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MAXIMUM PRINCIPLE FOR A FOURTH ORDER BOUNDARY VALUE PROBLEM

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Abstract. We consider a boundary value problem for the beam equation. Some upper and lower bounds for positive solutions of the boundary value problem are obtained. As an application, some new sufficient conditions for the existence and nonexistence of positive solutions for the boundary value problem are established.

1. Introduction

In this paper, we consider the fourth order differential equation

$$u^{(4)}(t) = g(t)f(u(t)), \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

together with boundary conditions

$$u(0) = \alpha u'(0) - \beta u''(0) = \gamma u'(1) + \delta u''(1) = u'''(1) = 0. \tag{1.2}$$

Throughout this paper, we assume that

(H1) $\alpha, \beta, \gamma, \delta$ are nonnegative constants such that $\rho := \alpha\gamma + \alpha\delta + \gamma\beta > 0$, $f : [0,\infty) \to [0,\infty)$ and $g : [0,1] \to [0,\infty)$ are continuous functions, and $g(t) \not\equiv 0$ on $[0,1]$.

Equation (1.1) is often referred to as the beam equation because it describes the deflection or deformation of an elastic beam under a force. The boundary conditions (1.2) include several important cases. When $\alpha = \delta = 1$ and $\beta = \gamma = 0$, the conditions (1.2) reduce to

$$u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{1.3}$$

which mean that the beam is embedded at the end $t = 0$ and free at the other end $t = 1$. When $\alpha = \delta = 0$ and $\beta = \gamma = 1$, the conditions (1.2) reduce to

$$u(0) = u''(0) = u'(1) = u'''(1) = 0, \tag{1.4}$$

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which mean that the beam is fulcrum supported at $t=0$ and supported by a sliding clamp at the other end $t=1$. When $\alpha=\gamma=1$ and $\beta=\delta=0$, the conditions (1.2) reduce to

$$u(0)=u'(0)=u'(1)=u'''(1)=0, \tag{1.5}$$

which mean that the beam is embedded at $t=0$ and supported by a sliding clamp at the other end $t=1$.

In 2005, Yang [21] considered the boundary value problem that consists of (1.1) and (1.3), and proved the following maximum principle.

THEOREM 1. *If $u \in C^4[0,1]$ satisfies the boundary conditions (1.3), and is such that*

$$u^{(4)}(t) \geqslant 0, \quad 0 \leqslant t \leqslant 1, \tag{1.6}$$

then

$$\frac{t^2}{2}(3-t)u(1) \leqslant u(t) \leqslant tu(1), \quad 0 \leqslant t \leqslant 1.$$

In 2000, Graef and Yang [10] considered the boundary value problem that consists of (1.1) and (1.4), and proved the following maximum principle.

THEOREM 2. *If $u \in C^4[0,1]$ satisfies (1.6) and the boundary conditions (1.4), then*

$$\frac{t}{2}(3-t^2)u(1) \leqslant u(t) \leqslant t(2-t)u(1), \quad 0 \leqslant t \leqslant 1.$$

In 2005, Yang [22] considered the boundary value problem that consists of (1.1) and (1.5), and proved the following maximum principle.

THEOREM 3. *If $u \in C^4[0,1]$ satisfies (1.6) and the boundary conditions (1.5), then*

$$t^2(3-2t)u(1) \leqslant u(t) \leqslant t(2-t)u(1), \quad 0 \leqslant t \leqslant 1.$$

In this paper, we will study positive solutions of the problem (1.1), (1.2) by a unified approach. Here, by a positive solution, we mean a solution $u(t)$ such that $u(t) > 0$ for $t \in (0,1)$. It is now well known that maximum principles, like the ones mentioned above, play an important role in finding positive solution for boundary value problems of the beam equation. The main purpose of this paper is to prove a maximum principle, which gives some upper and lower bounds for positive solutions of the problem (1.1), (1.2). These upper and lower bounds include Theorems 1, 2, and 3 as special cases.

For some other results on boundary value problems of the beam equation, we refer the reader to the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 23].

This paper is organized as follows. In Section 2, we give the Green function for the problem (1.1), (1.2), state the Krasnosel'skii's fixed point theorem, and fix some notations. In Section 3, we present some upper and lower estimates to positive solutions to the problem (1.1), (1.2). In Section 4, we establish some existence and nonexistence results for positive solutions to the problem (1.1), (1.2).

2. Preliminaries

The Green function $G:[0,1]\times[0,1]\to[0,\infty)$ for the problem (1.1), (1.2) is

$$G(t,s)=\frac{ts(2\beta+\alpha t)}{4\rho}(2\delta+2\gamma-\gamma s)-\frac{t^3}{6}+\frac{(t-s)^3}{6}H(t-s).$$

Here $H:\mathbb{R}\to\mathbb{R}$ is the unit step function given by

$$H(t)=\begin{cases}1, & \text{if } t\geqslant 0,\\ 0, & \text{if } t<0.\end{cases}$$

In this paper, we denote by $\mathbb{R}$ the set of real numbers. We leave it to the reader to verify that $G(t,s)$ is the Green function for problem (1.1), (1.2). Now problem (1.1), (1.2) is equivalent to the integral equation

$$u(t)=\int_0^1 G(t,s)g(s)f(u(s))\,ds,\quad 0\leqslant t\leqslant 1. \tag{2.1}$$

It is easy to verify that G is a continuous function. It is also easy to see that

$$G(1,s)=\frac{s(2\beta+\alpha)}{4\rho}(2\delta+2\gamma-\gamma s)-\frac{1}{6}+\frac{(1-s)^3}{6},\quad 0\leqslant s\leqslant 1.$$

We can rewrite $G(1,s)$ into the following form:

$$G(1,s)\ =\ \frac{s}{12\rho}\left(4\gamma\beta+12\beta\delta+\alpha s(\gamma+4\delta)+2\gamma\beta(1-s)+2\rho s(1-s)\right).$$

Therefore, we have $G(1,s)>0$ for $0<s\leqslant 1$.

We will need the following fixed point theorem, which is due to Krasnosel'skii [13], to prove our existence results in section 4.

THEOREM 4. *Let $(X,\|\cdot\|)$ be a Banach space over the reals, and let $P\subset X$ be a cone in X. Let H_1 and H_2 be real numbers such that $H_2>H_1>0$, and let*

$$\Omega_i=\{v\in X\mid \|v\|<H_i\},\quad i=1,2.$$

If $L:P\cap(\overline{\Omega_2}-\Omega_1)\to P$ is a completely continuous operator such that, either

(K1) $\|Lv\|\leqslant\|v\|$ *if* $v\in P\cap\partial\Omega_1$, *and* $\|Lv\|\geqslant\|v\|$ *if* $v\in P\cap\partial\Omega_2$, *or*

(K2) $\|Lv\|\geqslant\|v\|$ *if* $v\in P\cap\partial\Omega_1$, *and* $\|Lv\|\leqslant\|v\|$ *if* $v\in P\cap\partial\Omega_2$.

Then L has a fixed point in $P\cap(\overline{\Omega_2}-\Omega_1)$.

For the rest of this paper, we let $X=C[0,1]$ be equipped with the supremum norm

$$\|v\|=\max_{t\in[0,1]}|v(t)|,\quad \forall v\in X.$$

Clearly X is a Banach space. We define

$$Y = \{v \in X \,|\, v(t) \geqslant 0 \text{ for } 0 \leqslant t \leqslant 1\},$$

and define the operator $T : Y \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \leqslant t \leqslant 1. \tag{2.2}$$

It is clear that if (H1) holds, then $T(Y) \subset Y$ and $T : Y \to Y$ is a completely continuous operator. We also define the constants

$$F_0 = \limsup_{x \to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x \to 0^+} \frac{f(x)}{x},$$
$$F_\infty = \limsup_{x \to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x \to +\infty} \frac{f(x)}{x}.$$

These constants, which are associated with the function f, will be used in Section 4.

3. Upper and lower estimates for positive solutions

In this section, we shall give some upper and lower estimates for positive solutions of the problem (1.1), (1.2). To this purpose, we define the functions $a : [0,1] \to \mathbb{R}$ and $b : [0,1] \to \mathbb{R}$ by

$$a(t) = \frac{t(12\beta\delta + 6\gamma\beta + 6\alpha\delta t + 3\alpha\gamma t - 2\rho t^2)}{12\beta\delta + 4\gamma\beta + 4\alpha\delta + \alpha\gamma}, \quad 0 \leqslant t \leqslant 1,$$

$$b(t) = \frac{t(2\delta + 2\gamma - \gamma t)}{2\delta + \gamma}, \quad 0 \leqslant t \leqslant 1.$$

It is easy to see that $a(0) = b(0) = 0$ and $a(1) = b(1) = 1$. Since

$$a'(t) = \frac{6(2\beta\delta + \alpha\delta t + (\gamma\beta + \rho t)(1-t))}{12\beta\delta + 4\gamma\beta + 4\alpha\delta + \alpha\gamma} \geqslant 0, \quad 0 \leqslant t \leqslant 1,$$

$$b'(t) = \frac{2(\delta + \gamma(1-t))}{2\delta + \gamma} \geqslant 0, \quad 0 \leqslant t \leqslant 1,$$

we have $a(t) > 0$ on $(0,1]$ and $b(t) > 0$ on $(0,1]$. Since

$$b(t) - a(t) = \frac{2t(1-t)(2\delta + (\gamma + 2\delta)(1-t))(\alpha\gamma + \alpha\delta + \gamma\beta)}{(2\delta + \gamma)(12\beta\delta + 4\gamma\beta + 4\alpha\delta + \alpha\gamma)} \geqslant 0, \quad 0 \leqslant t \leqslant 1,$$

$$a(t) - t^2 = \frac{2t(1-t)(6\beta\delta + 3\gamma\beta + \rho t)}{12\beta\delta + 4\gamma\beta + 4\alpha\delta + \alpha\gamma} \geqslant 0, \quad 0 \leqslant t \leqslant 1,$$

we have $b(t) \geqslant a(t) \geqslant t^2$ for $0 \leqslant t \leqslant 1$.

The next lemma provides a lower bound for the Green function $G(t,s)$.

LEMMA 1. *We have* $G(t,s) \geqslant a(t)G(1,s)$ *for* $0 \leqslant t,s \leqslant 1$.

Proof. It suffices to show that $G(t,s) - a(t)G(1,s) \geqslant 0$ for $0 \leqslant t,s \leqslant 1$. We take two cases to prove the lemma. Our strategy is to decompose $G(t,s) - a(t)G(1,s)$ into several pieces, so that each piece is nonnegative.

If $0 \leqslant s \leqslant t \leqslant 1$, then

$$\begin{aligned}
&G(t,s) - a(t)G(1,s) \\
&= \frac{s(1-t)}{6(12\beta\delta + 4\gamma\beta + 4\alpha\delta + \alpha\gamma)}[\alpha\gamma s(1-t)(3t - s - 2ts) \\
&\quad + 2\gamma\beta(1-t)(3t - ts^2 - 2s^2) + 12\beta\delta(2t - t^2 - s^2) \\
&\quad + 2\alpha\delta s(s(1-s)(4-s) + (3-s)(2-s-t)(t-s))] \\
&\geqslant 0.
\end{aligned}$$

If $0 \leqslant t \leqslant s \leqslant 1$, then

$$\begin{aligned}
&G(t,s) - a(t)G(1,s) \\
&= \frac{t(1-s)}{6(12\beta\delta + 4\gamma\beta + 4\alpha\delta + \alpha\gamma)}[\alpha\gamma t(1-s)(3s - t - 2ts) \\
&\quad + 2\gamma\beta(1-s)(3s - st^2 - 2t^2) + 12\beta\delta(2s - s^2 - t^2) \\
&\quad + 2\alpha\delta t(t(1-t)(4-t) + (3-t)(2-s-t)(s-t))] \\
&\geqslant 0.
\end{aligned}$$

The proof is complete. $\square$

The next lemma provides an upper bound for the Green function $G(t,s)$.

LEMMA 2. *We have* $G(t,s) \leqslant b(t)G(1,s)$ *for* $0 \leqslant t,s \leqslant 1$.

Proof. We take two cases to prove the lemma. If $0 \leqslant s \leqslant t \leqslant 1$, then

$$b(t)G(1,s) - G(t,s) = \frac{s^3(1-t)(2\delta + \gamma(1-t))}{6(2\delta + \gamma)} \geqslant 0.$$

If $0 \leqslant t \leqslant s \leqslant 1$, then

$$\begin{aligned}
&b(t)G(1,s) - G(t,s) \\
&= \frac{t}{6(2\delta + \gamma)}[(s-t)(2\delta(2s-t) + (s-t)\gamma + s(1-s^2)\gamma) \\
&\quad + s^2(1-s)(2\delta + \gamma(1-s))] \\
&\geqslant 0.
\end{aligned}$$

The proof is complete. $\square$

LEMMA 3. *We have* $G(t,s) \geqslant 0$ *for* $0 \leqslant t,s \leqslant 1$.

Proof. The lemma follows immediately from Lemma 1 and the positivity of $a(t)$ and $G(1,s)$. The proof is complete. □

The next theorem is the main result of this section.

THEOREM 5. *If* $u \in C^4[0,1]$ *satisfies the boundary conditions* (1.2)*, and*

$$u^{(4)}(t) \geqslant 0 \quad \textit{for} \quad 0 \leqslant t \leqslant 1, \tag{3.1}$$

then

$$u(t) \geqslant 0, \quad 0 \leqslant t \leqslant 1, \tag{3.2}$$

$$a(t)u(1) \leqslant u(t) \leqslant b(t)u(1), \quad 0 \leqslant t \leqslant 1, \tag{3.3}$$

and $u(1) = \|u\|$.

Proof. Suppose $u \in C^4[0,1]$ satisfies (1.2) and (3.1). If $0 \leqslant t \leqslant 1$, then

$$u(t) = \int_0^1 G(t,s)u^{(4)}(s)ds \geqslant 0,$$

$$u(t) = \int_0^1 G(t,s)u^{(4)}(s)ds \geqslant a(t)\int_0^1 G(1,s)u^{(4)}(s)ds = a(t)u(1),$$

and

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)u^{(4)}(s)ds \leqslant b(t)\int_0^1 G(1,s)u^{(4)}(s)ds \\ &= b(t)u(1) \leqslant b(1)u(1) = u(1). \end{aligned}$$

The proof of the lemma is complete. □

The next theorem follows immediately from Theorem 5.

THEOREM 6. *Suppose that (H1) holds. If* $u \in C^4[0,1]$ *is a non-negative solution to the problem* (1.1)*,* (1.2)*, then* $u(t)$ *satisfies* (3.2) *and* (3.3).

We now define

$$P = \left\{ v \in X : v(1) \geqslant 0,\ a(t)v(1) \leqslant v(t) \leqslant b(t)v(1) \text{ on } [0,1] \right\}.$$

Clearly P is a positive cone in X. It is obvious that if $u \in P$, then $u(1) = \|u\|$. We see from Theorem 6 that if $u(t)$ is a nonnegative solution to the problem (1.1), (1.2), then $u \in P$. In a similar fashion to Theorem 5, we can show that $T(P) \subset P$. To find a positive solution to the problem (1.1), (1.2), we need only to find a fixed point u of T such that $u \in P$ and $u(1) = \|u\| > 0$.

4. Existence and nonexistence results

First, we define some important constants:

$$A = \int_0^1 G(1,s)g(s)a(s)\,ds, \quad B = \int_0^1 G(1,s)g(s)b(s)\,ds.$$

The next two theorems provide sufficient conditions for the existence of at least one positive solution for the problem (1.1), (1.2).

THEOREM 7. *Suppose that (H1) holds. If $BF_0 < 1 < Af_\infty$, then the problem* (1.1), (1.2) *has at least one positive solution.*

Proof. First, we choose $\varepsilon > 0$ such that $(F_0+\varepsilon)B \leqslant 1$. By the definition of F_0, there exists $H_1 > 0$ such that $f(x) \leqslant (F_0+\varepsilon)x$ for $0 < x \leqslant H_1$. Now for each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\leqslant \int_0^1 G(1,s)g(s)(F_0+\varepsilon)u(s)\,ds \\ &\leqslant (F_0+\varepsilon)\|u\| \int_0^1 G(1,s)g(s)b(s)\,ds \\ &= (F_0+\varepsilon)\|u\|B \leqslant \|u\|,\end{aligned}$$

which means $\|Tu\| \leqslant \|u\|$. Thus, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$\|Tu\| \leqslant \|u\| \ \text{ for } \ u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\delta > 0$ and $\tau \in (0, 1/4)$ such that

$$\int_\tau^1 G(1,s)g(s)a(s)\,ds \cdot (f_\infty - \delta) \geqslant 1.$$

There exists $H_3 > 2H_1$ such that $f(x) \geqslant (f_\infty - \delta)x$ for $x \geqslant H_3$. Let $H_2 = H_3/\tau^2$. If $u \in P$ such that $\|u\| = H_2$, then for each $t \in [\tau, 1]$, we have

$$u(t) \geqslant H_2 a(t) \geqslant H_2 t^2 \geqslant H_2 \tau^2 = H_3.$$

Therefore, for each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}(Tu)(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &\geqslant \int_\tau^1 G(1,s)g(s)f(u(s))\,ds \\ &\geqslant \int_\tau^1 G(1,s)g(s)(f_\infty - \delta)u(s)\,ds \\ &\geqslant \int_\tau^1 G(1,s)g(s)a(s)\,ds \cdot (f_\infty - \delta)\|u\| \geqslant \|u\|,\end{aligned}$$

which means $\|Tu\| \geqslant \|u\|$. Thus, if we let $\Omega_2 = \{u \in X \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$\|Tu\| \geqslant \|u\| \quad \text{for} \quad u \in P \cap \partial\Omega_2.$$

Now that the condition (K1) of Theorem 4 is satisfied, there exists a fixed point of T in $P \cap (\overline{\Omega_2} - \Omega_1)$. The proof is now complete. □

THEOREM 8. *Suppose that (H1) holds. If $BF_\infty < 1 < Af_0$, then the problem* (1.1)*,* (1.2) *has at least one positive solution.*

The proof of Theorem 8 is very similar to that of Theorem 7 and is therefore omitted. The next two theorems provide sufficient conditions for the nonexistence of positive solutions to the problem (1.1), (1.2).

THEOREM 9. *Suppose (H1) holds. If $Bf(x) < x$ for all $x > 0$, then the problem* (1.1)*,* (1.2) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (1.1), (1.2). Then $u \in P$, $u(t) > 0$ for $0 < t \leqslant 1$, and

$$\begin{aligned} u(1) &= \int_0^1 G(1,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(1,s)g(s)u(s)\,ds \\ &\leqslant B^{-1}\int_0^1 G(1,s)g(s)b(s)\,ds \cdot u(1) \\ &= B^{-1}Bu(1) = u(1), \end{aligned}$$

which is a contradiction. The proof is complete. □

THEOREM 10. *Suppose (H1) holds. If $Af(x) > x$ for all $x > 0$, then the problem* (1.1)*,* (1.2) *has no positive solutions.*

The proof of Theorem 10 is very similar to that of Theorem 9 and is therefore left to the reader.

We conclude the paper with an example.

EXAMPLE 1. Consider the fourth order equation

$$u^{(4)}(t) = \lambda(t+t^2)\frac{u(t)(1+2u(t))}{1+u(t)}, \quad 0 \leqslant t \leqslant 1, \tag{4.1}$$

where $\lambda > 0$ is a parameter, together with the boundary conditions

$$u(0) = u'(0) - u''(0) = u'(1) + u''(1) = u'''(1) = 0. \tag{4.2}$$

In this example, $g(t) = (t+t^2)$ and $f(u) = \lambda u(1+2u)/(1+u)$. It is easy to see that $f_0 = F_0 = \lambda$, $f_\infty = F_\infty = 2\lambda$, and

$$\lambda x < f(x) < 2\lambda x, \quad x > 0.$$

We also have

$$G(1,s) = \frac{s}{12}(6+3s-2s^2), \quad 0 \leqslant s \leqslant 1.$$

Calculations indicate that $A = 14167/52920$ and $B = 851/3024$. By Theorem 7, if

$$1.8677 \approx 1/(2A) < \lambda < 1/B \approx 3.553,$$

then the problem (4.1), (4.2) has at least one positive solution. From Theorems 9 and 10 we see that if

$$\lambda \leqslant 1/(2B) \approx 1.7767 \quad \text{or} \quad \lambda \geqslant 1/A \approx 3.7354,$$

then the problem (4.1), (4.2) has no positive solutions.

This example shows that our existence and nonexistence results are quite sharp.

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