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A COUPLED HYBRID FIXED POINT THEOREM INVOLVING THE SUM OF TWO COUPLED OPERATORS IN A PARTIALLY ORDERED BANACH SPACE WITH APPLICATIONS

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Abstract. In this paper we prove a coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space on the lines of Dhage [Math. Student **61** (1992), 81-88] which improve a coupled hybrid fixed point theorem of Dhage [J. Fixed Point Theory Appl. **19** (2017), 3231–3264] under a little stronger condition and correct and improve the hybrid fixed point theorems of Yang *et. al* [J. Fixed Point Theory Appl. **19** (2017), 1661–1678] involving the sum of two operators under weaker conditions. We apply our main abstract coupled hybrid fixed point result to a nonlinear first order coupled linearly perturbed hybrid differential equations with the periodic boundary conditions for proving the existence and approximation of solutions under certain mixed hybrid conditions. The abstract existence result of the coupled periodic boundary value problems is also illustrated by furnishing a numerical example.

1. Introduction

Throughout this paper, unless otherwise mentioned, let $(E, \leqslant, \|\cdot\|)$ denote a partially ordered Banach space with the order relation $\leqslant$ and the norm $\|\cdot\|$ defined on it. Given an operator $\mathscr{T} : E \times E \to E$, consider a pair of operator equations

$$x = \mathscr{T}(x,y) \tag{1.1}$$

and

$$y = \mathscr{T}(y,x) \tag{1.2}$$

which are called the coupled operator equations and the operator $\mathscr{T}$ involved in them is called the coupled operator on $E \times E$.

A pair (x^*, y^*) of elements in E is called a coupled fixed point the coupled operator $\mathscr{T}$ or a coupled solution of the coupled operator equations (1.1) and (1.2) if

$$x^* = \mathscr{T}(x^*, y^*) \quad \text{and} \quad y^* = \mathscr{T}(y^*, x^*).$$

A coupled fixed point (x^*, y^*) is called *unique comparable* if there does not exist another coupled fixed point (u^*, v^*) which is comparable to it. A coupled fixed point

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(x^*, y^*) is called *unique* if it is the only coupled solution of the coupled operator equations (1.1)–(1.2) in $E \times E$. Finally, a point (x^*, y^*) is called a *fixed point* if $x^* = y^*$, i.e., $x^* = \mathscr{T}(x^*, x^*)$ and $y^* = \mathscr{T}(y^*, y^*)$.

The coupled fixed point theorems for mixed monotone partially condensing coupled mappings on a partially ordered metric space guaranteeing the existence of coupled fixed points have been proved by Dhage [17] which includes the coupled fixed point theorems of Bhaskar and Laksmikatham [4], Berinde [3], Dhage and Dhage [21] and Dhage [16] as special cases. Bhaskar and Lakshmikathan [4] used a contraction type condition on the mixed monotone coupled operator $\mathscr{T}$ which is further generalized by Berinde [3] by generalizing the contraction condition to get the same conclusion via constructive method. However, Dhage [16] used a compactness type topological arguments on the mixed monotonic coupled operator $\mathscr{T}$ and obtained an algorithm for the coupled solution of the coupled operator equations (1.1)–(1.2). Sometimes it may happen that the mixed monotone operator $\mathscr{T}$ neither satisfies contraction condition nor the compactness type condition, but the splitting of the coupled operator $\mathscr{T}$ into two coupled operators $\mathscr{F}$ and $\mathscr{G}$ into the form $\mathscr{T} = \mathscr{F} + \mathscr{G}$ satisfy the above criteria. See Dhage [9, 10, 11, 12] and the references therein. So in this case it is interesting to establish the coupled hybrid fixed point theorem involving the sum of two operators in a partially ordered Banach space (cf. Dhage [13, 14, 15]).

The rest of the paper is organized as follows: Section 2 deals with the preliminaries and auxiliary results that will be used in the subsequent part of the paper. Section 3 deals with the main coupled hybrid fixed point theorem and its various consequences. Section 4 consists of coupled hybrid PBVPs and the related results to be used in the subsequent section of the paper. Finally an application of the abstract coupled hybrid fixed point theorem for proving the existence and approximation theorem is given in Section 5. We claim that the results of this paper are new to the literature on nonlinear analysis and applications.

2. Preliminaries and auxiliary results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered Banach space. Two elements x and y in E are said to be *comparable* if either the relation $x \preceq y$ or $y \succeq x$ holds. A non-empty subset C of E is called a *chain* or *totally ordered* if all the elements of C are comparable. It is known that E is *regular* if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E and $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikantham [24] and the references therein. Similarly a few details of a partially ordered normed linear space are given in Dhage [9] while orderings defined by different order cones are given in Deimling [7], Guo and Lakshmikantham [24], Heikkilä and Lakshmikantham [25], Carl and Heikkilä [5] and the references therein.

We need the following definitions (see Dhage [12, 13, 14, 15] and the references therein) in what follows.

A mapping $\mathscr{T} : E \to E$ is called *isotone* or *monotone nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ for all $x, y \in E$. Similarly, $\mathscr{T}$

is called *monotone nonincreasing* if $x \leqslant y$ implies $\mathscr{T}x \geqslant \mathscr{T}y$ for all $x, y \in E$. Finally, $\mathscr{T}$ is called *monotonic* or simply *monotone* if it is either monotone nondecreasing or monotone nonincreasing on E. A mapping $\mathscr{T} : E \to E$ is called *partially continuous* at a point $a \in E$ if for given $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(\mathscr{T}x, \mathscr{T}a) < \varepsilon$ whenever x is comparable to a and $d(x,a) < \delta$. $\mathscr{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered metric space E is called *partially bounded* if every chain C in S is bounded. A mapping $\mathscr{T}$ on a partially ordered metric space E into itself is called *partially bounded* if $\mathscr{T}(E)$ is a partially bounded subset of E. $\mathscr{T}$ is called *uniformly partially bounded* if all chains C in $\mathscr{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered metric space E is called *partially compact* if every chain C in S is a compact subset of E. A mapping $\mathscr{T} : E \to E$ is called *partially compact* if $\mathscr{T}(E)$ is a partially relatively compact subset of E. $\mathscr{T}$ is called *uniformly partially compact* if $\mathscr{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathscr{T}$ is called *partially totally bounded* if for any bounded subset S of E, $\mathscr{T}(S)$ is a partially totally bounded subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on E.

REMARK 2.1. Suppose that $\mathscr{T}$ is a nondecreasing operator on E into itself. Then $\mathscr{T}$ is a partially bounded or partially compact on E if $\mathscr{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

DEFINITION 2.1. (Dhage [13, 14], Dhage and Dhage [21]) The order relation $\leqslant$ and the metric d on a non-empty set E are said to be *$\mathscr{D}$-compatible* if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \leqslant, \|\cdot\|)$, the order relation $\leqslant$ and the norm $\|\cdot\|$ are said to be $\mathscr{D}$-compatible if $\leqslant$ and the metric d defined through the norm $\|\cdot\|$ are $\mathscr{D}$-compatible. A subset S of E is called *Janhavi* if the order relation $\leqslant$ and the metric d or the norm $\|\cdot\|$ are $\mathscr{D}$-compatible in it. In particular, if $S = E$, then E is called a *Janhavi metric* or *Janhavi Banach space*.

There do exist several examples of the regular and Janhavi Banach spaces in the literature. In fact, every finite dimensional Euclidean space $\mathbb{R}^n$ is regular as well as Janhavi with respect to the usual componentwise order relation and the standard norm in $\mathbb{R}^n$. The following results are of fundamental importance concerning the regularity of a partially ordered Banach space and the Janhavi sets whereby which it is possible to extend the utility or applicability of the abstract coupled fixed point theorems of this paper to the variety of nonlinear problems in a natural way.

We recall that a non-empty closed and convex subset K of the Banach space E is called a cone if i) $K + K \subseteq K$, ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$, and iii) $\{-K\} \bigcap K = \{\theta\}$, where θ is a zero element of E. The details of cones and their properties may be found in Guo and Lakshmikantham [24], Heikkilä and Lakshmikantham [25] and references therein. We define an order relation $\preceq$ in E by

$$x \preceq y \iff y - x \in K \tag{2.1}$$

for all $x, y \in E$. The Banach space E together with the order relation $\preceq$ becomes a partially ordered or simply ordered Banach space and it is denoted by (E, K). Note that every ordered Banach space (E, K) is not a Janhavi Banach space as against the claim made in Yang *et.al* [30]. The following two useful lemmas are recently proved in Dhage [19] play a crucial role in this connection. Since the proofs of these lemmas are not well-known, we give the details of them for completeness and ready reference.

LEMMA 2.1. *Every ordered Banach space (E, K) is regular.*

Proof. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in a partially ordered Banach space (E, K). Then,

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{$*$}$$

Suppose that the sequence $\{x_n\}$ converges to a point x^*, that is, $x_n \to x^*$ as $n \to \infty$. Then, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ also converges to the same limit point x^*, that is, $x_{n_k} \to x^*$ as $k \to \infty$. Since $\{x_n\}$ is nondecreasing, for any given positive integer n, we have $x_n \preceq x_{n_k}$ for each $k \geqslant n \in \mathbb{N}$. This further by definition of the order relation $\preceq$ implies that $x_{n_k} - x_n \in K$. As the cone K is closed and convex set in E, one has

$$\lim_{k\to\infty} \left(x_{n_k} - x_n \right) = x^* - x_n \in K$$

for each $n \in \mathbb{N}$. Therefore, $x_n \preceq x^*$ for all $n \in \mathbb{N}$. Similarly, if $\{x_n\}$ is monotone noincreasing sequence of points in E, then using the similar arguments, it can be proved that $x^* \preceq x_n$ for all $n \in \mathbb{N}$. As a result, (E, K) is a regular ordered Banach space and the proof of the lemma is complete. $\square$

LEMMA 2.2. *Every partially compact subset S of an ordered Banach space (E, K) is Janhavi.*

Proof. Let C be an arbitrary chain in a partially compact subset S of an ordered Banach space E. Then $C = \overline{C}$ is a compact set in E. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in the chain C, that is,

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{2.2}$$

Then $\{x_n\}$ is a relatively compact set in E. Therefore, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ converging to a point x^*. We show that $\{x_n\}$ also converges to x^*. Suppose not. Then for $\varepsilon > 0$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\|x_{n_i} - x^*\| \geqslant \varepsilon \quad \text{for each } i = 1, 2, \ldots. \tag{2.3}$$

Now, by relative compactness of $\{x_{n_i}\}$, there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \to x'$ as $j \to \infty$. Hence for any given positive integer k, by nondecreasing nature of $\{x_n\}$ it follows that when j is large enough ($j \geqslant k$), we have that $x_{n_k} \leqslant x_{n_{i_j}}$. Then $x_{n_{i_j}} - x_{n_k} \in K$. As K is closed and convex, taking the limit first as $j \to \infty$ and then as $k \to \infty$, we obtain

$$x' - x^* \in K \quad \Longrightarrow \quad x^* \leqslant x'.$$

Similarly, it can be shown that $x' \leqslant x^*$. As a result, we have $x' = x^*$ and that $x_{n_{i_j}} \to x^*$ as $j \to \infty$. Therefore, we get

$$\|x_{n_{i_j}} - x^*\| < \varepsilon \tag{2.4}$$

for large j. This is a contradiction to (2.3) and the proof of the lemma is complete. □

If C is a chain in E, then C' denotes the set of all limit points of C in E. The symbol $\overline{C}$ stands for the closure of C in E defined by $\overline{C} = C \cup C'$. The set $\overline{C}$ is called a closed chain in E. Thus, $\overline{C}$ is the intersection of all chains containing C. Clearly, $\inf C, \sup C \in \overline{C}$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z \in E$ such that for every $\varepsilon > 0$ there exists a $c \in C$ such that $d(c,z) < \varepsilon$ and $x \leqslant z$ for all $x \in C$. Similarly, $\inf C$ is defined essentially in an analogous way.

In what follows, we denote by $\mathscr{P}_{cl}(E)$, $\mathscr{P}_{bd}(E)$, $\mathscr{P}_{rcp}(E)$, $\mathscr{P}_{cn}(E)$, $\mathscr{P}_{bd,cn}(E)$, $\mathscr{P}_{rcp,cn}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively. Now we introduce the concept of a partial measure of noncompactness of the chains in E on the lines of Dhage [13, 14, 15]. The related idea of classical measure of noncompactness may be found in Appell [1], Banas and Goebel [2] and references therein.

DEFINITION 2.2. A mapping $\mu_p : \mathscr{P}_{bd,cn}(E) \to \mathbb{R}_+ = [0,\infty)$ is said to be a partial measure of noncompactness in E if it satisfies the following properties:

(P_1) $\emptyset \neq (\mu_p)^{-1}(\{0\}) \subset \mathscr{P}_{rcp,cn}(E)$. (kernel compactivity)

(P_2) $\mu_p(\overline{C}) = \mu_p(C)$. (closure invariance)

(P_3) μ_p is nondecreasing, i.e., if $C \subset D \Rightarrow \mu_p(C) \leqslant \mu_p(D)$. (monotonicity)

(P_4) $\mu_p(\lambda C) = |\lambda|\, \mu_p(C)$. (scalar multiplicativity)

(P_5) $\mu_p(C+D) \leqslant \mu_p(C) + \mu_p(D)$. (subadditivity)

(P_6) If $\{C_n\}$ is a sequence of closed chains from $\mathscr{P}_{bd,cn}(E)$ such that $C_{n+1} \subset C_n$, $n \in \mathbb{N}$ and if $\lim\limits_{n\to\infty} \mu_p(C_n) = 0$, then $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty. (limit intersection property)

The family of sets described in (P_1) is said to be the *kernel of the partial measure of noncompactness* μ_p and is defined as

$$\ker \mu_p = \{C \in \mathscr{P}_{bd,cn}(E) \,|\, \mu_p(C) = 0\}. \tag{2.5}$$

Clearly, $\ker \mu_p \subset \mathscr{P}_{rcp,cn}(E)$. Observe that the intersection set C_∞, from condition (P_3) is a member of the family $\ker \mu_p$. In fact, since $\mu_p(C_\infty) \leqslant \mu_p(C_n)$ for any n, we infer that $\mu_p(C_\infty) = 0$. This yields that $C_\infty \in \ker \mu_p$. This simple observation will be essential in our further investigations.

The partial measure μ_p of noncompactness is called *full* if it satisfies

(P_7) $\ker \mu_p = \mathscr{P}_{rcp,cn}(E)$.

Finally, μ_p is said to satisfy *maximum property* if

(P_8) $\mu_p\Big(C_1 \bigcup C_2\Big) = \max\{\mu_p(C_1), \mu_p(C_2)\}$.

EXAMPLE 2.1. Define two functions $\alpha_p, \beta_p : \mathscr{P}_{bd,cn}(E) \to \mathbb{R}_+$ by

$$\alpha_p(C) = \inf\left\{ r > 0 \,\middle|\, C = \bigcup_{i=1}^{n} C_i,\ \operatorname{diam}(C_i) \leqslant r\ \forall i \right\}, \tag{2.6}$$

where $C \in \mathscr{P}_{bd,cn}(E)$ and $\operatorname{diam}(C_i) = \sup\{d(x,y) : x,y \in C_i\}$, and

$$\beta_p(C) = \inf\left\{ r > 0 \,\middle|\, C \subset \bigcup_{i=1}^{n} \mathscr{B}(x_i, r) \text{ for some } x_i \in E \right\}, \tag{2.7}$$

where $\mathscr{B}(x_i, r) = \{x \in E : d(x_i, x) < r\}$. It is easy to prove that α_p and β_p are partial measures of noncompactness called respectively the diametric partial and ball partial measures of noncompactness which are full and enjoy the maximum property in E.

DEFINITION 2.3. (Dhage [17]) A coupled operator $\mathscr{T} : E \times E \to E$ is called partially condensing if

$$\mu_p(\mathscr{T}(C \times D)) + \mu_p(\mathscr{T}(D \times C)) < \mu_p(C) + \mu_p(D) \tag{2.8}$$

for all $C, D \in \mathscr{P}_{bd,cn}(E)$ for which $\mu_p(C) + \mu_p(D) > 0$, where μ_p is a full partial measure of noncompactness satisfying the maximum property on $\mathscr{P}_{bd,cn}(E)$.

The following coupled fixed point theorem for partially condensing mixed monotone coupled operators is proved in Dhage [17].

THEOREM 2.1. (Dhage [17]) *Let* $(E, \leqslant, \|\cdot\|)$ *be a complete and regular partially ordered normed linear space and let every compact chain* C *in* E *be Janhavi. Suppose that* $\mathscr{T} : E^2 \to E$ *is a partially continuous, partially bounded and partially condensing mixed monotone coupled operator. If there exists an element* $(x_0, y_0) \in E \times E$ *such that*

$x_0 \leqslant \mathscr{T}(x_0,y_0)$ and $y_0 \geqslant \mathscr{T}(y_0,x_0)$ or $x_0 \geqslant \mathscr{T}(x_0,y_0)$ and $y_0 \leqslant \mathscr{T}(y_0,x_0)$, then $\mathscr{T}$ has a coupled fixed point (x^*,y^*) and the sequences $\{\mathscr{T}^n(x_0,y_0)\}$ and $\{\mathscr{T}^n(y_0,x_0)\}$ of successive iterations converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled fixed points is compact.

As mentioned in Dhage [17], the above coupled hybrid fixed point theorem is useful to obtain other coupled hybrid fixed point theorems involving the sum and product of two or three coupled operators in a partially ordered Banach space. In the following section we prove our main coupled hybrid fixed point theorem of this paper on this line.

3. A coupled hybrid fixed point theorem

Given two mappings $\mathscr{F},\mathscr{G} : E\times E \to E$, consider a couple of operator equations

$$x = \mathscr{F}(x,y)+\mathscr{G}(x,y) \tag{3.1}$$

and

$$y = \mathscr{F}(y,x)+\mathscr{G}(y,x) \tag{3.2}$$

for all $(x,y) \in E\times E$, where the operators $\mathscr{F}$ and $\mathscr{G}$ are not necessarily continuous.

The operators $\mathscr{F}$ and $\mathscr{G}$ involved in the coupled operator equations (1.1)–(1.2) are called the *coupled operators* on $E\times E$ into E. A pair of elements $(x^*,y^*)\in E\times E$ is called a *coupled fixed point* of the sum $\mathscr{F}+\mathscr{G}$ of two coupled operators $\mathscr{F}$ and $\mathscr{G}$ or *coupled solution* of the coupled operator equations (1.1) and (1.2) if

$$x^* = \mathscr{F}(x^*,y^*)+\mathscr{G}(x^*,y^*) \quad \text{and} \quad y^* = \mathscr{F}(y^*,x^*)+\mathscr{G}(y^*,x^*). \tag{3.3}$$

The existence of such coupled fixed points for coupled operators is generally obtained under certain monotonic condition of the coupled operator $\mathscr{T}$ on $E\times E$. See Heikkilä and Lakshmikantham [25], Chang and Ma [6], Sun [29], Bhaskar and Lakshmikantham [4] and Dhage and Dhage [21] and the references therein. A coupled operator $\mathscr{T}(x,y)$ is called *mixed monotone* if the map $x\mapsto \mathscr{T}(x,y)$ is nondecreasing for each $y\in E$ and the map $y\mapsto \mathscr{T}(x,y)$ is nonincreasing for each $x\in E$.

Before proving to the main coupled hybrid fixed point theorem, we give some useful definitions in what follows.

DEFINITION 3.1. (Dhage [9, 10]) An upper semi-continuous and nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is called a $\mathscr{D}$-function if $\psi(0)=0$. The class of all $\mathscr{D}$-functions is denoted by $\mathfrak{D}$.

DEFINITION 3.2. (Dhage [12, 13]) An operator $\mathfrak{A}: E\to E$ is called nonlinear partial $\mathscr{D}$-contraction if there exists a $\mathscr{D}$-function $\psi\in\mathfrak{D}$ such that

$$\|\mathfrak{A}x-\mathfrak{A}y\| \leqslant \psi\big(\|x-y\|\big) \tag{3.4}$$

for all comparable elements $x,y\in E$, where ψ satisfies $\psi(r)<r$, $r>0$. If $\psi(r)=kr$, $0\leqslant k<1$, $\mathfrak{A}$ is called a partial contraction on $E\times E$ with the contraction constant k.

DEFINITION 3.3. A coupled operator $\mathscr{T}: E\times E \to E$ is called nonlinear partial $\mathscr{D}$-contraction if there exists a $\mathscr{D}$-function $\psi \in \mathfrak{D}$ such that

$$\|\mathscr{T}(x,y)-\mathscr{T}(u,v)\| \leqslant \frac{1}{2}\cdot \psi\big(\|x-u\|+\|y-v\|\big) \tag{3.5}$$

for all comparable elements $(x,y),(u,v)\in E\times E$, where ψ satisfies $\psi(r)<r$, $r>0$. If $\psi(r)=kr$, $0\leqslant k<1$, $\mathscr{T}$ is called a partial contraction on $E\times E$ with the contraction constant $k/2$.

DEFINITION 3.4. (Dhage [17]) A coupled operator $\mathscr{T}: E\times E\to E$ is called nonlinear symmetric partial $\mathscr{D}$-contraction if there exists a $\mathscr{D}$-function $\psi\in\mathfrak{D}$ such that

$$\|\mathscr{T}(x,y)-\mathscr{T}(u,v)\|+\|\mathscr{T}(y,x)-\mathscr{T}(v,u)\| \leqslant \psi\big(\|x-u\|+\|y-v\|\big) \tag{3.6}$$

for all comparable elements $(x,y),(u,v)\in E\times E$, where ψ satisfies $\psi(r)<r$, $r>0$. If $\psi(r)=kr$, $0\leqslant k<1$, $\mathscr{T}$ is called a symmetric partial contraction on $E\times E$ with the contraction constant k.

REMARK 3.1. It is clear that every nonlinear partial $\mathscr{D}$-contraction is nonlinear symmetric partial $\mathscr{D}$-contraction, but the converse may not be true.

THEOREM 3.1. *Let $(E,\leqslant,\|\cdot\|)$ be a complete regular partially ordered normed linear space and let every compact chain C in E be Janhavi. Let $\mathscr{F},\mathscr{G}: E\times E\to E$ be two mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ is partially bounded and nonlinear partial $\mathscr{D}$-contraction, and

(b) $\mathscr{G}$ is partially continuous and partially compact.

If there exists an element $(x_0,y_0)\in E\times E$ such that $x_0\leqslant \mathscr{F}(x_0,y_0)+\mathscr{G}(x_0,y_0)$ and $y_0\geqslant \mathscr{F}(y_0,x_0)+\mathscr{G}(y_0,x_0)$ or $x_0\geqslant \mathscr{F}(x_0,y_0)+\mathscr{G}(x_0,y_0)$ and $y_0\leqslant \mathscr{F}(y_0,x_0)+\mathscr{G}(y_0,x_0)$, then the coupled operator equations (3.1) *and* (3.2) *have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by*

$$x_{n+1}=\mathscr{F}(x_n,y_n)+\mathscr{G}(x_n,y_n) \tag{3.7}$$

and

$$y_{n+1}=\mathscr{F}(y_n,x_n)+\mathscr{G}(y_n,x_n) \tag{3.8}$$

converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Proof. Define a coupled operator $\mathscr{T}: E\times E\to E$ by

$$\mathscr{T}(x,y)=\mathscr{F}(x,y)+\mathscr{G}(x,y) \tag{3.9}$$

so that we have

$$\mathscr{T}(y,x)=\mathscr{F}(y,x)+\mathscr{G}(y,x).$$

Since $\mathscr{F}$ is a partial nonlinear $\mathscr{D}$-contraction, it is partially continuous on $E \times E$. As a result, the coupled operator $\mathscr{T} = \mathscr{F} + \mathscr{G}$ is well defined and partially continuous on $E \times E$ into E. Again, the coupled operators $\mathscr{F}$ and $\mathscr{G}$ are mixed monotone and partially bounded, so the coupled operator $\mathscr{T}$ is mixed monotone and partially bounded on $E \times E$ into E. We show that $\mathscr{T}$ satisfies the condition (2.8) of Theorem 2.1.

Let $\varepsilon > 0$ be given and let $C = \bigcup_{i=1}^{n} C_i$ and $D = \bigcup_{j=1}^{m} D_j$ be any two chains in E such that

$$\operatorname{diam}(C_i) < \alpha_p(C) + \frac{\varepsilon}{2} \text{ for each } i, \tag{3.10}$$

and

$$\operatorname{diam}(D_j) < \alpha_p(D) + \frac{\varepsilon}{2} \text{ for each } j. \tag{3.11}$$

Now, let

$$\mathscr{F}(C \times D) = F = \bigcup_{l=\lambda}^{m_1} F_\lambda.$$

Then,

$$\bigcup_{\lambda=1}^{m_1} \mathscr{F}^{-1}(F_\lambda) = C \times D.$$

Similarly, if

$$\mathscr{F}(D \times C) = F' = \bigcup_{\gamma=1}^{n_1} F'_\gamma,$$

then,

$$\bigcup_{\gamma=1}^{n_1} \mathscr{F}^{-1}(F'_\gamma) = D \times C.$$

Since $\mathscr{G}$ is partially compact, $\mathscr{G}(C \times D)$ is a relatively compact or totally bounded subset of E. Therefore, for $\varepsilon > 0$, there exist subsets $G_1, G_1, \ldots, G_{m_2}$ of E such that

$$\mathscr{G}(C \times D) = G = \bigcup_{\mu=1}^{m_2} G_\mu$$

and

$$\bigcup_{\mu=1}^{m_2} \mathscr{G}^{-1}(G_\mu) = C \times D$$

satisfying

$$\operatorname{diam}(G_\mu) < \frac{\varepsilon}{2} \text{ for each } \mu. \tag{3.12}$$

Similarly, there exist subsets $G'_1, G'_1, \ldots, G'_{n_2}$ of E such that

$$\mathscr{G}(D \times C) = G' = \bigcup_{\nu=1}^{n_2} G'_\nu$$

and

$$\bigcup_{\nu=1}^{n_2} \mathscr{G}^{-1}(G'_\nu) = D \times C$$

satisfying

$$\operatorname{diam}(G'_\nu) < \frac{\varepsilon}{2} \text{ for each } \nu. \tag{3.13}$$

Denote

$$\mathscr{C}_{\lambda,\mu,,i,j} = \mathscr{F}^{-1}(F_\lambda) \bigcap \mathscr{G}^{-1}(G_\mu) \bigcap (C_i \times D_j)$$

and

$$\mathscr{C}'_{\gamma,\nu,j,i} = \mathscr{F}^{-1}(F'_\gamma) \bigcap \mathscr{G}^{-1}(G'_\nu) \bigcap (D_j \times C_i).$$

Then, we have

$$\bigcup_{\lambda,\mu,i,j} \mathscr{C}_{\lambda,\mu,i,j} = C \times D \quad \text{and} \quad \bigcup_{\gamma,\nu,j,i} \mathscr{C}'_{\gamma,\nu,j,i} = D \times C.$$

Next, if $Z = (x,y), W = (u,v) \in \mathscr{C}_{\lambda,\mu,i,j}$ is such that $Z \succeq W$, then by definition of the diametric partial measure α_p of noncompactness, we get

$$\begin{aligned}
\alpha_p\big(\mathscr{T}(C \times D)\big) &\leqslant \operatorname{diam}\Big(\mathscr{T}\big(\mathscr{C}_{\lambda,\mu,i,j}\big)\Big) \\
&= \sup_{(x,y),(u,v)\in\mathscr{C}_{\lambda,\mu,i,j}} \|\mathscr{T}(x,y) - \mathscr{T}(u,v)\| \\
&\leqslant \sup_{(x,y),(u,v)\in\mathscr{C}_{\lambda,\mu,i,j}} \Big\{\|\mathscr{F}(x,y) - \mathscr{F}(u,v)\| + \|\mathscr{G}(x,y) - \mathscr{G}(u,v)\|\Big\}
\end{aligned} \tag{3.14}$$

Again, if $Z' = (y,x), W' = (v,u) \in \mathscr{C}'_{\gamma,\nu,j,i}$ is such that $Z' \preceq W'$, then we obtain

$$\begin{aligned}
\alpha_p\big(\mathscr{T}(D \times C)\big) &\leqslant \operatorname{diam}\Big(\mathscr{T}\big(\mathscr{C}'_{\gamma,\nu,j,i}\big)\Big) \\
&= \sup_{(y,x),(v,u)\in\mathscr{C}'_{\gamma,\nu,j,i}} \|\mathscr{T}(y,x) - \mathscr{T}(v,u)\| \\
&\leqslant \sup_{(y,x),(v,u)\in\mathscr{C}'_{\gamma,\nu,j,i}} \Big\{\|\mathscr{F}(y,x) - \mathscr{F}(v,u)\| + \|\mathscr{G}(y,x) - \mathscr{G}(v,u)\|\Big\}.
\end{aligned} \tag{3.15}$$

Adding the expressions (3.14) and (3.15) together implies that

$$\begin{aligned}
&\alpha_p\big(\mathscr{T}(C \times D)\big) + \alpha_p\big(\mathscr{T}(D \times C)\big) \\
&\quad\leqslant \sup_{(x,y),(u,v)\in\mathscr{C}_{\lambda,\mu,i,j}} \|\mathscr{F}(x,y) - \mathscr{F}(u,v)\| + \sup_{(y,x),(v,u)\in\mathscr{C}'_{\gamma,\nu,j,i}} \|\mathscr{F}(y,x) - \mathscr{F}(v,u)\| \\
&\qquad + \sup_{(x,y),(u,v)\in\mathscr{C}_{\lambda,\mu,i,j}} \|\mathscr{G}(x,y) - \mathscr{G}(u,v)\| + \sup_{(y,x),(v,u)\in\mathscr{C}'_{\gamma,\nu,j,i}} \|\mathscr{G}(y,x) - \mathscr{G}(v,u)\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sup_{(x,y),(u,v)\in C_i\times D_j} \frac{\psi\big(\|(x-u\|+\|v-y\|\big)}{2} + \sup_{(y,x),(v,u)\in D_j\times C_i} \frac{\psi\big(\|(x-u\|+\|v-y\|\big)}{2} \\
&\quad + \sup_{(x,y),(u,v)\in \mathscr{G}^{-1}(G_\mu)} \|\mathscr{G}(x,y)-\mathscr{G}(u,v)\| + \sup_{(y,x),(v,u)\in \mathscr{G}^{-1}(G'_\nu)} \|\mathscr{G}(y,x)-\mathscr{G}(v,u)\| \\
&\leqslant \sup_{(x,y),(u,v)\in C_i\times D_j} \psi\big(\|(x-u\|+\|v-y\|\big) + \Big[\operatorname{diam}\big(G_\mu\big)+\operatorname{diam}\big(G'_\nu\big)\Big] \\
&< \psi\big(\operatorname{diam}(C_i)+\operatorname{diam}(D_j)\big)+\varepsilon \\
&\leqslant \sup\Big\{\psi(r) \,:\, r\in\big[\alpha_p(C)+\alpha_p(D)\,,\,\alpha_p(C)+\alpha_p(D)+\varepsilon\big]\Big\}+\varepsilon. \qquad (3.16)
\end{aligned}
$$

Since ε is arbitrary, from above inequality (3.16), we obtain

$$
\alpha_p\big(\mathscr{T}(C\times D)\big)+\alpha_p\big(\mathscr{T}(D\times C)\big)\leqslant \psi\big(\alpha_p(C)+\alpha_p(D)\big)
$$

for all bounded chains C and D in E, where $\psi\in\mathfrak{D}$ satisfies the inequality $\psi(r)<r$, $r>0$. Hence the coupled operator $\mathscr{T}$ is a nonlinear partial $\mathscr{D}$-set-contraction and consequently a partial condensing on $E\times E$ with the same method with respect to the diametric partial measure of noncompactness $\mu_p=\alpha_p$ in E.

Next, by hypothesis, there exists the element $(x_0,y_0)\in E\times E$ such that

$$
x_0\leqslant \mathscr{F}(x_0,y_0)+\mathscr{G}(x_0,y_0)=\mathscr{T}(x_0,y_0)
$$

and

$$
y_0\geqslant \mathscr{F}(y_0,x_0)+\mathscr{G}(y_0,x_0)=\mathscr{T}(y_0,x_0)
$$

or

$$
x_0\geqslant \mathscr{F}(x_0,y_0)+\mathscr{G}(x_0,y_0)=\mathscr{T}(x_0,y_0)
$$

and

$$
y_0\leqslant \mathscr{F}(y_0,x_0)+\mathscr{G}(y_0,x_0)=\mathscr{T}(y_0,x_0).
$$

Thus, the coupled operator $\mathscr{T}$ satisfies all the conditions of Theorem 2.1 and therefore, the coupled operator equations $x=\mathscr{T}(x,y)$ and $y=\mathscr{T}(y,x)$ have a coupled fixed point $(x^*,y^*)\in E\times E$. Consequently the coupled operator equations (3.1) and (3.2) have a coupled solution $(x^*,y^*)\in E\times E$ and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.7) and (3.8) converge monotonically to x^* and y^* respectively. This completes the proof. □

REMARK 3.2. Note that Theorem 3.1 is also obtained in Dhage [19] under the condition that the coupled operator $\mathscr{A}$ is nonlinear symmetric partial contraction, but with that method the proof goes wrong. In this connection, Theorem 3.1 is an improvement of a coupled hybrid fixed point theorem proved in Dhage [19] containing the sum of two coupled operators on $E\times E$. Again, the regularity of the partially ordered metric space E in above Theorem 3.1 may be relaxed and compensated with the continuity of the mapping $\mathscr{F}$ on $E\times E$. See Dhage [8, 15, 16, 17] and the references therein.

REMARK 3.3. If $x = y$ in the coupled operator equations (3.1) and (3.2), then they reduce to the operator equation $\mathfrak{A}x + \mathfrak{B}x = x$, where $\mathfrak{A}x = \mathscr{F}(x,x)$ and $\mathfrak{B}x = \mathscr{G}(x,x)$, and consequently the Theorem 3.1 reduces to a hybrid fixed point theorem for the sum of two operators in a partially ordered Banach space E proved in Dhage [13, 14, 15].

In view of the Lemmas 2.1 and 2.2 we obtain the following interesting applicable coupled hybrid fixed point results in an ordered Banach space (E,K).

COROLLARY 3.1. *Let (E,K) be an ordered Banach space and let $\mathscr{T} : E^2 \to E$ be a partially continuous, partially bounded and partially condensing mixed monotone coupled operator. If there exists an element $(x_0,y_0) \in E \times E$ such that $x_0 \leqslant \mathscr{T}(x_0,y_0)$ and $y_0 \geqslant \mathscr{T}(y_0,x_0)$ or $x_0 \geqslant \mathscr{T}(x_0,y_0)$ and $y_0 \leqslant \mathscr{T}(y_0,x_0)$, then $\mathscr{T}$ has a coupled fixed point (x^*,y^*) and the sequences $\{\mathscr{T}^n(x_0,y_0)\}$ and $\{\mathscr{T}^n(y_0,x_0)\}$ of successive iterations converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled fixed points is compact.*

COROLLARY 3.2. *Let (E,K) be an ordered Banach space and let $\mathscr{F},\mathscr{G} : E \times E \to E$ be two mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ is partially bounded and nonlinear partial $\mathscr{D}$-contraction, and

(b) $\mathscr{G}$ is partially continuous and partially compact.

If there exists an element $(x_0,y_0) \in E \times E$ such that $x_0 \leqslant \mathscr{F}(x_0,y_0) + \mathscr{G}(x_0,y_0)$ and $y_0 \geqslant \mathscr{F}(y_0,x_0) + \mathscr{G}(y_0,x_0)$ or $x_0 \geqslant \mathscr{F}(x_0,y_0) + \mathscr{G}(x_0,y_0)$ and $y_0 \leqslant \mathscr{F}(y_0,x_0) + \mathscr{G}(y_0,x_0)$, then the coupled operator equations (3.1) and (3.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.7) and (3.8) converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Again, when $x = y$ and $\mathfrak{T}x = \mathscr{T}(x,x)$, Corollary 3.2 reduces to the following well-known hybrid fixed point theorem in an ordered Banach space (E,K).

COROLLARY 3.3. *Let (E,K) be an ordered Banach space and let $\mathfrak{T} : E \to E$ be a partially continuous, partially bounded and partially condensing nondecreasing operator. If there exists an element $x_0 \in E$ such that $x_0 \leqslant \mathfrak{T}x_0$ or $x_0 \geqslant \mathfrak{T}x_0$, then $\mathfrak{T}$ has a fixed point x^* and the sequence $\{\mathfrak{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the set of all comparable fixed points is compact.*

Similarly, if $x = y$ and $\mathfrak{A}x = \mathscr{F}(x,x)$ and $\mathfrak{B}x = \mathscr{G}(x,x)$, then Corollary 3.3 reduces to the following hybrid fixed point theorem involving the sum of two operators in an ordered Banach space (E,K).

COROLLARY 3.4. *Let (E,K) be an ordered Banach space and let $\mathfrak{A},\mathfrak{B} : E \to E$ be two nondecreasing operators satisfying the following conditions.*

(a) $\mathfrak{A}$ is partially bounded and nonlinear partial $\mathscr{D}$-contraction, and

(b) $\mathfrak{B}$ *is partially continuous and partially compact.*

If there exists an element $x_0 \in E$ *such that* $x_0 \leqslant \mathfrak{A}x_0 + \mathfrak{B}x_0$ *or* $x_0 \geqslant \mathfrak{A}x_0 + \mathfrak{B}x_0$, *then the operator equation* $\mathfrak{A}x + \mathfrak{B}x = x$ *has a solution* x^* *and the sequence* $\{x_n\}$ *defined by* $x_{n+1} = \mathfrak{A}x_n + \mathfrak{B}x_n$ *converges monotonically to* x^*. *Moreover, the set of all comparable solutions is compact.*

REMARK 3.4. The hybrid fixed point corollaries, Corollaries 3.1, 3.2, 3.3 and 3.4 are new to the literature on the theory of fixed point theorems in ordered Banach spaces and applications. In fact Corollary 3.4 is a corrected improved version of the hybrid fixed point theorems of Yang *et.al* [30] involving the sum of two operators in an ordered Banach space (E,K) under weaker normality condition of the order cone K. Note that the above mentioned hybrid fixed point theorems are very much useful in the subject of nonlinear analysis for proving the existence and approximation theorems for nonlinear differential and integral equations in Banach spaces.

The periodic boundary value problems are often times discussed for different aspects of the solutions via applications of the tools from nonlinear functional analysis. In the following section we state a coupled periodic boundary value problem of first order linearly perturbed nonlinear differential equations to be discussed by an application of Theorem 3.1.

4. Periodic boundary value problems

Given a closed and bounded interval $J = [0,T]$ of the real line $\mathbb{R}$, we consider the coupled hybrid periodic boundary value problems (in short coupled HPBVPs) of nonlinear first order ordinary differential equations,

$$\left.\begin{array}{l} x'(t) + \lambda x(t) = f(t,x(t),y(t)) + g(t,x(t),y(t)),\ t \in J, \\ x(0) = x(T), \end{array}\right\} \tag{4.1}$$

and

$$\left.\begin{array}{l} y'(t) + \lambda y(t) = f(t,y(t),x(t)) + g(t,y(t),x(t)),\ t \in J, \\ y(0) = y(T), \end{array}\right\} \tag{4.2}$$

for $\lambda \in \mathbb{R}$, $\lambda > 0$, where $f,g : J \times \mathbb{R} \times \mathbb{R}$ are continuous functions.

By a coupled solution of the coupled HPBVPs (4.1) and (4.2) we mean a pair of differentiable functions $(x^*,y^*) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ that satisfies the equations (4.1) and (4.2) where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The special case of the coupled HPBVPs of the form

$$\left.\begin{array}{l} x'(t) + \lambda x(t) = f(t,x(t),y(t)),\ t \in J, \\ x(0) = x(T), \end{array}\right\} \tag{4.3}$$

and

$$\left.\begin{array}{l} y'(t) + \lambda y(t) = f(t,y(t),x(t)),\ t \in J, \\ y(0) = y(T), \end{array}\right\} \tag{4.4}$$

has been discussed by Bhaskar and Lakshmikantham [4] and Berinde [3] for the existence and uniqueness theorem if the nonlinearity f satisfies a Lipschitz type condition and when f satisfies a compactness type condition it has been discussed in Dhage [16] for the existence and approximation of coupled solutions on J. The purpose of the present study is to establish an existence result and develop an algorithm for approximating the coupled solutions of the coupled HPBVPs (4.1) and (4.2) under some mixed hybrid conditions on the nonlinearities f and g.

The following useful lemma is obvious and may be found in Dhage [16, 18] and the references therein. The details are also found in Nieto [26], Nieto and Lopez [27, 28].

LEMMA 4.1. *For any function $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t)+\lambda x(t) &= \sigma(t),\ t\in J,\\ x(0) &= x(T), \end{aligned}\right\} \tag{4.5}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_\lambda(t,s)\,\sigma(s)\,ds,\ t\in J, \tag{4.6}$$

where, the Green's function $G(t,s)$ is given by

$$G_\lambda(t,s) = \begin{cases} \dfrac{e^{\lambda s-\lambda t+\lambda T}}{e^{\lambda T}-1}, & \text{if} \quad 0\leqslant s\leqslant t\leqslant T,\\[2ex] \dfrac{e^{\lambda s-\lambda t}}{e^{\lambda T}-1}, & \text{if} \quad 0\leqslant t< s\leqslant T. \end{cases} \tag{4.7}$$

Notice that the Green's function G_λ is continuous and nonnegative on $J\times J$ and therefore, the number

$$K_\lambda := \max\{\,|G_\lambda(t,s)| : t,s\in[0,T]\,\}$$

exists for all $\lambda\in\mathbb{R}^+$. For the sake of convenience, we write $G_\lambda(t,s)=G(t,s)$ and $K_\lambda = K$.

Other useful results for establishing the main result are as follows.

LEMMA 4.2. *If there exists a differentiable function $u\in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t)+\lambda u(t) &\leqslant \sigma(t),\ t\in J,\\ u(0) &\leqslant u(T), \end{aligned}\right\} \tag{4.8}$$

for all $t\in J$, where $\lambda\in\mathbb{R}$, $\lambda>0$ and $\sigma\in L^1(J,\mathbb{R})$, then

$$u(t)\leqslant \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{4.9}$$

for all $t\in J$, where $G(t,s)$ is the Green's function given by the expression (4.7) on $J\times J$.

Proof. The proof of the lemma appears in Dhage [13, 14, 15, 18] and Dhage and Dhage [22, 23]. Since the proof is not well-known, for the sake of completeness we give the details of it. Suppose that the function $u \in C(J,\mathbb{R})$ satisfies the inequalities given in (4.8). Multiplying the first inequality in (4.8) by $e^{\lambda t}$,

$$\left(e^{\lambda t}u(t)\right)' \leqslant e^{\lambda t}\sigma(t).$$

A direct integration of above inequality from 0 to t yields

$$e^{\lambda t}u(t) \leqslant u(0) + \int_0^t e^{\lambda s}\sigma(s)\,ds, \tag{4.10}$$

for all $t \in I$. Therefore, in particular,

$$e^{\lambda T}u(T) \leqslant u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds. \tag{4.11}$$

Now $u(0) \leqslant u(T)$, so one has

$$u(0)e^{\lambda T} \leqslant u(T)e^{\lambda T}. \tag{4.12}$$

From (4.11) and (4.12) it follows that

$$e^{\lambda T}u(0) \leqslant u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds \tag{4.13}$$

which further yields

$$u(0) \leqslant \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T}-1)}\sigma(s)\,ds. \tag{4.14}$$

Substituting (4.14) in (4.10) we obtain

$$u(t) \leqslant \int_0^T G(t,s)\sigma(s)\,ds,$$

for all $t \in J$. This completes the proof. □

Similarly, we have the following result of differential inequality related to the first order periodic boundary value problems defined on J.

LEMMA 4.3. *If there exists a differentiable function $v \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} v'(t)+\lambda v(t) &\geqslant \sigma(t),\ t\in J,\\ v(0) &\geqslant v(T), \end{aligned}\right\} \tag{4.15}$$

for all $t \in J$, where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$v(t) \geqslant \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{4.16}$$

for all $t \in J$, where $G(t,s)$ is the Green's function given by the expression (4.7) on $J \times J$.

Now we are ready to apply our abstract coupled hybrid fixed point theorem to coupled HPBVPs (4.1) and (4.2) under suitable natural conditions. In the following section we prove our main existence and approximation theorem for coupled solutions of the coupled HPBVPs (4.1) and (4.2) defined on J.

5. Existence and approximation results

The equivalent integral forms of the coupled HPBVPs (4.1) and (4.2) are considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leqslant$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{5.1}$$

and

$$x \leqslant y \quad \text{if and only if} \quad x(t) \leqslant y(t) \quad \text{for all} \quad t \in J. \tag{5.2}$$

Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and is also partially ordered w.r.t. the above partial order relation $\leqslant$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and is a lattice, so every pair of elements in the space has an upper and a lower bound in the space. The next lemma concerning the $\mathscr{D}$-compatibility of sets in $C(J,\mathbb{R})$ follows by an application of the Arzelá-Ascoli theorem.

LEMMA 5.1. *Let $\big(C(J,\mathbb{R}),\leqslant,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leqslant$ defined by* (5.1) *and* (5.2) *respectively. Then, every partially compact subset of $C(J,\mathbb{R})$ possesses $\mathscr{D}$-compatibility property w.r.t. $\|\cdot\|$ and $\leqslant$ and so is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [13, 14, 15, 16] and Dhage and Dhage [21, 22]. Here we give the proof of the lemma using somewhat different arguments via cones in a Banach space $C(J,\mathbb{R})$. Define a subset K of $C(J,\mathbb{R})$ by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geqslant 0 \;\text{ for all } t \in J\}. \tag{5.3}$$

Clearly K is a non-empty, closed and convex subset of the Banach space $C(J,\mathbb{R})$ satisfying the properties (i)- (iii) of a cone. So K is a cone in $C(J,\mathbb{R})$. Now, the order relation $\leqslant$ given by (3.2) is equivalent to the order relation $\leqslant$ defined by the cone K in $C(J,\mathbb{R})$. Therefore, the desired conclusion follows by an application of Lemma 2.2. This completes the proof. □

We need the following definition in what follows.

DEFINITION 5.1. A pair of differentiable functions $(u,v) \in C(J,\mathbb{R})\times C(J,\mathbb{R})$ is said to be a lower coupled solution of the coupled equations (4.1) and (4.2) if

$$\left.\begin{aligned} u'(t)+\lambda u(t) \leqslant f(t,u(t),v(t))+g(t,u(t),v(t)),\; t\in J,\\ u(0)\leqslant u(T), \end{aligned}\right\}$$

and

$$\left.\begin{aligned} v'(t)+\lambda v(t) \geqslant f(t,v(t),u(t))+g(t,v(t),u(t)),\ t\in J,\\ v(0)\geqslant v(T). \end{aligned}\right\}$$

Similarly, a pair of differentiable functions $(w,z)\in C(J,\mathbb{R})\times C(J,\mathbb{R})$ is called an upper coupled solution of the coupled HPBVPs (4.1) and (4.2) if the above inequalities are satisfied with reverse sign.

The coupled HPBVPs (4.1)–(4.2) will be considered under the following assumptions:

(H_1) The function f is bounded on $J\times\mathbb{R}\times\mathbb{R}$ with bound M_f.

(H_2) There exists a $\mathscr{D}$-function $\varphi\in\mathfrak{D}$ such that

$$0\leqslant f(t,x_1,y_1)-f(t,x_2,y_2)\leqslant \frac{1}{2}\cdot\varphi\big(x_1-x_2+y_2-y_1\big)$$

for all $(x_1,y_1),(x_2,y_2)\in\mathbb{R}\times\mathbb{R}$ with $x_1\geqslant x_2$ and $y_2\geqslant y_1$. Moreover $KT\varphi(r)<r$, $r>0$.

(H_3) $g(t,x,y)$ is nondecreasing in x and nonincreasing in y for each $t\in J$.

(H_4) The function g is bounded on $J\times\mathbb{R}\times\mathbb{R}$ with bound M_g.

(H_5) The coupled HPBVPs (4.1)–(4.2) have a lower coupled solution $(u,v)\in C(J,\mathbb{R})\times C(J,\mathbb{R})$.

The hypotheses (H_1) through (H_5) are standard and have been widely used in the literature on nonlinear differential and integral equations. The special case of the hypothesis (H_2) with $\varphi(r)=\dfrac{Lr}{K+r}$, $L\leqslant K$ is considered recently in Dhage [9, 12]. Now we formulate the main existence and approximation result for the coupled HPBVPs (4.1)–(4.2) under above mentioned natural conditions.

THEOREM 5.1. *Assume that the hypotheses (H_1) through (H_5) hold. Then the coupled HPBVPs* (4.1)–(4.2) *have a coupled solution* (x^*,y^*) *and the sequences* $\{x_n\}$ *and* $\{y_n\}$ *of successive approximations defined by*

$$\begin{aligned} x_0&=u,\\ x_{n+1}(t)&=\int_0^T G(t,s)f(s,x_n(s),y_n(s))\,ds\\ &\quad+\int_0^T G(t,s)g(s,x_n(s),y_n(s))\,ds,\ n\geqslant 0, \end{aligned}\tag{5.4}$$

and

$$\begin{aligned} y_0&=v,\\ y_{n+1}(t)&=\int_0^T G(t,s)f(s,y_n(s),x_n(s))\,ds\\ &\quad+\int_0^T G(t,s)g(s,y_n(s),x_n(s))\,ds,\ n\geqslant 0, \end{aligned}\tag{5.5}$$

for $t \in J$, converge monotonically to x^* and y^* respectively.

Proof. Set $E = C(J,\mathbb{R})$. Then, by Lemma 5.1, every compact chain C in E is Janhavi. Next, by Lemma 4.1, the coupled HPBVPs (4.1) and (4.2) are equivalent to the nonlinear coupled integral equations of Fredholm type,

$$x(t) = \int_0^T G(t,s)f(s,x(s),y(s))\,ds + \int_0^T G(t,s)g(s,x(s),y(s))\,ds,\ t \in J, \tag{5.6}$$

and

$$y(t) = \int_0^T G(t,s)f(s,y(s),x(s))\,ds + \int_0^T G(t,s)g(s,y(s),x(s))\,ds,\ t \in J. \tag{5.7}$$

Now, consider the two coupled operators $\mathscr{F},\mathscr{G} : E \times E \to E$ defined by

$$\mathscr{F}(x,y)(t) = \int_0^T G(t,s)f(s,x(s),y(s))\,ds,\ t \in J, \tag{5.8}$$

and

$$\mathscr{G}(x,y)(t) = \int_0^T G(t,s)g(s,x(s),y(s))\,ds,\ t \in J. \tag{5.9}$$

Then the nonlinear coupled integral equations (5.6) and (5.7) are equivalent to the coupled operator equations,

$$x(t) = \mathscr{F}(x,y)(t) + \mathscr{G}(x,y)(t),\ t \in J, \tag{5.10}$$

and

$$y(t) = \mathscr{F}(y,x)(t) + \mathscr{G}(y,x)(t),\ t \in J. \tag{5.11}$$

We shall show that the coupled operators $\mathscr{F}$ and $\mathscr{G}$ satisfy all the conditions of Theorem 3.1 on $E \times E$ into E. This will be done in a series of following steps:

Step I: $\mathscr{F}$ and $\mathscr{G}$ are mixed monotone.

Let $(x,y),(u,v) \in E \times E$ be arbitrary and let $(x,y) \succeq (u,v)$. Then by definition of $\preceq$, we get $x \geqslant u$ and $v \geqslant y$. Now, by hypotheses (H_2) and (H_3),

$$\begin{aligned}\mathscr{F}(x,y)(t) &= \int_0^T G(t,s)f(s,x(s),y(s))\,ds\\ &\geqslant \int_0^T G(t,s)g(s,u(s),v(s))\,ds\\ &= \mathscr{F}(u,v)(t)\end{aligned}$$

and

$$\begin{aligned}\mathscr{G}(x,y)(t) &= \int_0^T G(t,s)g(s,x(s),y(s))\,ds\\ &\geqslant \int_0^T G(t,s)g(s,u(s),v(s))\,ds\\ &= \mathscr{G}(u,v)(t)\end{aligned}$$

for all $t \in J$. Hence $\mathscr{F}$ and $\mathscr{G}$ are mixed monotone operators on $E \times E$ into E.

Step II: $\mathscr{F}$ is partially bounded and nonlinear partial $\mathscr{D}$-contraction.

Let $(x,y) \in E \times E$ be arbitrary. Then,

$$|\mathscr{F}(x,y)(t)| \leqslant \int_0^T G(t,s)|f(s,x(s),y(s))|\,ds \leqslant KTM_f$$

for all $t \in J$. Taking the supremum over t in the above inequality yields $\|\mathscr{F}(x,y)\| \leqslant KTM_f$ for all $x,y \in E$. So the coupled operator $\mathscr{F}$ is bounded and consequently partially bounded on $E \times E$.

Next, let $(x,y),(u,v) \in E \times E$ be any two elements such that $(x,y) \succeq (u,v)$. Then, by hypothesis (H_2),

$$\begin{aligned}
\big|\mathscr{F}(x,y)(t) - \mathscr{F}(u,v)(t)\big| &\leqslant \int_0^T G(t,s)|f(s,x(s),y(s)) - f(s,u(s),v(s))|\,ds \\
&\leqslant \int_0^T G(t,s)\big[f(s,x(s),y(s)) - f(s,u(s),v(s))\big]\,ds \\
&\leqslant \frac{1}{2}\int_0^T G(t,s)\varphi\big(x(s)-u(s)+v(s)-y(s)\big)\,ds \\
&\leqslant \frac{1}{2}\int_0^T G(t,s)\varphi\big(|x(s)-u(s)|+|v(s)-y(s)|\big)\,ds \\
&\leqslant \frac{1}{2}KT\varphi\big(\|x-u\|+\|v-y\|\big).
\end{aligned}$$

Taking the supremum over t in the above inequality yields,

$$\|\mathscr{F}(x,y) - \mathscr{F}(u,v)\| \leqslant \frac{1}{2}KT\varphi\big(\|x-u\|+\|v-y\|\big) \tag{5.12}$$

for all comparable elements $(x,y),(u,v) \in E \times E$. This shows that the coupled operator $\mathscr{F}$ is a nonlinear partial $\mathscr{D}$-contraction on $E \times E$.

Step III: $\mathscr{G}$ is partially continuous on $E \times E$.

Let C and D be any two chains in E and let $\{x_n\}$ and $\{y_n\}$ be two sequences in C and D respectively such that $x_n \to x$ and $y_n \to y$. Then, by continuity of the function g, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{G}(x_n,y_n)(t) &= \lim_{n\to\infty}\int_0^T G(t,s)g(s,x_n(s),y_n(s))\,ds \\
&= \int_0^T G(t,s)\Big[\lim_{n\to\infty} g(s,x_n(s),y_n(s))\Big]\,ds \\
&= \int_0^T G(t,s)g(s,x(s),y(s))\,ds \\
&= \mathscr{G}(x,y)(t)
\end{aligned}$$

for all $t \in J$. This shows that the sequence $\{\mathscr{G}(x_n,y_n)\}$ converges to $\mathscr{G}(x,y)$ pointwise on J. We show that the convergence is uniform. To do so, it is enough to show that the sequence $\{\mathscr{G}(x_n,y_n)\}$ is equicontinuous set of functions in E. Let $t_1,t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}\big|\mathscr{G}(x_n,y_n)(t_1) - \mathscr{G}(x_n,y_n)(t_2)\big| \\ &\leqslant \int_0^T |G(t_1,s) - G(t_2,s)|\,|g(s,x_n(s),y_n(s))|\,ds \\ &\leqslant M_g \int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\ &\to 0 \quad \text{as} \quad t_1 \to t_2,\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This proves the equicontinuity of the sequence $\{\mathscr{G}(x_n,y_n)\}$ of functions in E. As a result, $\mathscr{G}(x_n,y_n) \to \mathscr{G}(x,y)$ uniformly. Hence $\mathscr{G}$ is continuous coupled operator on $C \times D$. Consequently, $\mathscr{G}$ is partially continuous on $E \times E$.

Step IV: $\mathscr{G}$ is partially compact on $E \times E$.

Let C and D be any two chains in E. We show that $\mathscr{G}(C \times D)$ is a partially compact subset of E. First we show that $\mathscr{G}(C \times D)$ is a uniformly bounded subset of E. Let $z \in \mathscr{G}(C \times D)$ be a fixed element. Then there exists a point $(x,y) \in C \times D$ such that $z = \mathscr{G}(x,y)$. Then,

$$\big|z(t)\big| = \big|\mathscr{G}(x,y)(t)\big| \leqslant \int_0^T G(t,s)|g(s,x(s),y(s))|\,ds \leqslant M_g KT$$

for all $t \in J$. Taking the supremum over t, $\|z\| \leqslant M_g KT$ for all $z \in \mathscr{G}(C \times D)$. Hence $\mathscr{G}(C \times D)$ is a uniformly bounded subset of E.

Next, we show that $\mathscr{G}(C \times D)$ is an equicontinuous subset of E. Let $t_1,t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}|z(t_1) - z(t_2)| &= \big|\mathscr{G}(x,y)(t_1) - \mathscr{G}(x,y)(t_2)\big| \\ &\leqslant \int_0^T |G(t_1,s) - G(t_2,s)|\,|g(s,x(s),y(s))|\,ds \\ &\leqslant M_g \int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\ &\to 0 \quad \text{as} \quad t_1 \to t_2,\end{aligned}$$

uniformly for all $z \in \mathscr{G}(C \times D)$. This proves the equicontinuity of the set $\mathscr{G}(C \times D)$ in E. As a result, $\mathscr{G}(C \times D)$ is compact and hence relatively compact in view of Arzelá-Ascoli theorem. Hence $\mathscr{G}$ is a partially compact coupled operator on $E \times E$ into E.

Step V: Coupled equations (5.10)–(5.11) *have a lower coupled solution.*

Now, by hypothesis (H$_5$), there exists an element $(u,v) \in E \times E$ such that

$$\left.\begin{aligned} u'(t) + \lambda u(t) \leqslant f(t,u(t),v(t)) + g(t,u(t),v(t)), \\ u(0) \leqslant u(T), \end{aligned}\right\}$$

and

$$\left.\begin{aligned} v'(t) + \lambda v(t) \geqslant f(t,v(t),u(t)) + g(t,v(t),u(t)), \\ v(0) \geqslant v(T). \end{aligned}\right\}$$

for all $t \in J$. This further in view of Lemmas 4.2 and 4.3 implies that

$$u(t) \leqslant \int_0^T G(t,s)f(s,u(s),v(s))\,ds + \int_0^T G(t,s)g(s,u(s),v(s))\,ds$$

and

$$v(t) \geqslant \int_0^T G(t,s)f(s,v(s),u(s))\,ds + \int_0^T G(t,s)g(s,v(s),u(s))\,ds$$

for all $t \in J$. Again, from the definition of the coupled operators $\mathscr{F}$ and $\mathscr{G}$ it follows that

$$u(t) \leqslant \mathscr{F}(u,v)(t) + \mathscr{G}(u,v)(t),\ t \in J,$$

and

$$v(t) \geqslant \mathscr{F}(v,u)(t) + \mathscr{G}(v,u)(t),\ t \in J.$$

Therefore, the coupled operator equations (5.10)–(5.11) have a lower coupled solution (u,v) in $E \times E$. Thus the coupled operators $\mathscr{F}$ and $\mathscr{G}$ satisfy all the conditions of Theorem 3.1 and hence the coupled operator equations and consequently the coupled HPBVPs (4.1)–(4.2) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (5.4) and (5.5) converge monotonically to x^* and y^* respectively. □

REMARK 5.1. The conclusion of Theorem 5.1 also remains true if we replace the hypothesis (H_5) by the following one:

(H_6) The coupled HPBVPs (4.1)–(4.2) have a upper coupled solution $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

The proof under this hypothesis is obtained by giving similar arguments with appropriate modifications.

EXAMPLE 5.1. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, we consider the coupled hybrid periodic boundary value problems (in short coupled HPBVPs) of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} x'(t) + x(t) &= f_1(t,x(t),y(t)) + g_1(t,x(t),y(t)),\ t \in J,\\ x(0) &= x(1), \end{aligned}\right\} \tag{5.13}$$

and

$$\begin{aligned} y'(t) + y(t) &= f_1(t,y(t),x(t)) + g_1(t,y(t),x(t)),\ t \in J,\\ y(0) &= y(1), \end{aligned} \tag{5.14}$$

where $f_1, g_1 : J \times \mathbb{R} \times \mathbb{R}$ are continuous functions defined by

$$f_1(t,x,y) = \begin{cases} 0 & \text{if } -\infty < x,y \leqslant 0,\\ \frac{1}{8} \cdot \frac{x}{1+x} & \text{if } y \leqslant 0 < x < \infty,\\ -\frac{1}{8} \cdot \frac{y}{1+y} & \text{if } x \leqslant 0 < y < \infty,\\ \frac{1}{8} \cdot \left[\frac{x}{1+x} - \frac{y}{1+y}\right] & \text{if } 0 < x,y < \infty, \end{cases}$$

and

$$g_1(t,x,y) = \tanh x - \tanh y$$

for all $t \in [0,1]$.

It is easy to verify that the real-valued functions f_1 and g_1 are continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$. We shall show that the nonlinearities f_1 and g_1 satisfy the hypotheses (H_1) through (H_4). Clearly the function f_1 is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ by $M_{f_1} = 1$. Next let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be such that $x_1 \geqslant x_2 > 0$ and $0 < y_1 \leqslant y_2$. Then, we have

$$\begin{aligned}
0 &\leqslant f(t,x_1,y_1) - f_1(t,x_2,y_2) \\
&= \frac{1}{8}\left[\frac{x_1}{1+x_1} - \frac{y_1}{1+y_1} - \frac{x_2}{1+x_2} + \frac{y_2}{1+y_2}\right] \\
&= \frac{1}{8}\left[\frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} + \frac{y_2}{1+y_2} - \frac{y_1}{1+y_1}\right] \\
&= \frac{1}{8}\left[\frac{x_1 - x_2}{(1+x_1)(1+x_2)} + \frac{y_2 - y_1}{(1+y_2)(1+y_1)}\right] \\
&= \frac{1}{8}\left[\frac{x_1 - x_2}{1+x_1+x_2+x_1x_2} + \frac{y_2 - y_1}{1+y_2+y_1+y_2y_1}\right] \\
&\leqslant \frac{1}{8}\left[\frac{x_1 - x_2}{1+x_1+x_2} + \frac{y_2 - y_1}{1+y_2+y_1}\right] \\
&\leqslant \frac{1}{8}\left[\frac{x_1 - x_2}{1+x_1-x_2} + \frac{y_2 - y_1}{1+y_2-y_1}\right] \\
&\leqslant \frac{1}{4} \cdot \frac{x_1 - x_2 + y_2 - y_1}{1+x_1-x_2+y_2-y_1} \\
&= \frac{1}{2} \cdot \varphi(x_1 - x_2 + y_2 - y_1)
\end{aligned}$$

where, $\varphi(r) = \dfrac{1}{2} \cdot \dfrac{r}{1+r}$ for $r > 0$ and that $\varphi \in \mathfrak{D}$. Again, if $0 \geqslant x_1 \geqslant x_2$ and $y_1 \leqslant y_2 \leqslant 0$, then also the above inequality is satisfied. Similarly, if $x_1 \geqslant x_2 > 0$ and $y_1 \leqslant y_2 < 0$, then also the above inequality is satisfied. Therefore, in all cases, the function f_1 satisfies the hypothesis (H_2).

Next, the function g_1 is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_{g_1} = 2$. Again, $g_1(t,x,y)$ is nondecreasing in x and nonincreasing in y for each $t \in [0,1]$. Finally, the pair of functions (u,v) given by $u(t) = -3\int_0^1 G(t,s)\,ds$ and $v(t) = 3\int_0^1 G(t,s)\,ds$ is a lower coupled solution of the coupled HPBVPs (5.13) and (5.14) defined on $J = [0,1]$, where the Green's function G is defined by

$$G(t,s) = \begin{cases} \dfrac{e^{s-t+1}}{e-1}, & \text{if} \quad 0 \leqslant s \leqslant t \leqslant 1, \\[2ex] \dfrac{e^{s-t}}{e-1}, & \text{if} \quad 0 \leqslant t < s \leqslant 1. \end{cases}$$

Further, $KT\varphi(r) \leqslant \frac{1}{2} \cdot \frac{e}{e-1} \cdot \frac{r}{1+r} < r$ for $r > 0$. Thus the functions f_1 and g_1 satisfy all the hypotheses (H_1) through (H_4) of Theorem 3.1 and therefore, the coupled HPBVPs (5.13) and (5.14) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$
\begin{aligned}
x_0(t) &= -3 \int_0^1 G(t,s)\,ds, \\
x_{n+1}(t) &= \int_0^1 G(t,s) f_1(s, x_n(s), y_n(s))\,ds \\
&\quad + \int_0^1 G(t,s) g_1(s, x_n(s), y_n(s))\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
y_0(t) &= 3 \int_0^1 G(t,s)\,ds, \\
y_{n+1}(t) &= \int_0^1 G(t,s) f_1(s, y_n(s), x_n(s))\,ds \\
&\quad + \int_0^1 G(t,s) g_1(s, y_n(s), x_n(s))\,ds
\end{aligned}
$$

for all $t \in [0,1]$, converge monotonically to x^* and y^* respectively.

REMARK 5.2. In the present study we have applied the newly developed coupled hybrid fixed point theorem to a very simple and well-known PBVP of first order ordinary differential equations. However, the technique can also be extended and applied to other BVPs of nonlinear second or higher order differential equations under suitable boundary conditions for proving the existence as well as approximation result and developing the algorithms for the solutions. Finally, while concluding this paper we mention that the study of this section may be extended with similar arguments to the coupled hybrid periodic boundary value problems of nonlinear second order ordinary differential equations,

$$
\left.
\begin{array}{c}
-x''(t) + \lambda^2 x(t) = f(t, x(t), y(t)) + g(t, x(t), y(t)),\ t \in J, \\
x(0) = x(T),\ x'(0) = x'(T),
\end{array}
\right\}
\tag{5.15}
$$

and

$$
\left.
\begin{array}{c}
-y''(t) + \lambda^2 y(t) = f(t, y(t), x(t)) + g(t, y(t), x(t)),\ t \in J, \\
y(0) = y(T),\ y'(0) = y'(T),
\end{array}
\right\}
\tag{5.16}
$$

for $\lambda \in \mathbb{R},\ \lambda > 0$, where the functions $f, g : J \times \mathbb{R} \times \mathbb{R}$ are continuous and satisfy certain mixed hybrid conditions from algebra, geometry and topology.

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