$, then $\lim_{t\to\infty} x(t;\alpha,\beta) > 0$ and $\lim_{t\to\infty} y(t;\alpha,\beta) = 0$;

(ii) if $\beta_1 \leqslant \beta \leqslant \beta_2$, then $\lim_{t\to\infty} x(t;\alpha,\beta) = \lim_{t\to\infty} y(t;\alpha,\beta) = 0$;

(iii) if $\beta > \beta_2$, then $\lim_{t\to\infty} x(t;\alpha,\beta) = 0$ and $\lim_{t\to\infty} y(t;\alpha,\beta) > 0$.

REMARK 2. (i) For the typical system (S_0) we have already shown that the critical numbers introduced in Theorem 1 are the same: $\beta_1 = \beta_2$. So we conjecture that $\beta_1 = \beta_2$ in Theorem 1 generally. One of our next aims is to see this fact.

(ii) It is impossible for solutions (x,y) of (S) to satisfy $\lim_{t\to\infty} x(t) > 0$ as well as $\lim_{t\to\infty} y(t) > 0$. In fact if this is the case, then

$$\infty > -x(\infty) + x(0) = \int_0^\infty a(s)x(s)y(s)ds \geqslant x(\infty)y(\infty)\int_0^\infty a(s)ds,$$

which is a contradiction because of $\int_0^\infty a(s)ds = \infty$ by assumption (A_2).

(iii) In [9], system (S) was considered mainly under more restrictive conditions. However, the main objective in [9] is different from ours.

This paper is organized as follows. In Section 2 we give several preliminary results. In Sections 3 we give firstly several propositions forming part of the proof of Theorem 1, and then we give the proof of the main result Theorem 1.

2. Preliminary results

LEMMA 3. *A vector function $(x(t), y(t))$ is the solution of initial value problem (S)-(I) if and only if it solves the system of integral equations*

$$x(t) = \alpha \exp\left(-\beta \int_0^t a(s) \exp\left(-\int_0^s b(r)x(r)dr \right) ds \right), \tag{1}$$

$$y(t) = \beta \exp\left(-\alpha \int_0^t b(s) \exp\left(-\int_0^s a(r)y(r)dr \right) ds \right). \tag{2}$$

Proof. The initial value problem (S)-(I) is equivalent to the system of integral equations

$$x(t) = \alpha \exp\left(-\int_0^t a(s)y(s)ds \right), \tag{3}$$

$$y(t) = \beta \exp\left(-\int_0^t b(s)x(s)ds \right). \tag{4}$$

Substituting (4) into the integrand of (3) we find that $x(t)$ satisfies (1). Similarly we can get (2) for $y(t)$.

Conversely, suppose that $(x(t),y(t))$ satisfies (1) and (2). Let us introduce the auxiliary functions $\tilde{x}$ and $\tilde{y}$ by

$$\tilde{x}(t)=\alpha\exp\left(-\int_0^t a(s)y(s)ds\right) \quad \text{and} \quad \tilde{y}(t)=\beta\exp\left(-\int_0^t b(s)x(s)ds\right).$$

Then, (1) and (2), respectively, can be rewritten as

$$x(t)=\alpha\exp\left(-\int_0^t a(s)\tilde{y}(s)ds\right), \quad \text{and} \quad y(t)=\beta\exp\left(-\int_0^t b(s)\tilde{x}(s)ds\right).$$

Therefore, $x(t),\tilde{x}(t),y(t)$ and $\tilde{y}(t)$ satisfy

$$\begin{cases} x'=-a(t)x\tilde{y}, & x(0)=\alpha, \\ \tilde{y}'=-b(t)x\tilde{y}, & \tilde{y}(0)=\beta, \end{cases}$$

and

$$\begin{cases} \tilde{x}'=-a(t)\tilde{x}y, & \tilde{x}(0)=\alpha, \\ y'=-b(t)\tilde{x}y, & y(0)=\beta. \end{cases}$$

So by the uniqueness of solutions of initial value problems of ordinary differential systems, we find that $(x,\tilde{y})\equiv(\tilde{x},y)$; that is, (x,y) is the solution of (S)-(I). This completes the proof. $\square$

LEMMA 4. (Comparison Lemma) *Let* $\overline{x}_0 \geqslant \underline{x}_0 > 0$ *and* $0<\underline{y}_0<\overline{y}_0$. *Then*

$$x(t;\overline{x}_0,\underline{y}_0) > x(t;\underline{x}_0,\overline{y}_0) \quad in \ (0,\infty);$$

and

$$y(t;\overline{x}_0,\underline{y}_0) < y(t;\underline{x}_0,\overline{y}_0) \quad on \ [0,\infty). \tag{5}$$

REMARK 5. An analogous result to this lemma also holds when $\overline{x}_0>\underline{x}_0$ and $\underline{y}_0 \leqslant \overline{y}_0$.

Proof. We will show that $y(t;\overline{x}_0,\underline{y}_0)<y(t;\underline{x}_0,\overline{y}_0)$ on $[0,\infty)$ by contradiction. Suppose the contrary. Since in some right neighborhood of 0 we have $y(t;\overline{x}_0,\underline{y}_0)<y(t;\underline{x}_0,\overline{y}_0)$, there is a T satisfying

$$y(t;\overline{x}_0,\underline{y}_0)<y(t;\underline{x}_0,\overline{y}_0) \quad \text{on} \quad [0,T);$$

and

$$y(T;\overline{x}_0,\underline{y}_0)=y(T;\underline{x}_0,\overline{y}_0).$$

On the other hand, from Lemma 3 we have

$$\begin{aligned}\frac{1}{y(T;\overline{x}_0,\underline{y}_0)} &= \frac{1}{\underline{y}_0}\exp\left(\overline{x}_0\int_0^T \frac{b(s)}{\exp\left(\int_0^s a(r)y(r;\overline{x}_0,\underline{y}_0)dr\right)}ds\right) \\ &> \frac{1}{\overline{y}_0}\exp\left(\underline{x}_0\int_0^T \frac{b(s)}{\exp\left(\int_0^s a(r)y(r;\underline{x}_0,\overline{y}_0)dr\right)}ds\right) = \frac{1}{y(T;\underline{x}_0,\overline{y}_0)}.\end{aligned}$$

This is a contradiction. So (5) holds. The other inequality is a direct consequence of (3) and (5). This completes the proof. □

In what follows we put $B(t) = \int_0^t b(s)ds$ and $A(t) = \int_0^t a(s)ds$. Note that, by assumption (A_2), $A(t)$ and $B(t)$ have polynomial growths as $t \to \infty$.

LEMMA 6. *Let K, $k > 0$, and $y_0 > 0$ be constants satisfying $K > k$ and*

$$k\exp\left(y_0\int_0^\infty a(s)e^{-kB(s)}ds\right) \leqslant K;$$

and

$$ky_0\left(\int_0^\infty a(s)B(s)e^{-kB(s)}ds\right)\exp\left(y_0\int_0^\infty a(s)e^{-kB(s)}ds\right) < 1.$$

Then, system (S) has a solution $(x(t),y(t))$ such that

$$k \leqslant x(t) \leqslant K, \quad \lim_{t\to\infty} x(t) = k;$$

and

$$y(0) = y_0, \quad \lim_{t\to\infty} y(t) = 0.$$

REMARK 7. An analogous result to Lemma 6 holds if $a(t)$, $b(t)$, x and y are replaced by $b(t)$, $a(t)$, y and x, respectively.

Proof of Lemma 6. A solution $(x(t),y(t))$ of system (S) satisfies $\lim_{t\to\infty} x(t) = k$ and $y(0) = y_0$ if and only if it solves the system of integral equations

$$x(t) = k\exp\left(\int_t^\infty a(s)y(s)ds\right),$$

$$y(t) = y_0\exp\left(-\int_0^t b(s)x(s)ds\right).$$

By substituting the formula for $y(t)$ into the formula for $x(t)$, it suffices to find a solution $x(t)$ of the single integral equation

$$x(t) = k\exp\left(y_0\int_t^\infty \frac{a(s)}{\exp\left(\int_0^s b(r)x(r)dr\right)}ds\right) \tag{6}$$

satisfying $k \leqslant x(t) \leqslant K$, $t \geqslant 0$. We will solve this nonlinear integral equation via a fixed point theorem.

Let X be the Banach space defined by

$$X = \left\{ x \in C[0,\infty) \,\middle|\, \sup_{t \geqslant 0} |x(t)| < \infty \right\}$$

equipped with the norm $\|x\| = \sup_{t \geqslant 0} |x(t)|$ for $x \in X$, and we introduce the subset $S \subset X$ given by

$$S = \{x \in X | k \leqslant x(t) \leqslant K \text{ on } [0,\infty)\}.$$

Let us define the operator $\Phi : X \to X$ by

$$\Phi[x](t) = k \exp\left(y_0 \int_t^\infty \Psi[x](s)ds \right), \quad t \geqslant 0,$$

for $x \in X$, where $\Psi[x](s)$ denotes

$$\Psi[x](s) = a(s) \exp\left(-\int_0^s b(r)x(r)dr \right), \quad s \geqslant 0.$$

We will show that Φ is a contractive mapping on S below.

To see $\Psi(S) \subset S$, let $x \in S$. Then, it is easy to see that

$$0 \leqslant \Psi[x](s) \leqslant a(s) \exp\left(-k \int_0^s b(r)dr \right) \equiv a(s)e^{-kB(t)}, \quad s \geqslant 0. \tag{7}$$

So,

$$k \leqslant \Phi[x](t) \leqslant k \exp\left(y_0 \int_0^\infty a(s)e^{-kB(s)}ds \right) \leqslant K$$

by the assumptions. Therefore, $\Phi(S) \subset S$ as desired.

For x_1, $x_2 \in S$, we have

$$|\Psi[x_1](s) - \Psi[x_2](s)| \leqslant a(s) \int_0^s b(r)|x_1(r) - x_2(r)|dr \cdot e^{-\eta(s)},$$

where $\eta(s)$ is a number between $\int_0^s b(r)x_1(r)dr$ and $\int_0^s b(r)x_2(r)dr$, and so

$$\begin{aligned} |\Psi[x_1](s) - \Psi[x_2](s)| &\leqslant a(s) \int_0^s b(r)dr \cdot e^{-kB(s)} \|x_1 - x_2\| \\ &\equiv a(s)B(s)e^{-kB(s)} \|x_1 - x_2\|, \quad s \geqslant 0. \end{aligned} \tag{8}$$

Then

$$|\Phi[x_1](t) - \Phi[x_2](t)| \leqslant ky_0 \int_t^\infty |\Psi[x_1](s) - \Psi[x_2](s)|ds \cdot e^{\xi(t)},$$

where $\xi(t)$ is a number between $y_0 \int_t^\infty \Psi[x_1](s)ds$ and $y_0 \int_t^\infty \Psi[x_2](s)ds$. Therefore, by (8)

$$|\Phi[x_1](t) - \Phi[x_2](t)| \leqslant ky_0 \int_t^\infty a(s)B(s)e^{-kB(s)}ds \cdot e^{\xi(t)} \|x_1 - x_2\|.$$

Since $0 \leqslant \xi(t) \leqslant y_0 \int_t^\infty a(s)e^{-kB(s)}ds$ by (7), we finally obtain

$$|\Phi[x_1](t) - \Phi[x_2](t)| \leqslant ky_0 \left(\int_t^\infty a(s)B(s)e^{-kB(s)}ds \right) \\ \times \exp\left(y_0 \int_t^\infty a(s)e^{-kB(s)}ds \right) \|x_1 - x_2\|, \quad t \geqslant 0.$$

Accordingly,

$$\|\Phi[x_1] - \Phi[x_2]\| \leqslant ky_0 \left(\int_0^\infty a(s)B(s)e^{-kB(s)}ds \right) \\ \times \exp\left(y_0 \int_0^\infty a(s)e^{-kB(s)}ds \right) \|x_1 - x_2\|.$$

By our assumptions Φ is a contraction on S. This completes the proof. $\square$

3. Proof of the main result

Recall that we fix $\alpha > 0$ arbitrarily. To prove Theorem 1, we further prepare two propositions.

PROPOSITION 8. *If $\beta > 0$ is sufficiently small, then*

$$\lim_{t\to\infty} x(t;\alpha,\beta) > 0 \quad \textit{and} \quad \lim_{t\to\infty} y(t;\alpha,\beta) = 0.$$

Proof. Let $K, k > 0$ be numbers satisfying $k < K < \alpha$, and we fix them. Then, there is a sufficiently small y_0 such that the assumptions of Lemma 6 hold. Therefore, we find an $x_0 \in (k, K)$ and a y_0 satisfying

$$\lim_{t\to\infty} x(t;x_0,y_0) = k > 0 \quad \text{and} \quad \lim_{t\to\infty} y(t;x_0,y_0) = 0.$$

Let β be sufficiently small so that $0 < \beta < y_0$. Since $x_0 < \alpha$, by Remark 5

$$x(t;\alpha,\beta) > x(t;x_0,y_0) \quad \text{and} \quad y(t;\alpha,\beta) < y(t;x_0,y_0), \quad t \geqslant 0.$$

So

$$\lim_{t\to\infty} x(t;\alpha,\beta) = \text{const} \geqslant k > 0; \quad \text{and} \quad \lim_{t\to\infty} y(t;\alpha,\beta) = 0.$$

This completes the proof. $\square$

PROPOSITION 9. *If $\beta > 0$ is sufficiently large, then*

$$\lim_{t\to\infty} x(t;\alpha,\beta) = 0 \quad \textit{and} \quad \lim_{t\to\infty} y(t;\alpha,\beta) > 0.$$

Proof. For $k \geqslant 1$ we can show easily that the inequality

$$kze^{-kz} \leqslant \frac{z}{e^z - 1}, \quad z > 0$$

holds. Therefore for $k \geqslant 1$ we have

$$k \int_0^\infty b(s)A(s)e^{-kA(s)} ds \leqslant \int_0^\infty b(s) \frac{A(s)}{e^{A(s)} - 1} ds < \infty.$$

So the Lebesgue dominated convergence theorem implies that

$$\lim_{k \to \infty} k \int_0^\infty b(s)A(s)e^{-kA(s)} ds = 0.$$

It follows that there is a sufficiently large $k > 0$ satisfying

$$k \exp\left(\alpha \int_0^\infty b(s)e^{-kA(s)} ds\right) \leqslant 2k;$$

and

$$k\alpha\left(\int_0^\infty b(s)A(s)e^{-kA(s)} ds\right) \exp\left(\alpha \int_0^\infty b(s)e^{-kA(s)} ds\right) < 1.$$

Then by Lemma 6 (and Remark 7), we find that for some $y_0 > k$

$$\lim_{t \to \infty} x(t;\alpha,y_0) = 0, \quad \lim_{t \to \infty} y(t;\alpha,y_0) = k > 0.$$

For β satisfying $\beta > y_0$, Lemma 4 implies that

$$x(t;\alpha,y_0) > x(t;\alpha,\beta) \quad \text{and} \quad y(t;\alpha,y_0) < y(t;\alpha,\beta).$$

So we get

$$\lim_{t \to \infty} x(t;\alpha,\beta) = 0; \quad \text{and} \quad \lim_{t \to \infty} y(t;\alpha,\beta) = \text{const} \geqslant k > 0.$$

This completes the proof. $\square$

Now, we are in a position to prove Theorem 1 on the basis of Propositions 8 and 9.

Proof of Theorem 1. Recall that $\alpha > 0$ be fixed arbitrarily. Define the sets $\overline{S} = \overline{S}(\alpha)$ and $\underline{S} = \underline{S}(\alpha)$ in $(0,\infty)$ by

$$\overline{S} = \left\{\beta > 0 \,\middle|\, \lim_{t \to \infty} x(t;\alpha,\beta) = 0 \text{ and } \lim_{t \to \infty} y(t;\alpha,\beta) > 0\right\}$$

and

$$\underline{S} = \left\{\beta > 0 \,\middle|\, \lim_{t \to \infty} x(t;\alpha,\beta) > 0 \text{ and } \lim_{t \to \infty} y(t;\alpha,\beta) = 0\right\},$$

respectively. Clearly $\overline{S} \cap \underline{S} = \emptyset$.

By Propositions 8 and 9, and Lemma 4, $\overline{S}$ and $\underline{S}$ are intervals containing a neighborhood of ∞ and a right neighborhood of 0, respectively. Furthermore, we find from Lemma 4 that $\sup \underline{S} \leqslant \inf \overline{S}$. Put $\beta_* = \inf \overline{S}$ and $\beta^* = \sup \underline{S}$. Then $\beta^* \leqslant \beta_*$. We will show that the conclusion of Theorem 1 holds by defining $\beta_1 = \beta^*$ and $\beta_2 = \beta_*$. The proof is divided into the following steps:

Step 1: Proof of $\lim_{t\to\infty} y(t;\alpha,\beta_*) = 0$.

Step 2: Proof of $\lim_{t\to\infty} x(t;\alpha,\beta_*) = 0$.

Step 3: Proof of $\lim_{t\to\infty} y(t;\alpha,\beta^*) = \lim_{t\to\infty} x(t;\alpha,\beta^*) = 0$.

Step 4: The final step.

Step 1. We claim that $\lim_{t\to\infty} y(t;\alpha,\beta_*) = 0$. The proof is done by contradiction. Suppose to the contrary that $\lim_{t\to\infty} y(t;\alpha,\beta_*) = 2k > 0$. We can find a sufficiently large $T_1 > 0$ satisfying

$$\exp\left(\alpha \int_{T_1}^{\infty} b(s) e^{-kA(s)} ds\right) < 2.$$

By the continuity on the initial data, we can find a sufficiently small $\delta \in (0,\beta_*)$ satisfying

$$y(t;\alpha,\beta_* - \delta) > 2k \quad \text{on} \quad [0,T_1].$$

Now, we claim that $y(t;\alpha,\beta_* - \delta) > k$ on $[0,\infty)$. In fact, if this is not the case, then there is a $T_2 > T_1$ satisfying

$$y(t;\alpha,\beta_* - \delta) > k \text{ on } [0,T_2); \quad \text{and} \quad y(T_2;\alpha,\beta_* - \delta) = k.$$

Then by Lemma 3

$$\begin{aligned}
\frac{1}{k} &= \frac{1}{y(T_2;\alpha,\beta_* - \delta)} \\
&= \frac{1}{\beta - \delta_*} \exp\left(\alpha \int_0^{T_1} \frac{b(s)}{\exp\left(\int_0^s a(r) y(r;\alpha,\beta_* - \delta) dr\right)} ds\right) \\
&\quad \times \exp\left(\alpha \int_{T_1}^{T_2} \frac{b(s)}{\exp\left(\int_0^s a(r) y(r;\alpha,\beta_* - \delta) dr\right)} ds\right) \\
&= \frac{1}{y(T_1;\alpha,\beta_* - \delta)} \exp\left(\alpha \int_{T_1}^{T_2} \frac{b(s)}{\exp\left(\int_0^s a(r) y(r;\alpha,\beta_* - \delta) dr\right)} ds\right) \\
&< \frac{1}{2k} \exp\left(\alpha \int_{T_1}^{\infty} b(s) e^{-kA(s)} ds\right) < \frac{1}{2k} \cdot 2 = \frac{1}{k}.
\end{aligned}$$

This is an obvious contradiction. Therefore we get $y(t;\alpha,\beta_* - \delta) > k$ on $[0,\infty)$, and so $\lim_{t\to\infty} y(t;\alpha,\beta_* - \delta) = \text{const} \geqslant k$. Furthermore by (3) we see that $\lim_{t\to\infty} x(t;\alpha,\beta_* - \delta) = 0$. So $\beta_* - \delta \in \overline{S}$. However, this is a contradiction to the definition of $\beta_* = \inf \overline{S}$. So $\lim_{t\to\infty} y(t;\alpha,\beta_*) = 0$ as desired. This completes the proof of Step 1.

Step 2. We claim that $\lim_{t\to\infty} x(t;\alpha,\beta_*) = 0$. The proof is done by contradiction as in Step 1. Suppose to the contrary that $\lim_{t\to\infty} x(t;\alpha,\beta_*) = 2k > 0$. Then, as before, we can find a sufficiently large number $T_1 > 0$ satisfying

$$\exp\left((\beta_*+1)\int_{T_1}^{\infty} a(s)e^{-kB(s)}ds\right) < 2.$$

By the continuity on the initial data, we can find a sufficiently small $\delta \in (0,1)$ such that the solution $x(t;\alpha,\beta_*+\delta)(< x(t;\alpha,\beta_*))$ fulfills

$$x(t;\alpha,\beta_*+\delta) > 2k \quad \text{on} \quad [0,T_1].$$

Now, we claim that $x(t;\alpha,\beta_*+\delta) > k$ on $[0,\infty)$. In fact, if not, then there is a $T_2 > T_1$ satisfying

$$x(t;\alpha,\beta_*+\delta) > k \text{ on } [0,T_2); \quad \text{and} \quad x(T_2;\alpha,\beta_*+\delta) = k.$$

Then

$$\begin{aligned}
\frac{1}{k} &= \frac{1}{x(T_2;\alpha,\beta_*+\delta)} \\
&= \frac{1}{\alpha}\exp\left((\beta_*+\delta)\int_0^{T_1} \frac{a(s)}{\exp\left(\int_0^s b(r)x(r;\alpha,\beta_*+\delta)dr\right)}ds\right) \\
&\quad \times \exp\left((\beta_*+\delta)\int_{T_1}^{T_2} \frac{a(s)}{\exp\left(\int_0^s b(r)x(r;\alpha,\beta_*+\delta)dr\right)}ds\right) \\
&= \frac{1}{x(T_1;\alpha,\beta_*+\delta)}\exp\left(\alpha\int_{T_1}^{T_2} \frac{a(s)}{\exp\left(\int_0^s b(r)x(r;\alpha,\beta_*+\delta)dr\right)}ds\right) \\
&< \frac{1}{2k}\exp\left((\beta_*+1)\int_{T_1}^{\infty} a(s)e^{-kB(s)}ds\right) < \frac{1}{2k}\cdot 2 = \frac{1}{k}.
\end{aligned}$$

This is an obvious contradiction. It follows that $x(t;\alpha,\beta_*+\delta) > k$ on $[0,\infty)$, and so $\lim_{t\to\infty} x(t;\alpha,\beta_*+\delta) = \text{const} \geqslant k$. Further by (4) we get $\lim_{t\to\infty} y(t;\alpha,\beta_*+\delta) = 0$. So $\beta_*+\delta \in \underline{S}$. This is, as before, a contradiction to the definition of $\beta_* = \inf\overline{S}$. So $\lim_{t\to\infty} x(t;\alpha,\beta_*) = 0$ as desired. This completes the proof of Step 2.

Step 3. We claim that $\lim_{t\to\infty} y(t;\alpha,\beta^*) = 0$ *and* $\lim_{t\to\infty} x(t;\alpha,\beta^*) = 0$. In fact, if $\lim_{t\to\infty} y(t;\alpha,\beta^*) > 0$, then, as in Step 1, we can find that, for sufficiently small $\delta > 0$, $\lim_{t\to\infty} y(t;\alpha,\beta^*-\delta) > 0$ and $\lim_{t\to\infty} x(t;\alpha,\beta^*-\delta) = 0$, that is, $\beta^*-\delta \in \overline{S}$. This is a contradiction; and so, $\lim_{t\to\infty} y(t;\alpha,\beta^*) = 0$ as desired.

Next suppose to the contrary that $\lim_{t\to\infty} x(t;\alpha,\beta^*) > 0$. Then, as in Step 1, we can find that, for sufficiently small $\delta > 0$, $\lim_{t\to\infty} x(t;\alpha,\beta^*+\delta) > 0$ and $\lim_{t\to\infty} y(t;\alpha,\beta^*+\delta) = 0$, that is, $\beta^*+\delta \in \underline{S}$. This is again a contradiction; and so, $\lim_{t\to\infty} y(t;\alpha,\beta^*) = 0$.

Step 4. The final step. We claim that $\lim_{t\to\infty} x(t;\alpha,\beta) = \lim_{t\to\infty} y(t;\alpha,\beta) = 0$ for $\beta \in [\beta^*,\beta_*]$. In fact, for $\beta \in [\beta^*,\beta_*]$, Lemma 4 and Remark 5 imply that

$$0 < x(t;\alpha,\beta) \leqslant x(t;\alpha,\beta^*),$$

and

$$0 < y(t;\alpha,\beta) \leqslant y(t;\alpha,\beta_*).$$

So by the results of Steps 1 and 3, we find that $\lim_{t\to\infty} x(t;\alpha,\beta) = \lim_{t\to\infty} y(t;\alpha,\beta) = 0$.

This completes the proof of Theorem 1. □

REMARK 10. From the close look at the arguments in the paper, we find that all results in this paper still hold if condition (A_2) is replaced by

$$0 < \liminf_{t\to\infty} \frac{a(t)}{t^{-1}} \leqslant \limsup_{t\to\infty} \frac{a(t)}{t^{-1}} < \infty;$$

and

$$0 < \liminf_{t\to\infty} \frac{b(t)}{t^{-1}} \leqslant \limsup_{t\to\infty} \frac{b(t)}{t^{-1}} < \infty.$$

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