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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF IMPULSIVE NEUTRAL DIFFERENTIAL EQUATIONS WITH CONSTANT JUMPS

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Abstract. In this paper, we investigate the asymptotic behavior of solutions for a class of mixed type impulsive neutral delay differential equations with constant jumps. Sufficient conditions are given to guarantee that every non-oscillatory solution of the system tends to zero as $t \to \infty$. An example illustrating the result is also presented.

1. Introduction

The asymptotic behavior of solutions of neutral delay differential equations has been studied by two basic methods, by construction of Lyapunov functionals, see [1]–[5] and by considering the asymptotic behavior of non-oscillatory and oscillatory solutions respectively, for example, see [6, 7] and the references therein.

The theory of impulsive differential equations is not only richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modeling of many real world phenomena, see the monographs [8, 9, 10].

In [11] Jiang and Shen investigated the following nonlinear neutral delay differential equation with constant impulsive jumps and forced term

$$\begin{cases} [x(t)-px(t-\tau)]' + \sum_{i=1}^{n} q_i(t) f(x(t-\sigma_i)) = h(t), \quad t \neq t_k, \\ x(t_k^+) - x(t_k^-) = \alpha_k, \quad k \in \mathbb{Z}_+, \end{cases} \tag{1.1}$$

and derived that every non-oscillatory/oscillatory solution tends to zero as $t \longrightarrow \infty$. These results were improved in [12].

In [13] Jiang and Sun considered the asymptotic behavior of every non-oscillatory/ oscillatory solution for the following forced nonlinear neutral differential equation in first-order Euler form with constant impulsive jumps and unbounded delay

$$\begin{cases} [x(t)-C(t)x(\gamma(t))]' + \sum_{i=1}^{n} \frac{P_i(t)}{t} f(x(\beta_i t)) = h(t), \quad t \neq t_k, \\ x(t_k^+) - x(t_k^-) = \alpha_k, \quad k \in \mathbb{Z}_+, \end{cases} \tag{1.2}$$

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and proved that every non-oscillatory/oscillatory solution tends to zero as $t \to \infty$.

The aim of this paper is to investigate the asymptotic behavior of solutions of the following mixed type impulsive neutral differential equation with constant jumps:

$$\begin{cases} [x(t) - bx(t-\tau) - C(t)x(\gamma(t))]' \\ \qquad + \sum_{i=1}^{n} \left\{ q_i(t) f(x(t-\sigma_i)) + \frac{P_i(t)}{t} g(x(\beta_i t)) \right\} = h(t), \quad t \neq t_k, \\ x(t_k^+) - x(t_k^-) = \alpha_k, \quad k = 1,2,3,\ldots, \end{cases} \tag{1.3}$$

where b, τ, σ_i are given constants such that $\tau > 0$, $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n$, γ is monotone increasing for $t > t_0$ and $\gamma(t) \leqslant t$, $0 < \beta_i < 1$ satisfying $\beta_1 < \beta_2 < \cdots < \beta_n$, $i \in \Lambda$; C, q_i, P_i, $h \in PC([t_0,\infty),\mathbb{R})$ where $\Lambda = \{1,2,\ldots,n\}$, $t_0 > 0$, $\mathbb{R}$ denotes the set of real numbers, for $J \subset \mathbb{R}$, $PC(J,\mathbb{R})$ denotes the set of all functions $\varphi : J \to \mathbb{R}$ such that φ is continuous everywhere except at some points t_k, $k \in \mathbb{Z}_+$ and the limits $\varphi(t_k^+) = \lim_{t \to t_k^+} \varphi(t)$, $\varphi(t_k^-) = \lim_{t \to t_k^-} \varphi(t)$ exist with $\varphi(t_k) = \varphi(t_k^-)$, the sequence $\{t_k\}$, $k \in \mathbb{Z}_+$ is impulsive points satisfying $0 < t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots \to \infty$ as $k \to \infty$, the notation $\{\alpha_k\}$, $k \in \mathbb{Z}_+$ is a constant impulsive sequence, and $\mathbb{Z}_+$ denotes the set of positive integers. Notice that problem (1.3) reduces to the problem (1.1) for $C = 0$, $P_i = 0$ and to problem (1.2) for $b = 0, q_i = 0$.

In this paper we derive sufficient conditions such that every non-oscillatory solution of system (1.3) tends to zero as $t \to \infty$. The rest of the paper is organized as follows. In the next section, we present some preliminaries. In Section 3, we give and prove our main result by a technique of construction. Finally, in Section 4, as an application of our results, we present an example to illustrate the usefulness of the obtained results.

2. Preliminaries

Before going to prove our main result, we would like to state the hypotheses. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that:

(H_1) There exists two constants $M > 0$ and $N > 0$ such that

$$|f(x)| \leqslant M|x| \quad \text{for } x \in \mathbb{R}; \quad xf(x) > 0 \quad \text{for} \quad x \neq 0,$$

and

$$|g(x)| \leqslant N|x| \quad \text{for } x \in \mathbb{R}; \quad xg(x) > 0 \quad \text{for} \quad x \neq 0.$$

(H_2) For all $0 < t_0 \leqslant t$, the integral

$$G(t) = \int_t^\infty h(s)ds \quad \text{is convergent.}$$

(H_3) $t_k - \tau$, $\gamma(t_k)$ are not impulsive points for all $k \in \mathbb{Z}_+$ and the limit $\lim_{t\to\infty} \alpha_k^+ = 0$ where $\alpha_k^+ = \max\{\alpha_k, 0\}$.

To set the initial function, we define $\rho_1 = \max\{\tau, \sigma_n\}$, $\rho_2 = \min\left\{\frac{\gamma(t_0)}{t_0}, \beta_1\right\}$, $0 < \rho = \min\{t_0 - \rho_1, \rho_2 t_0\}$. Also, we define an initial value function

$$x(t) = \varphi(t), \quad t \in [\rho, t_0], \tag{2.1}$$

where $\varphi \in PC([\rho, t_0], \mathbb{R}) = \{\varphi : [\rho, t_0] \to R : \varphi$ is continuous everywhere except at some points t_k, $k \in \mathbb{Z}_+$ and $\varphi(t_k^-) = \lim_{t \to t_k^-} \varphi(t)$, $\varphi(t_k^+) = \lim_{t \to t_k^+} \varphi(t)$ exist with $\varphi(t_k^-) = \varphi(t_k)\}$.

The solution of problem (1.3) is defined as follows.

DEFINITION 1. A function $x(t)$ is said to be a solution of system (1.3) satisfying the initial value condition (2.1) if

(1). $x(t) = \varphi(t)$ for $0 < \rho \leqslant t \leqslant t_0$ and $x(t)$ is continuous for $t \geqslant t_0$, $t \neq t_k$, $k \in \mathbb{Z}_+$;

(2). $x(t) - bx(t-\tau) - C(t)x(\gamma(t))$ is continuously differentiable for $t > t_0$, $t \neq t_k$, $k \in \mathbb{Z}_+$ and satisfies equation (1.3);

(3). $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$ for all $k \in \mathbb{Z}_+$ and satisfies equation (1.3).

The oscillatory and non-oscillatory solutions of system (1.3) are defined as follows.

DEFINITION 2. A solution $x(t)$ of system (1.3) is said to be eventually positive (negative) if it is positive (negative) for all sufficiently large t. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be non-oscillatory.

Throughout this paper, we introduce the function $H(t)$ defined by

$$H(t) = \begin{cases} \displaystyle\int_t^\infty h(s)ds, & t \in (t_k, t_{k+1}], \\ \displaystyle\int_t^\infty h(s)ds + \alpha_{k-1}^+, & t = t_k,\ k \in Z_+, \end{cases} \tag{2.2}$$

where $\alpha_k^+ = \max\{\alpha_k, 0\}$, $k \in \mathbb{Z}_+ \cup \{0\}$ and $\alpha_0 = 0$.

3. Main result

THEOREM 1. *Let the conditions (H_1)–(H_3) hold. Assume that for some $\xi_1, \xi_2 > 0$, there exist two constants $\theta_1, \theta_2 > 0$ such that*

$$|f(x)| \geqslant \theta_1,\ |x| \geqslant \xi_1 \quad \textit{and} \quad |g(x)| \geqslant \theta_2,\ |x| \geqslant \xi_2. \tag{3.1}$$

Suppose that

$$|b| = B < 1, \quad \lim_{t \to \infty} |C(t)| = C < 1 \textit{ such that } B + C < 1, \tag{3.2}$$

and

$$\sum_{i=1}^{n} q_i(t+\sigma_i) \geqslant 0, \quad \int_{t_0}^{\infty} \sum_{i=1}^{n} q_i(s+\sigma_i)ds = \infty, \tag{3.3}$$

$$\sum_{i=1}^{n} \frac{P_i(t/\beta_i)}{t} \geqslant 0, \quad \int_{t_0}^{\infty} \sum_{i=1}^{n} \frac{P_i(s/\beta_i)}{s} ds = \infty. \tag{3.4}$$

In addition, for sufficiently large t, assume that there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\sum_{\sigma_i<r} \int_{t-r}^{t-\sigma_i} q_i^-(s+\sigma_i)ds + \sum_{\sigma_i>r} \int_{t-\sigma_i}^{t-r} q_i^+(s+\sigma_i)ds \leqslant \lambda_1, \tag{3.5}$$

$$\sum_{\beta_i<u} \int_{\beta_i t}^{ut} \left(\frac{P_i(s/\beta_i)}{s}\right)^+ ds + \sum_{\beta_i>u} \int_{ut}^{\beta_i t} \left(\frac{P_i(s/\beta_i)}{s}\right)^- ds \leqslant \lambda_2, \tag{3.6}$$

where a fixed constant $u \in (0,\beta_n]$, $\lambda < (1-|b|-C)/(M+N)$ and $\lambda = \max\{\lambda_1,\lambda_2\}$, $r \in [0,\sigma_n]$, $q_i^+(s) = \max\{q_i(s),0\}$, $q_i^-(s) = \max\{-q_i(s),0\}$ and

$$((P_i(s/\beta_i))/(s))^+ = \max\{(P_i(s/\beta_i))/(s),0\},$$

$$((P_i(s/\beta_i))/(s))^- = \max\{(-P_i(s/\beta_i))/(s),0\}.$$

Then every non-oscillatory solution of equation (1.3) tends to zero as $t \to \infty$.

Proof. Firstly, we choose a positive integer N sufficiently large enough such that there exists a positive integer m large enough satisfying $\gamma(t_m)$, $t-\tau > t_N$ and (3.5)–(3.6) hold for $t \geqslant t_N$, where N is the largest subscript satisfying $\gamma(t_m)$, $t-\tau > t_N$. Let $x(t)$ be a non-oscillatory solution of equation (1.3). Without loss of generallity, we will assume that $x(t)$ is eventually positive solution. For the case $x(t)$ is eventually negative, the proof is similar and we omit it. Let $x(t) > 0$ for $t \geqslant t_N$. For all $t \geqslant t_N$, we set

$$\alpha(t) = \begin{cases} \alpha_{N^t}^+; & t > t_{N+1}, \\ 0; & t \in [t_N, t_{N+1}], \end{cases} \tag{3.7}$$

where N^t corresponds to the largest subscript of impulsive points in the interval $t \geqslant t_N$.

Next, we define

$$\begin{aligned} y(t) &= x(t) - bx(t-\tau) - C(t)x(\gamma(t)) \\ &\quad - \sum_{i=1}^{n} \left[\int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i) f(x(s))ds + \int_{\beta_i t}^{ut} \frac{P_i(s/\beta_i)}{s} g(x(s))ds \right] \\ &\quad + H(t) - \alpha(t), \end{aligned} \tag{3.8}$$

where $H(t)$ is as in (2.2). Now, we derive the derivative of a function $\alpha(t)$. For $t \neq t_k$, we choose Δt sufficiently small such that there is no impulsive point in the interval $(t, t+\Delta t)$. Then we have

$$\alpha'(t) = \lim_{\Delta t \to 0} \frac{\alpha(t+\Delta t) - \alpha t}{\Delta t} = 0, \quad t \neq t_k.$$

From (3.8) and (H_2)–(H_3), we get that for $t \neq t_k$, $t \neq t_k + \sigma_i$, $t \neq t_k/\beta_i$, $i \in \Lambda$, $k \in \mathbb{Z}_+$,

$$\begin{aligned} y'(t) &= [x(t) - bx(t-\tau) - C(t)x(\gamma(t))]' - \sum_{i=1}^{n} \Big[q_i(t-r+\sigma_i) f(x(t-r)) \\ &\quad - q_i(t) f(x(t-\sigma_i)) + \frac{P_i(ut/\beta_i)}{rt} g(x(ut)) r - \frac{P_i(t)}{\beta_i t} g(x(\beta_i t))\beta_i \Big] - h(t) \\ &= [x(t) - bx(t-\tau) - C(t)x(\gamma(t))]' + \sum_{i=1}^{n} \Big[q_i(t) f(x(t-\sigma_i)) + \frac{P_i(t)}{t} g(x(\beta_i t)) \Big] \\ &\quad - \sum_{i=1}^{n} \Big[q_i(t-r+\sigma_i) f(x(t-r)) + \frac{P_i(ut/\beta_i)}{t} g(x(ut)) \Big] - h(t) \\ &= - \sum_{i=1}^{n} \Big[q_i(t-r+\sigma_i) f(x(t-r)) + \frac{P_i(ut/\beta_i)}{t} g(x(ut)) \Big] \leqslant 0. \end{aligned} \tag{3.9}$$

For $t = t_k$, $k = N+1, N+2, \ldots$, we have

$$H(t_k^+) - H(t_k) = -\alpha_{k-1}^+. \tag{3.10}$$

In addition, for $t = t_k$, $k = N+1, N+2, \ldots$, we obtain

$$\begin{aligned} y(t_k^+) - y(t_k) &= x(t_k^+) - bx(t_k^+ - \tau) - C(t_k^+)x(\gamma(t_k^+)) \\ &\quad - \sum_{i=1}^{n} \Big[\int_{t_k^+ - \sigma_i}^{t_k^+ - r} q_i(s+\sigma_i) f(x(s)) ds + \int_{\beta_i t_k^+}^{r t_k^+} \frac{P_i(s/\beta_i)}{s} g(x(s)) ds \Big] \\ &\quad + H(t_k^+) - \alpha(t_k^+) - x(t_k) + bx(t_k - \tau) + C(t_k)x(\gamma(t_k)) \\ &\quad + \sum_{i=}^{n} \Big[\int_{t_k - \sigma_i}^{t_k - r} q_i(s+\sigma_i) f(x(s)) ds + \int_{\beta_i t_k}^{r t_k} \frac{P_i(s/\beta_i)}{s} g(x(s)) ds \Big] \\ &\quad - H(t_k) + \alpha(t_k) \\ &= x(t_k^+) - x(t_k) + H(t_k^+) - H(t_k) - \alpha(t_k^+) + \alpha(t_k) \\ &= \alpha_k - \alpha_{k-1}^+ - \alpha_k^+ + \alpha_{k-1}^+ = \alpha_k - \alpha_k^+ \leqslant 0. \end{aligned}$$

From (3.9) and the above inequality, we have $y(t)$ is nonincreasing on $[\frac{t_N}{u} + r, \infty)$.

Now, we will claim that $y(t)$ is convergence. Let $L = \lim_{t\to\infty} y(t)$, we will show that $L \in \mathbb{R}$. Otherwise, $L = -\infty$, then $x(t)$ is unbounded. Indeed, if $x(t)$ is bounded then it follows from (3.8) and (H_1) that for some constants $D>0$, $E>0$,

$$\begin{aligned}
y(t) &= x(t)-bx(t-\tau)-C(t)x(\gamma(t))\\
&\quad -\sum_{i=1}^{n}\left[\int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i)f(x(s))ds+\int_{\beta_i t}^{ut}\frac{P_i(s/\beta_i)}{s}g(x(s))ds\right]+H(t)-\alpha(t)\\
&\geqslant x(t)-bx(t-\tau)-C(t)x(\gamma(t))\\
&\quad -D\left[\sum_{\sigma_i<r}\int_{t-r}^{t-\sigma_i} q_i^-(s-\sigma_i)ds+\sum_{\sigma_i>r}\int_{t-\sigma_i}^{t-r} q_i^+(s+\sigma_i)ds\right]\\
&\quad -E\left[\sum_{\beta_i<u}\int_{\beta_i t}^{ut}\left(\frac{P_i(s/\beta_i)}{s}\right)^+ds+\sum_{\beta_i>u}\int_{ut}^{\beta_i t}\left(\frac{P_i(s/\beta_i)}{s}\right)^-ds\right]+H(t)-\alpha(t).
\end{aligned}$$

From the conditions $(H_2)-(H_3)$ and (3.5)–(3.6), we have that $L=-\infty\geqslant K$. This is a contradiction and then $x(t)$ is unbounded.

On the other hand, from $x(t)$ is unbounded and $\lim_{t\to\infty}y(t)=-\infty$, we can choose $t^*\geqslant\max\{t_N+\sigma_n, t_N+\tau, t_N/\beta_1, \gamma(t_m)\}$ such that $y(t^*)-H(t^*)+\alpha(t^*)<0$ and $x(t^*)=\max\{x(t): t_N\leqslant t\leqslant t^*\}$. Therefore, it follows from (3.5)–(3.6) that

$$\begin{aligned}
0 &> y(t^*)-H(t^*)+\alpha(t^*)\\
&= x(t^*)-bx(t^*-\tau)-C(t^*)x(\gamma(t))\\
&\quad -\sum_{i=1}^{n}\left[\int_{t^*-\sigma_i}^{t^*-r} q_i(s+\sigma_i)f(x(s))ds+\int_{\beta_i t^*}^{ut^*}\frac{P_i(s/\beta_i)}{s}g(x(s))ds\right]\\
&\quad +H(t^*)-\alpha(t^*)-H(t^*)+\alpha(t^*)\\
&\geqslant x(t^*)-|b|x(t^*)-Cx(t^*)\\
&\quad -Mx(t^*)\left[\sum_{\sigma_i<r}\int_{t^*-r}^{t^*-\sigma_i} q_i^-(s-\sigma_i)ds+\sum_{\sigma_i>r}\int_{t^*-\sigma_i}^{t^*-r} q_i^+(s+\sigma_i)ds\right]\\
&\quad -Nx(t^*)\left[\sum_{\beta_i<u}\int_{\beta_i t^*}^{ut^*}\left(\frac{P_i(s/\beta_i)}{s}\right)^+ds+\sum_{\beta_i>u}\int_{ut^*}^{\beta_i t^*}\left(\frac{P_i(s/\beta_i)}{s}\right)^-ds\right]\\
&\geqslant x(t^*)\{1-|b|-C-M\lambda_1-N\lambda_2\}\\
&\geqslant x(t^*)\{1-|b|-C-\lambda(M+N)\}>0,
\end{aligned}$$

which is a contradiction and therefore $L\in\mathbb{R}$.

Integrating both sides of (3.9) from $t_N/\beta_1+\sigma_n$ to t, we obtain

$$
\begin{aligned}
&\int_{\frac{t_N}{\beta_1}+\sigma_n}^{t}\left[\sum_{i=1}^{n} q_i(s-r+\sigma_i)f(x(s-r))+\sum_{i=1}^{n}\frac{P_i(us/\beta_i)}{s}g(x(us))\right]ds\\
&=-\int_{\frac{t_N}{\beta_1}+\sigma_n}^{t} y'(s)ds\\
&=-y(t)+y\left(\frac{t_N}{\beta_1}+\sigma_n\right)+\sum_{\frac{t_N}{\beta_1}+\sigma_n<t_k\leqslant t}[y(t_k^+)-y(t_k)]\\
&<y\left(\frac{t_N}{\beta_1}+\sigma_n\right)-L. \qquad (3.11)
\end{aligned}
$$

From (3.3), (3.4) and (3.11), we have

$$
f(x(t))\in L^1\left(\left[\frac{t_N}{\beta_1}+\sigma_n,\infty\right),\mathbb{R}\right) \quad\text{and}\quad g(x(t))\in L^1\left(\left[\frac{t_N}{\beta_1}+\sigma_n,\infty\right),\mathbb{R}\right).
$$

Hence, $\liminf_{t\to\infty} f(x(t))=0$ and $\liminf_{t\to\infty} g(x(t))=0$.

Now, we claim that

$$
\liminf_{t\to\infty} x(t)=0. \qquad (3.12)
$$

Let $\{S_m\}$ be a sequence such that $S_m\to\infty$ as $m\to\infty$ with $\lim_{m\to\infty} f(x(S_m))=0$ and $\lim_{m\to\infty} g(x(S_m))=0$. We must show that $\liminf_{m\to\infty} x(S_m)=c=0$. If $c>0$, then there exists a subsequence $\{S_{m_k}\}$ of $\{S_m\}$ such that $x(S_{m_k})\geqslant c/2$ for some k sufficiently large. From (3.1), we have $f(x(S_{m_k}))\geqslant\theta_{1_c}$ and $g(x(S_{m_k}))\geqslant\theta_{2_c}$ for some θ_{1_c}, $\theta_{2_c}>0$ and sufficiently large k. Then it is a contradiction because $\lim_{k\to\infty} f(x(S_{m_k}))=0$ and $\lim_{k\to\infty} g(x(S_{m_k}))=0$. Thus, (3.12) holds.

Observe that (3.11) implies

$$
\int_{t_0}^{\infty}\sum_{i=1}^{n} q_i(s-r+\sigma_i)f(x(s-r))ds+\int_{t_0}^{\infty}\sum_{i=1}^{n}\frac{P_i(rs/\beta_i)}{s}f(x(sr))ds<\infty. \qquad (3.13)
$$

Set

$$
z(t)=y(t)+\sum_{i=1}^{n}\left[\int_{t-\sigma_i}^{t-r} q_i(s+\sigma_i)f(x(s))ds+\int_{\beta_i t}^{ut}\frac{P_i(s/\beta_i)}{s}f(x(s))\right]ds-H(t)+\alpha(t).
$$

From (H_2)–(H_3) and (3.13), we have

$$
\lim_{t\to\infty} z(t)=\mu \quad \text{exists.}
$$

Then, from (3.8), we obtain

$$
\lim_{t\to\infty}[x(t)-bx(t-\tau)-C(t)x(\gamma(t))]=\mu. \qquad (3.14)
$$

Next, we will show that $\lim_{t\to\infty} x(t) = 0$. From the condition (3.2), we can choose a sufficiently large T_1 such that $|b| + |C(t)| < 1$ for $t > T_1$. Set

$$\eta = \limsup_{t\to\infty} x(t).$$

By $\liminf_{t\to\infty} x(t) = 0$, we get that there exist two sequences $\{u_n\}$ and $\{v_n\}$ with $u_n \to \infty, v_n \to \infty$ as $n \to \infty$ such that

$$\lim_{t\to\infty} x(u_n) = 0, \quad \lim_{t\to\infty} x(v_n) = \eta.$$

For all $t > T_1$, we divide the following nine possible cases to discuss.

Case 1. If $b = 0$ and $\lim_{t\to\infty} C(t) = 0$ for $t > T_1$, then we get

$$\lim_{t\to\infty} x(t) = \mu = 0.$$

Since $\lim_{t\to\infty} x(t)$ exists and $\liminf_{t\to\infty} x(t) = 0$.

Case 2. If $b = 0$ and $-1 < C(t) < 0$ for $t > T_1$, then we have

$$\mu = \lim_{n\to\infty} [x(u_n) - C(u_n)x(\gamma(u_n))] \leqslant C\eta,$$

and

$$\mu = \lim_{n\to\infty} [x(v_n) - C(v_n)x(\gamma(v_n))] \geqslant \eta,$$

which imply that $\eta \leqslant C\eta$. It follows from $\eta \geqslant 0$ and $0 < C < 1$ that $\eta = 0$. This shows $\lim_{t\to\infty} x(t) = 0$.

Case 3. If $b = 0$ and $0 < C(t) < 1$ for $t > T_1$, then we obtain

$$\mu = \lim_{n\to\infty} [x(u_n) - C(u_n)x(\gamma(u_n))] \leqslant 0,$$

and

$$\mu = \lim_{n\to\infty} [x(v_n) - C(v_n)x(\gamma(v_n))] \geqslant \eta - C\eta,$$

which imply that $\eta(1 - C) \leqslant 0$. It follows that $\eta \geqslant 0$ and $0 < C < 1$ which imply $\eta = 0$. This shows $\lim_{t\to\infty} x(t) = 0$.

Case 4. If $\lim_{t\to\infty} C(t) = 0$ and $-1 < b < 0$ for $t > T_1$, then we get

$$\mu = \lim_{n\to\infty} [x(u_n) - bx(u_n - \tau)] \leqslant B\eta,$$

and

$$\mu = \lim_{n\to\infty} [x(v_n) - bx(v_n - \tau)] \geqslant \eta,$$

which imply that $\eta \leqslant B\eta$. It follows that $\eta \geqslant 0$ and $0 < B < 1$ which imply $\eta = 0$. This shows $\lim_{t\to\infty} x(t) = 0$.

Case 5. If $\lim_{t\to\infty} C(t) = 0$ and $0 < b < 1$ for $t > T_1$, then we obtain

$$\mu = \lim_{n\to\infty} [x(u_n) - bx(u_n - \tau)] \leqslant 0,$$

and

$$\mu = \lim_{n\to\infty}[x(v_n) - bx(v_n - \tau)] \geqslant \eta - B\eta,$$

which imply that $\eta(1-B) \leqslant 0$. It follows that $\eta \geqslant 0$ and $0 < B < 1$ which yield $\eta = 0$. This means $\lim_{t\to\infty} x(t) = 0$.

Case 6. If $0 < b < 1$ and $-1 < C(t) < 0$ for $t > T_1$, then we have

$$\mu = \lim_{n\to\infty}[x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leqslant C\eta,$$

and

$$\mu = \lim_{n\to\infty}[x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geqslant \eta - B\eta,$$

which imply that $\eta[1-(B+C)] \leqslant 0$. It follows that $\eta \geqslant 0$ and $0 < B + C < 1$ which lead to $\eta = 0$. This implies $\lim_{t\to\infty} x(t) = 0$.

Case 7. If $0 < b < 1$ and $0 < C(t) < 1$ for $t > T_1$, then we obtain

$$\mu = \lim_{n\to\infty}[x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leqslant 0,$$

and

$$\mu = \lim_{n\to\infty}[x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geqslant \eta - B\eta + C\eta,$$

which imply that $\eta[1-(B+C)] \leqslant 0$. It follows that $\eta \geqslant 0$ and $0 < B + C < 1$ which imply $\eta = 0$. This shows $\lim_{t\to\infty} x(t) = 0$.

Case 8. If $-1 < b < 0$ and $-1 < C(t) < 0$ for $t > T_1$, then we get

$$\mu = \lim_{n\to\infty}[x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leqslant B\eta + C\eta,$$

and

$$\mu = \lim_{n\to\infty}[x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geqslant \eta,$$

which imply that $\eta[1-(B+C)] \leqslant 0$. It follows that $\eta \geqslant 0$ and $0 < B + C < 1$. Thus $\eta = 0$. This shows $\lim_{t\to\infty} x(t) = 0$.

Case 9. If $-1 < b < 0$ and $0 < C(t) < 1$ for $t > T_1$, then we have

$$\mu = \lim_{n\to\infty}[x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leqslant B\eta,$$

and

$$\mu = \lim_{n\to\infty}[x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geqslant \eta - C\eta,$$

which imply that $\eta[1-(B+C)] \leqslant 0$. It follows that $\eta \geqslant 0$ and $0 < B + C < 1$. Thus $\eta = 0$. This shows $\lim_{t\to\infty} x(t) = 0$.

Therefore, we conclude that $\lim_{t\to\infty} x(t) = 0$, and so the proof is completed. □

4. An Example

EXAMPLE 1. Consider the following mixed type neutral differential equation with impulsive perturbations

$$\begin{cases} \left[x(t)-\frac{1}{4}x\left(t-\frac{1}{3}\right)-C(t)x\left(\frac{2t}{e}\right)\right]'+\left(\frac{3}{t+2}\right)\left(\frac{2x(t-1)}{1+(x(t-1))^2}\right) \\ \quad +\frac{1}{2\left(\ln\left(\frac{1}{7}t\right)-1\right)}3x\left(\frac{t}{7e}\right)+\left(\frac{2}{t+2}\right)\left(\frac{2x(t-2)}{1+(x(t-2))^2}\right) \\ \quad +\frac{1}{3\left(\ln\left(\frac{1}{5}t\right)-1\right)}3x\left(\frac{t}{5e}\right)+\left(\frac{1}{t+2}\right)\left(\frac{2x(t-3)}{1+(x(t-3))^2}\right) \\ \quad +\frac{1}{4\left(\ln\left(\frac{1}{3}t\right)-1\right)}3x\left(\frac{t}{3e}\right)=\frac{1}{t^3}, \qquad t\geqslant t_0=e,\ t\neq t_k, \\ x(t_k^+)-x(t_k)=(-1)^k\frac{2}{k}, \qquad t_k=k+2,\ k\in\mathbb{Z}_+, \end{cases} \tag{4.1}$$

where

$$C(t)=\frac{(k+2)\lfloor t\rfloor}{2k^2+2k+4}, \quad t\in(k,k+1],\ k=2,3,4,\ldots.$$

Here $b=1/4,\ \tau=1/3,\ \gamma(t)=2t/e,\ f(x)=2x/(1+x^2),\ g(x)=3x,\ h(t)=1/t^3$, $q_1(t)=3/(t+2),\ q_2(t)=2/(t+2),\ q_3(t)=1/(t+2),\ P_1(t)=1/\left(2\left(\ln\left(\frac{1}{7}t\right)-1\right)\right)$, $P_2(t)=1/\left(3\left(\ln\left(\frac{1}{5}t\right)-1\right)\right),\ P_3(t)=1/\left(4\left(\ln\left(\frac{1}{3}t\right)-1\right)\right),\ \sigma_1=1,\ \sigma_2=2,\ \sigma_3=3$, $\beta_1=1/(7e),\ \beta_2=1/(5e),\ \beta_3=1/(3e)$, when we choose $M=2,\ N=3,\ r=5/2\in[0,3],\ u=1/(6e)\in(0,1/(3e)]$. We can find that

(i) $|f(x)|=\left|\frac{2x}{1+x^2}\right|\leqslant 2|x|,\ x\in\mathbb{R},\quad x\left(\frac{2x}{1+x^2}\right)>0$ for $x\neq 0$ and

$$|g(x)|=|3x|\leqslant 3|x|,\ x\in\mathbb{R},\quad x(3x)>0 \text{ for } x\neq 0;$$

(ii) $G(t)=\int_t^\infty \frac{1}{s^3}ds=\frac{1}{2t^2}$ is convergent for $t\geqslant e$;

(iii) $t_k-(1/3)$ and $(2/e)t_k$ are not impulsive points for all $k\in\mathbb{Z}_+$ and $\lim_{k\to\infty}\alpha_k^+=\lim_{k\to\infty}\frac{1}{k}=0$;

(iv) $|b|=\frac{1}{4}=B<1,\ \lim_{t\to\infty}|C(t)|=\frac{1}{2}=\mu<1$ with $B+C=\frac{3}{4}<1$;

(v) $\sum_{i=1}^{3}q_i(t+\sigma_i)=\frac{3}{t+3}+\frac{2}{t+4}+\frac{1}{t+5}\geqslant 0$ for $t\geqslant e$ and

$$\int_e^\infty q_i(s+\sigma_i)ds=\int_e^\infty\left[\frac{3}{s+3}+\frac{2}{s+4}+\frac{1}{s+5}\right]ds=\infty;$$

(vi) $\sum_{i=1}^{3} \frac{P_i(t/\beta_i)}{t} = \frac{13}{12t\ln t} \geqslant 0$ for $t \geqslant e$ and

$$\int_e^{\infty} \sum_{i=1}^{3} \frac{P_i(s/\beta_i)}{s} ds = \int_e^{\infty} \left[\frac{13}{12s\ln s}\right] ds = \infty;$$

(vii) For large enough t, there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\sum_{\sigma_i<r} \int_{t-r}^{t-\sigma_i} q_i^-(s+\sigma_i)ds + \sum_{\sigma_i>r} \int_{t-\sigma_i}^{t-r} q_i^+(s+\sigma_i)ds = \int_{t-3}^{t-5/2} \frac{1}{s+5} ds$$
$$= \ln(s+5)\big|_{t-3}^{t-5/2} \longrightarrow 0;$$

and

$$\sum_{\beta_i<u} \int_{\beta_i t}^{ut} \left(\frac{P_i(s/\beta_i)}{s}\right)^+ ds + \sum_{\beta_i>u} \int_{ut}^{\beta_i t} \left(\frac{P_i(s/\beta_i)}{s}\right)^- ds = \int_{t/(7e)}^{t/(6e)} \frac{1}{2s\ln s} ds$$
$$= \frac{1}{2} \ln\ln(s)\big|_{t/(7e)}^{t/(6e)} \longrightarrow 0;$$

by L'Höpital's rule. Hence, by $(i)-(vii)$ all assumptions of Theorem 1 are satisfied. Therefore, we conclude that every non-oscillatory solution of (4.1) tends to zero as $t \longrightarrow \infty$.

REMARK 1. In this paper, by combining the impulsive neutral differential equations with bounded and unbounded delays (1.1) and (1.2), respectively, an asymptotic behavior of non-oscillatory solutions of equation (1.3) is proved. Notice that in [11] and [13] the authors proved the asymptotic behavior of oscillatory solutions by assuming that there exists a critical point ξ such that $y'(\xi) = 0$ (on page 11 of [11] and 9911 of [13], respectively), and $y(\xi)$ is the extremum value for oscillatory function $y' \in PC(\mathbb{R}_+, \mathbb{R})$ which does not satisfy the Definition 2.

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