Differential
Equations
& Applications
Volume 9, Number 4 (2017), 505–520 doi:10.7153/dea-2017-09-34

MULTIPLE POSITIVE SOLUTIONS FOR A CHOQUARD EQUATION INVOLVING BOTH CONCAVE–CONVEX AND HARDY–LITTLEWOOD–SOBOLEV CRITICAL EXPONENT

R. ECHARGHAOUI, M. KHIDDI AND S. M. SBAI

(*Communicated by Claudianor O. Alves*)

Abstract. In this paper, we consider a Choquard equation involving both concave-convex and Hardy-Littlewood-Sobolev critical exponent. By using the $\mathcal{N}$ehari manifold, fibering maps and the Lusternik-Schnirelman category, we prove that the problem has at least $\operatorname{cat}(\Omega)+1$ distinct positive solutions.

1. Introduction and main result

In this paper, we are concerned with the multiplicity of positive solutions of the following critical nonlocal problem

$$\begin{cases} -\Delta u = \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u|^{2^*_\mu-2}u + \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \in \Omega$ is a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, $0 < q < 1$, λ is a positive parameter, $0 < \mu < N$, and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical Sobolev exponent (in the sense of the Hardy-Littlewood-Sobolev inequality). This problem has a wide ring of application in physics and related sciences such as quantum theory of a polaron at rest by S. Pekar in [16] and the modeling of an electron trapped in its own hole in the work of P. Choquard, as well as a certain approximation to the Hartree-Fock theory of one-component plasma [13]. For a complete and updated discussion on the current literature of such problems, we refer the reader to the guide [15]. Recently, many papers have studied the multiplicity of positive solutions by way of $\mathcal{N}$ehari manifolds, fibering maps and the Lusternik-Schnirelman category for different semilinear, quasilinear, and nonlocal problems involving a critical exponent and concave and convex nonlinearities (see [4, 6, 10]). Our purpose here continue this line of work by relating the number of positive solutions of a nonlinear Choquard equation (1.1) to topology of Ω. Several works have been devoted to the study of nonlinear Choquard equations of the type (1.1). The reader can find a lot of papers in the literature involving this subject, we cite [1], [2], [6], [9], [8], [14]. The main result is the following.

Mathematics subject classification (2010): 35J25, 35J60, 35A15.
Keywords and phrases: Choquard equation, $\mathcal{N}$ehari manifold, Concave and convex nonlinearities, Lusternik-Schnirelman category.

THEOREM 1. *Let $N > 4$ and $\frac{2}{N-2} \leqslant q < 1$. Then, there exists $\wedge_* > 0$ such that if for each $\lambda^{\frac{2}{1-q}} \in (0, \wedge_*)$, problem (1.1) has a least $cat(\Omega) + 1$ distinct positive solutions.*

To establish our main result we follow, as in [1], [3], [5], a classical approach, some techniques employed in [18], and an argument developed in [8]. The paper is organized as follows. In Section 2, we fix some notations and give some preliminary results and known facts. In section 3, we show some technical lemmas which enable us to construct homotopies between Ω and certain sublevel set of the energy functional associated to (1.1). In section 4, we prove theorem 1.

2. Some notations and preliminaries

In this section, we recall some preliminary results that are required in the later sections.

We denote $|\cdot|_p$ as the standard $L^p(\Omega)$ norm with $1 < p < \infty$, and $\|\cdot\|$ for $H_0^1(\Omega)$ norm. we set $|\Omega|$ the Lebesgue measure of Ω and $\int_\Omega |u|^q dx \leqslant C_q \|u\|^q$. The following well-known Hardy-Littlewood-Sobolev inequality [12] is key in order to follow a variational approach for our problem (1.1).

Proof. Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N$ and $h \in L^r(\mathbb{R}^N$. There exists a sharp a constant $C(t, N, \mu, r)$, independent of f, h, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} dxdy \leqslant C(t, N, \mu, r)|f|_t |h|_r.$$

If $t = r = \frac{2N}{2N-\mu}$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{-1+\frac{\mu}{N}}$$

In this case there is equality in (6) if and only if $f = Ch$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{-\frac{2N-\mu}{2}}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$. Notice that, by Hardy-Littlewood-Sobolev inequality, the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dxdy$$

is well defined if

$$\frac{2N-\mu}{N} \leqslant q \leqslant \frac{2N-\mu}{N-2}$$

We say $\frac{2N-\mu}{N}$ is the lower critical exponent and $2_\mu^* = \frac{2N-\mu}{N-2}$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. From this inequality, for each $u \in \mathscr{D}^{1,2}(\mathbb{R}^N)$, we have

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy\right)^{\frac{1}{2_\mu^*}} \leqslant C(N, \mu)^{\frac{1}{2_\mu^*}} |u|_{2^*}^2$$

where $C(N,\mu)$ is a suitable constant defined in Proposition 2 and $2^* = \frac{2N}{N-2}$. □

We use $S_{H,L}$ to denote the best constant defined by

$$S_{H,L} = \inf_{u \in \mathscr{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy\right)^{\frac{1}{2^*_\mu}}}. \tag{2.1}$$

The constant $S_{H,L}$ defined in (2.1) is achieved if and only if

$$u(x) = C\left(\frac{t}{t^2+|x-a|^2}\right)^{\frac{N-2}{2}}$$

where $C>0$ is a fixed constant, $a \in \mathbb{R}^N$ and $t>0$ are parameters (refer Lemma 1.2 of [9]). Moreover,

$$S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}},$$

where S is the best Sobolev constant. Let

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}} \tag{2.2}$$

be a minimizer for S, see [18], then

$$\widetilde{U}(x) = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}} \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$$

is the unique minimizer for $S_{H,L}$ that satisfies

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right)|u|^{2^*_\mu-2}u \text{ in } \mathbb{R}^N$$

with

$$\int_{\mathbb{R}^N} |\nabla \widetilde{U}|^2 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\widetilde{U}(x)|^{2^*_\mu}|\widetilde{U}(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Moreover, Let $N \geqslant 3$, for every open subset Ω of $\mathbb{R}^N$,

$$S_{H,L}(\Omega) = \inf_{u \in \mathscr{D}^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy\right)^{\frac{1}{2^*_\mu}}} = S_{H,L}, \tag{2.3}$$

$S_{H,L}(\Omega)$ is never achieved except $\Omega = \mathbb{R}^N$, (see [9]).

The energy functional associated to equation (1.1) is defined by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{1}{22^*_\mu}\mathbb{D}(u) - \frac{\lambda}{q+1}|u|_{q+1}^{q+1}, \tag{2.4}$$

where

$$\mathbb{D}(u) := \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy, \text{ and } |u|_{q+1}^{q+1} = \int_\Omega u^{q+1} dx.$$

The Hardy-Littlewood-Sobolev inequality implies that I_λ is well defined on $H_0^1(\Omega)$ and belong to $\mathscr{C}^1(H_0^1(\Omega),\mathbb{R})$ with its derivative given by

$$\langle I'_\lambda(u),\varphi\rangle = \int_\Omega \nabla u \nabla\varphi dx - \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu-2}u(y)\varphi(y)}{|x-y|^\mu}|dxdy - \lambda\int_\Omega u^q\varphi dx,$$

for all u, φ in $H_0^1(\Omega)$.

Therefore, the solutions of (1.1) correspond to critical points of the energy I_λ. let us denote by $\mathcal{N}_\lambda$ the $\mathcal{N}$ehari manifold related to I_λ, given by

$$\mathcal{N}_\lambda := \{u \in H_0^1(\Omega), u \neq 0 : \langle I_\lambda^{'}(u), u\rangle = 0\},$$

namely

$$\mathcal{N}_\lambda := \{u \in H_0^1(\Omega), u \neq 0 : \|u\|^2 = \mathbb{D}(u) + \lambda |u|_{q+1}^{q+1}\}.$$

For $t > 0$, we define the fibering maps

$$\phi_u(t) := I_\lambda(u) = \frac{t^2}{2}\|u\|^2 - \frac{t^{22_\mu^*}}{22_\mu^*}\mathbb{D}(u) - \lambda \frac{t^{q+1}}{q+1}|u|_{q+1}^{q+1}.$$

Then we have

$$\phi_u'(t) = t\|u\|^2 - t^{22_\mu^*-1}\mathbb{D}(u) - \lambda t^q |u|_{q+1}^{q+1}.$$

and

$$\phi_u''(t) = \|u\|^2 - (22_\mu^* - 1)t^{22_\mu^*-2}\mathbb{D}(u) - q\lambda t^{q-1}|u|_{q+1}^{q+1}.$$

It is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\phi_u'(t) = 0$, and in particular for $t = 1$ we have $u \in \mathcal{N}_\lambda$. The elements in $\mathcal{N}_\lambda$ correspond to stationary of fibering maps $\phi(t)$. Thus, for $u \in \mathcal{N}_\lambda$, we have

$$\begin{aligned} \phi_u''(1) &= (1-q)\|u\|^2 - (22_\mu^* - 1 - q)\mathbb{D}(u), \\ &= (2 - 22_\mu^*)\|u\|^2 - (q + 1 - 22_\mu^*)\lambda |u|_{q+1}^{q+1}. \end{aligned}$$

Therefore, we can split the $\mathcal{N}$ehari manifold $\mathcal{N}_\lambda$ into three parts. Namely:

$$\begin{aligned} \mathcal{N}_\lambda^+ &:= \{u \in \mathcal{N}_\lambda : \phi_u''(1) > 0\} \\ \mathcal{N}_\lambda^- &:= \{u \in \mathcal{N}_\lambda : \phi_u''(1) < 0\} \\ \mathcal{N}_\lambda^0 &:= \{u \in \mathcal{N}_\lambda : \phi_u''(1) = 0\} \end{aligned}$$

LEMMA 1. *If u_0 is a local minimizer of I_λ on $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$. Then u_0 is a critical point of I_λ.*

Proof. The proof is the same as that in [10], we give it here for completeness. Set $J_\lambda(u) = \langle I_\lambda'(u), u\rangle$. Since u_0 is a local minimizer of I_λ under the constraint $I_\lambda(u_0) = 0$, by the theory of Lagrange multipliers, there exists $\sigma \in \mathbb{R}$ such that

$$I_\lambda'(u_0) = \sigma J_\lambda(u_0)$$

Thus implies

$$\langle I_\lambda'(u_0), u_0\rangle = \sigma\langle I_\lambda'(u_0), u_0\rangle = \sigma\phi_{u_0}''(1).$$

Since $u_0 \notin \mathcal{N}_\lambda^0$, so $\phi_{u_0}''(1) \neq 0$. Hence $\sigma = 0$. We complete the proof. □

LEMMA 2. *There exists $\Lambda_* > 0$ such that $\lambda^{\frac{2}{1-q}} \in (0, \Lambda_*)$, such that $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. We suppose that there exists $\lambda \in (0, \Lambda_*)$, such that $\mathcal{N}_\lambda^0 \neq \emptyset$ Let $u \in \mathcal{N}_\lambda^0$, we have

$$\phi_u''(1) = (1-q)\|u\|^2 + (22_\mu^* - 1 - q)\mathbb{D}(u) = 0,$$

then

$$\|u\|^2 = \frac{(22_\mu^* - 1 - q)}{(1-q)}\mathbb{D}(u). \tag{2.5}$$

And by definition of $S_{H,L}$ given in (2.3), we have

$$\|u\|^2 \geqslant S_{H,L}\mathbb{D}^{\frac{1}{2_\mu^*}}(u), \tag{2.6}$$

so with (2.5) and (2.6), we have

$$\|u\| \geqslant S_{H,L}^{\frac{2_\mu^*}{2(2_\mu^*-1)}}\left(\frac{1-q}{22_\mu^*-1-q}\right)^{\frac{1}{2(2_\mu^*-1)}}. \tag{2.7}$$

Since,

$$\phi_u''(1) = (2 - 22_\mu^*)\|u\|^2 - (q+1-22_\mu^*)\lambda|u|_{q+1}^{q+1} = 0,$$

and by using the above equality and the Sobolev inequality, we have

$$\|u\| \leqslant \left((22_\mu^* - 1 - q)S^{-\frac{q+1}{2}}|\Omega|^{\frac{2^*-1-q}{2}}\right)^{\frac{1}{1-q}}\lambda^{\frac{1}{1-q}} \tag{2.8}$$

by (2.6) and (2.8), we can deduce that

$$\lambda^{\frac{2}{1-q}} \geqslant \left[S_{H,L}^{2_\mu^*}\left(\frac{1-q}{22_\mu^*-1-q}\right)\right]^{\frac{1}{2(2_\mu^*-1)}}\left[(22_\mu^*-1-q)S^{-\frac{q+1}{2}}|\Omega|^{\frac{2^*-1-q}{2}}\right]^{\frac{-2}{1-q}} = \Lambda_*.$$

Which is a contradiction. □

Then, we can write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^+$ and define

$$c_\lambda = \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u),\ c_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u),\ c_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u).$$

LEMMA 3. *There exists $\Lambda_* > 0$ such that $\lambda^{\frac{2}{1-q}} \in (0, \Lambda_*)$, then*

i) $c_\lambda^+ < 0$

ii) there exist $\rho_0 > 0$ such that $I_\lambda(u) \geqslant \rho_0$ for all $u \in \mathcal{N}_\lambda^-$.

Proof. i) Let $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we have

$$I_\lambda(u) = \Big(\frac{1}{2} - \frac{1}{q+1}\Big)\|u\|^2 - \Big(\frac{1}{22_\mu^*} - \frac{1}{q+1}\Big)\mathbb{D}(u),$$

and

$$(1-q)\|u\|^2 - (22_\mu^* - 1 - q)\mathbb{D}(u) > 0,$$

this implies

$$I_\lambda(u) < (q-1)\frac{2_\mu^* - 1}{22_\mu^*(q+1)}\|u\|^2 < 0.$$

Then $c_\lambda^+ < 0$.

ii) Let $u \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, we have

$$(1-q)\|u\|^2 < (22_\mu^* - 1 - q)\mathbb{D}(u), \tag{2.9}$$

and by definition of $S_{H,L}$ given in (2.3), we have

$$S_{H,L}\mathbb{D}^{\frac{1}{2_\mu^*}}(u) \leqslant \|u\|^2. \tag{2.10}$$

By (2.9) and (2.10), we deduce that

$$\|u\| > \left[S_{H,L}^{2_\mu^*}\left(\frac{1-q}{22_\mu^*-1-q}\right)\right]^{\frac{1}{2(2_\mu^*-1)}}. \tag{2.11}$$

Then

$$\begin{aligned}
I_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{22_\mu^*}\right)\|u\|^2 - \lambda\left(\frac{1}{q+1} - \frac{1}{22_\mu^*}\right)|u|_{q+1}^{q+1} \\
&\geqslant \left(\frac{1}{2} - \frac{1}{22_\mu^*}\right)\|u\|^2 - \lambda\left(\frac{1}{q+1} - \frac{1}{22_\mu^*}\right)S^{-\frac{q+1}{2}}|\Omega|^{\frac{2^*-1-q}{2}}\|u\|^{q+1} \\
&\geqslant \|u\|^{q+1}\left[\left(\tfrac{1}{2} - \tfrac{1}{22_\mu^*}\right)\|u\|^{1-q} - \lambda\left(\tfrac{1}{q+1} - \tfrac{1}{22_\mu^*}\right)S^{-\frac{q+1}{2}}|\Omega|^{\frac{2^*-1-q}{2}}\right] \\
&\geqslant \left[S_{H,L}^{2_\mu^*}\left(\tfrac{1-q}{22_\mu^*-1-q}\right)\right]^{\frac{q+1}{2(2_\mu^*-1)}}\left\{\left(\frac{1}{2} - \frac{1}{22_\mu^*}\right)\left[S_{H,L}^{2_\mu^*}\left(\tfrac{1-q}{22_\mu^*-1-q}\right)\right]^{\frac{1-q}{2(2_\mu^*-1)}}\right. \\
&\quad \left. -\left(\frac{1}{q+1} - \frac{1}{22_\mu^*}\right)S^{-\frac{q+1}{2}}|\Omega|^{\frac{2^*-1-q}{2}}\right\}.
\end{aligned}$$

So, there exists $\Lambda_* > 0$ small enough and $\rho_0 > 0$ such that if $\lambda \in (0, \Lambda_*)$, $c_\lambda^+ \geqslant \rho_0 > 0$ for all $u \in \mathcal{N}_\lambda^-$. This completes this proof. □

For each $u \in H_0^1(\Omega)$, with $\mathbb{D}(u) > 0$, set

$$t_{\max}(u) = \left(\frac{\|u\|^2}{(22_\mu^*-1)\mathbb{D}(u)}\right)^{2(2_\mu^*-1)} > 0.$$

Then the following lemma holds. Its proof is similar to the lemma [10] (or see Tarantello [17]).

LEMMA 4. *For each $u \in H_0^1(\Omega)$ with $\mathbb{D}(u) > 0$, then there are unique $0 < t^+ < t_{\max}(u) < t^-$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$ and*

$$I_\lambda(t^+u) = \inf_{0 \leqslant t \leqslant t_{\max}(u)} I_\lambda(tu), \ I_\lambda(t^-u) = \sup_{t \geqslant 0} I_\lambda(tu).$$

We have the following Lemma.

LEMMA 5. *There exists a $C_0 > 0$ (depending only on N, μ and $|\Omega|$) such that $I_\lambda(u) \geqslant C_0\lambda^{\frac{2}{1-q}}$, for all $u \in \mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$, by the Sobolev embedding theorem and Young inequality, we have

$$\begin{aligned} I_\lambda(u) &= \Big(\frac{1}{2} - \frac{1}{22_\mu^*}\Big)\|u\|^2 - \lambda\big(\frac{1}{q+1} - \frac{1}{22_\mu^*}\big)|u|_{q+1}^{q+1} \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{22_\mu^*}\Big)\|u\|^2 - \lambda C\Big(\frac{1}{q+1} - \frac{1}{22_\mu^*}\Big)\|u\|^{q+1} \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{22_\mu^*} - \frac{N-\mu+2}{4(2N-\mu)}\Big)\|u\|^2 - C_0\lambda^{\frac{2}{1-q}} \\ &\geqslant -C_0\lambda^{\frac{2}{1-q}}, \end{aligned}$$

for some constant $C_0 > 0$, depending only on N, μ and $|\Omega|$ such that

$$\Big(\tfrac{1}{2} - \tfrac{1}{22_\mu^*} - \tfrac{N-\mu+2}{4(2N-\mu)} > 0\Big). \quad \square$$

Next we establish that I_λ satisfies the $(PS)_c$ (Palais-Smale condition) under some restriction on the level of $(PS)_c$-sequence in the following.

LEMMA 6. *I_λ satisfies the $(PS)_c$-condition for*

$$c \in \Big(-\infty, c_\lambda := \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} - K\lambda^{\frac{2}{1-q}}\Big),$$

where $K > 0$ is independent on λ.

Proof. The first step for the $(PS)_c$-sequence to hold is bounded.

$$I_\lambda(u_n) = c + o_n(1) \text{ and } I_\lambda'(u_n) = o_n(1) \text{ in } H^{-1}, \tag{2.12}$$

so, there exists $C_1 > 0$ such that

$$|I_\lambda(u_n)| \leqslant C_1 \text{ and } |\langle I_\lambda'(u_n), \frac{u_n}{\|u_n\|}\rangle| \leqslant C_1 \tag{2.13}$$

Let $\theta \in (\frac{1}{22_\mu^*}, \frac{1}{2})$. For n large enough, we have

$$\begin{aligned} C_1(1+\|u_n\|) &\geqslant I_\lambda(u_n) - \theta\langle I_\lambda'(u_n), u_n\rangle \\ &= \Big(\frac{1}{2} - \theta\Big)\|u_n\|^2 + \lambda\big(\theta - \frac{1}{q+1}\big)|u_n|_{q+1}^{q+1} + \Big(\theta - \frac{1}{22_\mu^*}\Big)\mathbb{D}(u_n) \\ &\geqslant \Big(\frac{1}{2} - \theta\Big)\|u_n\|^2 + \lambda\Big(\theta - \frac{1}{q+1}\Big)S^{-\frac{q+1}{2}}|\Omega|^{\frac{2^*-1-q}{2}}\|u\|^{q+1} \\ &\quad + \Big(\theta - \frac{1}{22_\mu^*}\Big)\mathbb{D}(u_n) \end{aligned}$$

Since $(\theta - \frac{1}{22_\mu^*}) > 0$, $(\frac{1}{2} - \theta) > 0$ and $0 < q < 1$, we know that $(u_n)_{n\geqslant 1}$ is bounded in $H_0^1(\Omega)$. Hence, we may extract a subsequence denoted again by (u_n) such that

$$\begin{aligned} u_n &\rightharpoonup u_0 && \text{in } H_0^1(\Omega), \\ u_n &\rightharpoonup u_0 && \text{in } L^{2^*}(\Omega), \\ |u_n|^{2_\mu^*} &\rightharpoonup |u_0|^{2_\mu^*} && \text{in } L^{2_\mu^*}(\Omega), \\ \lambda|u_n|_{q+1}^{q+1} &= \lambda|u_0|_{q+1}^{q+1} + o(1), \end{aligned}$$

as $n \to +\infty$. By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$, so we have

$$|x|^{-\mu} * |u_n|^{2^*_\mu} \rightharpoonup |x|^{-\mu} * |u_0|^{2^*_\mu} \text{ in } L^{\frac{2N}{\mu}}(\Omega)$$

as $n \to +\infty$, Combining with the fact that

$$|u_n|^{2^*_\mu-2}u_n \rightharpoonup |u_0|^{2^*_\mu-2}u_0 \text{ in } L^{\frac{2N}{N-\mu+2}}(\Omega)$$

as $n \to +\infty$, we have

$$|x|^{-\mu} * |u_n|^{2^*_\mu}|u_n|^{2^*_\mu-2}u_n \rightharpoonup |x|^{-\mu} * |u_0|^{2^*_\mu}|u_0|^{2^*_\mu-2}u_0 \text{ in } L^{\frac{2N}{N+2}}(\Omega)$$

as $n \to +\infty$, Since, for all $\varphi \in H^1_0(\Omega)$,

$$\langle I'_\lambda(u_n), \varphi\rangle \to 0$$

we obtain by passing to the limit as $n \to +\infty$

$$\langle I'_\lambda(u_0), \varphi\rangle = 0.$$

So that, we may apply Brézis-Lieb's Lemma[7], we obtain that

$$\|u_n - u_0\|^2 = \|u_n\|^2 - \|u_0\|^2 + o(1),$$

and

$$\mathbb{D}(u_n - u_0) = \mathbb{D}(u_n) - \mathbb{D}(u_0) + o(1).$$

Then, we have

$$c - I_\lambda(u_n) + o(1) = \frac{1}{2}\|u_n - u_0\|^2 - \frac{1}{22^*_\mu}\mathbb{D}(u_n - u_0), \tag{2.14}$$

and

$$0 = \langle I'_\lambda(u_n), (u_n - u_0)\rangle = \|u_n - u_0\|^2 - \mathbb{D}(u_n - u_0) + o(1)$$

Without loss of generality, we suppose that

$$\|u_n - u\|^2 = a + o(1).$$

So

$$\mathbb{D}(u_n - u_0) = a + o(1).$$

If $a = 0$, we complete the proof. On the contrary, we suppose that $a > 0$. Then by the definition of $S_{H,L}$, we have

$$a \geqslant S_{H,L} a^{\frac{1}{2^*_\mu}},$$

this implies

$$a \geqslant S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \tag{2.15}$$

By (2.14), (2.15) and $u_0 \in \mathcal{N}_\lambda$ such that $u_0 \neq 0$, we have

$$c = I_\lambda(u_0) + \frac{a}{2} - \frac{a}{22^*_\mu} = I_\lambda(u_0) + \frac{a(2^*_\mu - 1)}{22^*_\mu} \geqslant \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} - C_0\lambda^{\frac{2}{1-q}},$$

which contradicts $c < \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} - C_0\lambda^{\frac{2}{1-q}}$. So, we have $a = 0$, that is $u_n \to u$ strongly in $H^1_0(\Omega)$. $\square$

Then we have the following lemma

LEMMA 7. *There exists* $\Lambda_* > 0$ *such that if* $\lambda \in (0,\Lambda_*)$, *then* I_λ *has a minimizer* $u_\lambda^+ \in \mathcal{N}_\lambda^+$ *and its satisfies*

i) $I_\lambda(u_\lambda^+) = c_\lambda^+$

ii) u_λ^+ *is a positive solution of* (1.1).

iii) $I_\lambda(u_\lambda^+) \to 0$ *and* $\|u_\lambda^+\|^2 \to 0$ *as* $\lambda \to 0$.

Proof. The proof is almost the same as that [[11] Lemma 2.5] and is omitted here. □

LEMMA 8. *There exists* Λ_*, $\varepsilon_0 > 0$ *and* $\sigma_0 > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$, $\lambda^{\frac{2}{1-q}} \in (0,\Lambda_*)$, *we have*

$$\sup_{t\geqslant 0} I_\lambda(tu_\varepsilon) < c_\lambda - \sigma_0,$$

where $c_\lambda = \frac{N-\mu+2}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} - K\lambda^{\frac{2}{1-q}}$.

Proof. Let us consider $\rho_0 > 0$ such that $B(0,2\rho_0) \subset \Omega$ and define a cut function $\eta \in \mathscr{C}_0^\infty(\Omega)$ such that $0 \leqslant \eta \leqslant 1$, $|\nabla \eta| \leqslant C$, $\eta(x) = 1$ for $|x| \leqslant \rho_0$ and $\eta(x) = 0$ for $|x| \geqslant 2\rho_0$. For $\varepsilon > 0$ $U_\varepsilon = \varepsilon^{\frac{2-N}{2}} U(\frac{x}{\varepsilon})$ and $u_\varepsilon(x) = \eta U_\varepsilon(x)$ where $U(x)$ given in (2.2) is minimizer for S, the best Sobolev constant and also for $S_{H,L}$. From [9], we know that

$$\|u_\varepsilon\|^2 = C(N,\mu)^{\frac{N-2}{2N-\mu}\frac{N}{2}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2}), \tag{2.16}$$

and

$$\mathbb{D}(u_\varepsilon) \geqslant C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}}). \tag{2.17}$$

Moreover

$$\begin{aligned} |u_\varepsilon|_{q+1}^{q+1} = \int_\Omega |u_\varepsilon(x)|^{q+1} dx &= \varepsilon^{\frac{(2-N)}{2}(q+1)} \int_{B(0,\rho_0)} U^{q+1}(\frac{x}{\varepsilon}) dx \\ &\geqslant \varepsilon^{\frac{(2-N)}{2}(q+1)+N} \int_0^{\rho_0} U^{q+1}(r) r^{N-1} dr \\ &\geqslant C\varepsilon^{\frac{(2-N)}{2}(q+1)+N} \int_0^{\rho_0\varepsilon^{-1}} r^{2(q+1)-qN-1} dr \end{aligned}$$

Now, since $\frac{2}{N-2} \leqslant q < 1$, by suitable choice of $R_0 > 0$, it follows that

$$\begin{aligned} |u_\varepsilon|_{q+1}^{q+1} &\geqslant C\varepsilon^{\frac{(2-N)}{2}(q+1)+N} \int_{R_0}^{\rho_0\varepsilon^{-1}} r^{2(q+1)-qN-1} dx \\ &\geqslant \begin{cases} C\varepsilon^{\frac{(2-N)}{2}(q+1)+N} \text{ if } \frac{2}{N-2} < q < 1, \\ C\varepsilon^{\frac{(2-N)}{2}(q+1)+N} \ln(\varepsilon) \text{ if } q = \frac{2}{N-2} < 1, \end{cases} \end{aligned} \tag{2.18}$$

where C is a positive constant.

Let

$$g(t)=I_\lambda(tu_\varepsilon)=\frac{t^2}{2}\|u_\varepsilon\|^2-\lambda\frac{t^{q+1}}{q+1}|u_\varepsilon|_{q+1}^{q+1}-\frac{t^{22_\mu^*}}{22_\mu^*}\mathbb{D}(u_\varepsilon)$$

Since $g(0)=0$ and $\lim_{t\to+\infty}g(t)=-\infty$, so there exists t_ε such that $\sup_{t\geqslant 0}g(t)$ is attained at t_ε. This implies that t_ε satisfies

$$\|u_\varepsilon\|^2=t_\varepsilon^{22_\mu^*-1-q}\mathbb{D}(u_\varepsilon)+\lambda|u_\varepsilon|_{q+1}^{q+1}$$

then we deduce

$$t_\varepsilon\leqslant\left(\frac{\|u_\varepsilon\|^2}{\mathbb{D}(u_\varepsilon)}\right)^{\frac{1}{22_\mu^*-1-q}}$$

this implies that t_ε is bounded above for ε small enough. And by lemma 4 we have

$$t_\varepsilon\geqslant t_{\max}(u_\varepsilon)>0,$$

then, we can also suppose that t_ε is bounded below. So we conclude that there exist the positive constants $C_i\ (i=1,2)$ independent of ε, such that

$$0<C_1<t_\varepsilon<C_2<\infty. \tag{2.19}$$

Consider

$$h(t)=\frac{t^2}{2}\left((C(N,\mu))^{\frac{N-2}{2N-\mu}\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2})\right)-\frac{t^{22_\mu^*}}{22_\mu^*}\left((C(N,\mu))^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}})\right)$$

An easy computation implies that

$$\sup_{t\geqslant 0}h(t)=\frac{N-\mu+2}{4N-2\mu}\left(\frac{(C(N,\mu))^{\frac{N-2}{2N-\mu}\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2})}{\left((C(N,\mu))^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}})\right)^{\frac{N-2}{2N-\mu}}}\right)^{\frac{2N-\mu}{N-\mu+2}} \tag{2.20}$$

$$=\frac{N-\mu+2}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}+O(\varepsilon^{\min(N-2,N-\frac{\mu}{2})}). \tag{2.21}$$

We have, by (2.16), (2.17), (2.18) and (2.19)

$$g(t_\varepsilon)=I_\lambda(t_\varepsilon u_\varepsilon)\leqslant h(t_\varepsilon)-\lambda\begin{cases}C\varepsilon^{\frac{(2-N)}{2}(q+1)+N} \text{ if } \frac{2}{N-2}<q<1,\\ C\varepsilon^{\frac{(2-N)}{2}(q+1)+N}\ln(\varepsilon) \text{ if } q=\frac{2}{N-2}<1,\end{cases}$$

Then, by (2.20)

$$I_\lambda(t_\varepsilon u_\varepsilon)\leqslant\frac{N-\mu+2}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}+O(\varepsilon^{\min(N-2,N-\frac{\mu}{2})})$$
$$-\lambda\begin{cases}C\varepsilon^{\frac{(2-N)}{2}(q+1)+N} \text{ if } \frac{2}{N-2}<q<1,\\ C\varepsilon^{\frac{(2-N)}{2}(q+1)+N}\ln(\varepsilon) \text{ if } q=\frac{2}{N-2}<1,\end{cases}$$

In the case of $\frac{2}{N-2}<q<1$, and $N>4$, we have

$$0<\frac{(2-N)}{2}(q+1)+N<\min\left(N-2,N-\frac{\mu}{2}\right).$$

Then, there exists $\varepsilon_0 > 0$ small enough, Λ_* and $\sigma(\varepsilon) > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\lambda^{\frac{2}{1-q}} \in (0, \Lambda_*)$ and $\sigma \in (0, \sigma(\varepsilon))$

$$0 < O(\varepsilon^{\frac{(2-N)}{2}(q+1)+N}) - \lambda O(\varepsilon^{\min(N-2, N-\frac{\mu}{2})}) < -C\lambda^{\frac{2}{1-q}} - \sigma,$$

and then,

$$\sup_{t \geqslant 0} I_\lambda(tu_\varepsilon) < \frac{N-\mu+2}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} - C\lambda^{\frac{2}{1-q}} - \sigma = c_\lambda - \sigma.$$

If $\frac{2}{N-2} = q$, we can verify that

$$\frac{1}{2}\left(N - (q+1)\frac{N-2}{2}\right) < \frac{N-2}{2},$$

then it is easy to see that

$$\sup_{t \geqslant 0} I_\lambda(tu_\varepsilon) < c_\lambda - \sigma. \quad \square$$

Let the set

$$N_\lambda^-(c_\lambda - \sigma) = \{u \in N_\lambda^- : I_\lambda(u) \leqslant c_\lambda - \sigma\}.$$

COROLLARY 1. *By lemmas 8 and 7, the functional I_λ has a local minimizer in $N_\lambda^-(c_\lambda - \sigma)$, that is, there exists $u_\lambda \in N_\lambda^-(c_\lambda - \sigma)$ satisfying*

$$I_\lambda(u_\lambda) = c_\lambda - \sigma.$$

3. Some technical results

In this section, we shall introduce some useful results which are crucial for the proof of theorem 1

LEMMA 9. *Let $(u_n) \subset H_0^1(\Omega)$ be a non-negative function sequence with*

$$\mathbb{D}(u_n) = 1 \ \text{and} \ \|u_n\|^2 \to S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Then, there exists a sequences $(y_n, \varepsilon_n) \subset \mathbb{R}^N \times \mathbb{R}^+$ such that $v_n(x) = \varepsilon_n^{\frac{N-2}{2}} u_n(\varepsilon_n x + y_n)$ contains a convergent subsequence denoted again by $(v_n(x))$ such that $v_n(x) \to v(x)$ in $H_0^1(\Omega)$ Moreover, we have $\varepsilon_n \to 0$ and $y_n \to y \in \overline{\Omega}$ as $n \to +\infty$.

Proof. the proof of this lemma is standard, we refer the readers [18] for similar proofs. $\square$

LEMMA 10. *Suppose that X is a Hilbert manifold and $F \in \mathscr{C}^1(X, \mathbb{R})$. Assume that, for $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$:*

1. *F satisfies the $(PS)_c$ condition for $c \leqslant c_0$,*
2. *$cat(\{x \in X, F(x) \leqslant c_0\}) \geqslant k$.*

Then F has at least k critical points in $\{x \in X, F(x) \leqslant c_0\}$.

Proof. See section 5.3 in [18]. □

We consider
$$\mathcal{N}_\lambda^-(c_\lambda) := \{u \in \mathcal{N}_\lambda^- : I_\lambda(u) \leqslant c_\lambda\}$$
Now, let us introduce the following map $\beta : \mathcal{N}_\lambda^- \to \mathbb{R}^N$ given by
$$\beta(u) := \frac{\int_\Omega x|u|^2 dx}{\int_\Omega |u|^2 dx}$$
Then we have the following result.

LEMMA 11. *There exists $\Lambda_* > 0$ such that for each $\lambda \in (0, \Lambda_*)$ we have*
$$\beta(\mathcal{N}_\lambda^-(c_\lambda)) \subset \Omega_r^+.$$

Proof. We argue by contradiction and suppose that there exist $(t_n) \subset [0,1]$, $\lambda_n \to 0$ and $(u_n) \subset N_\lambda^-(c_\lambda)$ such that
$$\beta(u_n) \notin \Omega_r^+ \text{ for all } n \in \mathbb{N}.$$
We can assume that, up to a subsequence, $t_n \to t_0 \in [0,1]$. As in the proof of Lemma 6, it is easy to verify that the sequence (u_n) is bounded in $H_0^1(\Omega)$ and by this we obtain $\lambda_n|u_n^+|_{q+1}^{q+1} \to 0$ as $n \to +\infty$. Then,
$$I_{\lambda_n}(u_n) = \Big(\frac{1}{2} - \frac{1}{22_\mu^*}\Big)\|u_n\|^2 + o(1) \leqslant c_{\lambda_n} + o(1)$$
and
$$\frac{1}{2}\Big(\frac{N-\mu+2}{2N-\mu}\Big)\|u_n\|^2 \leqslant c_{\lambda_n} + o(1) \leqslant \frac{1}{2}\Big(\frac{N-\mu+2}{2N-\mu}\Big)(S_{H,L})^{\frac{2N-\mu}{N+2-\mu}} + o(1).$$
Since $(u_n) \subset \mathcal{N}_{\lambda_n}^-(c_{\lambda_n}) \subset \mathcal{N}_{\lambda_n}^-$, we have
$$\|u_n\|^2 = \mathbb{D}(u_n) + o(1),$$
and by definition of $S_{H,L}$, we obtain
$$S_{H,L} \leqslant \frac{\|(u_n)\|^2}{\mathbb{D}^{\frac{1}{2_\mu^*}}(u_n)} \leqslant \|u_n\|^{2(\frac{N-\mu+2}{2N-\mu})} \leqslant S_{H,L} + o(1).$$
Thus
$$\|u_n\|^2 \to (S_{H,L})^{\frac{2N-\mu}{N-\mu+2}} \text{ and } \mathbb{D}(u_n) \to (S_{H,L})^{\frac{2N-\mu}{N-\mu+2}}.$$
Now, it is easy to see that the sequence $(\widetilde{u}_n)$ given by
$$\widetilde{u}_n = \frac{u_n}{(S_{H,L})^{\frac{2N-\mu}{2(N+2-\mu)}}}$$
verifies
$$\mathbb{D}(\widetilde{u}_n) = 1 \text{ and } \|\widetilde{u}_n\|^2 \to S_{H,L}.$$
Then, by using lemma 9, there exists a sequences $(y_n) \subset \mathbb{R}^N$ and $(\varepsilon_n) \subset \mathbb{R}^+$ such that $\varepsilon_n \to 0$, $y_n \to y \in \overline{\Omega}$ and $v_n = \varepsilon_n^{\frac{N-2}{2}} \widetilde{u}_n(\varepsilon_n x + y_n) \to v_1$ with $v_1 > 0$ in $\mathbb{R}^N$ as $n \to +\infty$.

Considering $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$ such that $\chi(x) = x$ in $\overline{\Omega}$, we infer

$$\beta(u_n) = \frac{\int_\Omega x|u_n|^2dx}{\int_\Omega |u_n|^2dx} = \frac{\int_\Omega x|\widetilde{u}_n|^2dx}{\int_\Omega |\widetilde{u}_n|^2dx} = \frac{\int_\Omega \chi(\varepsilon_n x + y_n)|\widetilde{u}_n(\varepsilon_n x + y_n)|^2dx}{\int_\Omega |\widetilde{u}_n(\varepsilon_n x + y_n)|^2dx}$$
$$= \frac{\int_\Omega \chi(\varepsilon_n x + y_n)|v_n|^2dx}{\int_\Omega |v_n|^2dx}.$$

Moreover, by the Lebesgue theorem, we have

$$\frac{\int_\Omega \chi(\varepsilon_n x + y_n)|v_n|^2dx}{\int_\Omega |v_n|^2dx} \to y \in \overline{\Omega}$$

as $n \to +\infty$, so that $\lim_{n\to+\infty} \beta(u_n) = y \in \overline{\Omega}$, in contradiction with $\beta(u_n) \notin \Omega_r^+$. □

Note that Ω_r^- is compact, then by lemma 6 and corollary 1 we can easily get there exists $t^- > 0$ such that

$$t^- u_\varepsilon(x-y) \in \mathscr{N}_\lambda^-(c_\lambda - \sigma)$$

uniformly in $y \in \Omega_r^-$. Moreover, by lemma 11,

$$\beta(t^- u_\varepsilon(x-y)) \in \Omega_r^+.$$

Then, we can define the map $\eta : \Omega_r^- \to \mathscr{N}_\lambda^-(c_\lambda - \sigma)$ given by

$$\eta(y)(x) := \begin{cases} t^- u_\varepsilon(x-y) & \text{if } x \in B_r(y), \\ 0 & \text{if } x \notin B_r(y). \end{cases}$$

Below we denote by β_λ the restriction of β on $\mathscr{N}_\lambda^-(c_\lambda - \sigma)$. Taking into account that u_ε is radial, we have for each $y \in \Omega_r^-$

$$(\beta_\lambda \circ \eta)(y) = \frac{\int_\Omega x|t^- u_\varepsilon(x-y)|^2dx}{\int_\Omega |t^- u_\varepsilon(x-y)|^2dx} = \frac{\int_\Omega (z+y)|u_\varepsilon(z)|^2|dz}{\int_\Omega |u_\varepsilon(z)|^2dz} = y.$$

Next, we define the map $H_\lambda : [0,1] \times \mathscr{N}_\lambda^-(c_\lambda - \sigma) \to \mathbb{R}^N$ by

$$H_\lambda(t,u) := t\beta_\lambda(u) + (1-t)\beta_\lambda(u).$$

LEMMA 12. *There exists $\Lambda_* > 0$ such that for each $\lambda^{\frac{2}{1-q}} \in (0, \Lambda_*)$ we have*

$$H_\lambda\left([0,1] \times \mathscr{N}_\lambda^-(c_\lambda - \sigma)\right) \subset \Omega_r^+.$$

Proof. We argue by contradiction and suppose that there exist $(t_n) \subset [0,1]$, $\lambda_n \to 0$ and $(u_n) \subset \mathscr{N}_\lambda^-(c_\lambda - \sigma)$ such that

$$H_{\lambda_n}(t_n, u_n) \notin \Omega_r^+ \text{ for all } n \in \mathbb{N}.$$

We can assume that, up to a subsequence, $t_n \to t_0 \in [0,1]$. Then, by Lemma 3 and argument as in the proof of Lemma 11, we have

$$H_{\lambda_n}(t_n, u_n) \to y \in \overline{\Omega},$$

as $n \to +\infty$, which is a contradiction. □

LEMMA 13. *There exists $\Lambda_* > 0$ such that if $\lambda^{\frac{2}{1-q}} \in (0, \Lambda_*)$, we have*

$$cat(\mathscr{N}_\lambda^-(c_\lambda - \sigma)) \geqslant cat(\Omega).$$

Proof. Suppose that $\text{cat}(\mathcal{N}_\lambda^-(c_\lambda-\sigma))=n$, this means that n is the least integer such that

$$\mathcal{N}_\lambda^-(c_\lambda-\sigma)=A_1\cup\ldots\cup A_n,$$

where $A_j,\ j=1,\ldots,n,$ is closed and contractible in $\mathcal{N}_\lambda^-(c_\lambda-\sigma)$ that is; there exists a continuous function $h_j:[0,1]\times A_j\to\mathcal{N}_\lambda^-(c_\lambda-\sigma)$ such that for all $u,v\in A_j$

$$h_j(0,z)=z \texttt{ and } h_j(1,z)=w_j,$$

where $\omega\in A_j$ is fixed. Consider $B_j:=\psi^{-1}(A_j),\ j=1,\ldots,n.$ The sets B_j are closed and

$$\Omega_r^-=B_1\cup\ldots\cup B_n.$$

Noting Lemma 12, we define the deformation $g_j:[0,1]\times B_j\to\Omega_r^+$ by setting

$$g_j(t,y)=H_\lambda(t,h_j(t,\theta(y))).$$

So, we have for all $y\in B_j$

$$g_j(0,y)=H_\lambda(0,h_j(0,\eta(y)))=y$$

and

$$g_j(1,y)=H_\lambda(1,h_j(1,\eta(y)))=\beta_\lambda(\omega)\in\Omega_r^+,$$

Thus the sets B_j is contractible in Ω_r^+. It follows that

$$cat_{\Omega_r^+}(\Omega_r^-)=cat(\Omega)\leqslant n,$$

which completes the proof. □

4. The proof of Theorem 1

Denoting by $I_{\mathcal{N}_\lambda^-}$ the restriction of I_λ on $\mathcal{N}_\lambda^-$.

LEMMA 14. *There exists $\Lambda^*>0$ such that if $\lambda^{\frac{2}{1-q}}\in(0,\Lambda^*)$, then $I_{\mathcal{N}_\lambda^-}$ satisfies the $(PS)_c$ condition for $c\in(-\infty,c_\lambda)$.*

Proof. If (u_n) is a Palais-Smale sequence for $I_{\mathcal{N}_\lambda^-}$ at level c, by [[18], Proposition 5.12], there exists a sequence $\theta_n\subset\mathbb{R}$ such that

$$I'_\lambda(u_n)=\theta_nJ'_\lambda(u_n)+o(1), \tag{4.1}$$

where

$$J_\lambda(u_n)=\langle I'_\lambda(u_n),u_n\rangle=\|u_n\|^2-\mathbb{D}(u_n)-\lambda|u_n|_{q+1}^{q+1}$$

Recall that $u_n\in\mathcal{N}_\lambda^-$, so $\langle J'_\lambda(u_n),u_n\rangle<0.$

If $\langle J'_\lambda(u_n),u_n\rangle\to0,$ we see by the Sobolev embedding theorem that there are two positive numbers C_1,C_2 independent of n and λ, such that

$$\|u_n\|^2\leqslant C_1\|u_n\|^{22_\mu^*}+o(1),$$

and

$$\|u_n\|^2\leqslant C_2\lambda\|u_n\|^{q+1}+o(1),$$

we conclude that

$$\|u_n\| \geqslant C_2^{-\frac{1}{2(2_\mu^*-1)}} + o(1).$$

and

$$\|u_n\| \leqslant C_1^{\frac{1}{1-q}} \lambda^{\frac{2}{1-q}} + o(1),$$

if we choose Λ_* is small enough. This is impossible.

Thus we may assume that $\langle J'_\lambda(u_n), u_n\rangle \to l$, as $n \to +\infty$. Since $\langle I'_\lambda(u_n), u_n\rangle = 0$, we conclude that $\theta_n \to 0$ as $n \to +\infty$ and, consequently $I'_\lambda(u_n) \to 0$. Using this information, we have

$$I_\lambda(u_n) \to c \in (0, c_\lambda), \text{ and } I'_\lambda(u_n) \to 0, \tag{4.2}$$

so by Lemma 2.12 the proof is complete. □

LEMMA 15. *There exists $\Lambda^* > 0$ such that if $\lambda^{\frac{2}{1-q}} \in (0, \Lambda^*)$, then a critical point of $I_{\mathcal{N}_\lambda^-}$ on $\mathcal{N}_\lambda^-$ is a critical point of I_λ in $H_0^1(\Omega)$.*

Proof. For the proof of this lemma, is similar to lemma 14. □

Proof of Theorem 1. Applying Lemmas 6 and 3, $I_{\mathcal{N}_\lambda^-}$ satisfies $(PS)_c$ condition for all $c \in (0, c_\lambda)$. Then, by Lemmas 13 and 10, $I_{\mathcal{N}_\lambda^-}$ admits at least $\operatorname{cat}(\Omega)$ critical points in $\mathcal{N}_\lambda^-(c_\lambda - \sigma)$. Hence, we deduce from Lemma 15 that I_λ has at least $\operatorname{cat}(\Omega)$ critical points in $\mathcal{N}_\lambda^-$. Moreover, $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset$, I_λ at least $\operatorname{cat}(\Omega) + 1$ critical points in $H_0^1(\Omega)$. □

REFERENCES

[1] CLAUDIANOR O. ALVES, FASHUN GAO, MARCO SQUASSINA, AND MINBO YANG, *Singularly perturbed critical choquard equations*, Journal of Differential Equations **263** (7): 3943–3988, 2017.

[2] CLAUDIANOR O. ALVES AND MINBO YANG, *Multiplicity and concentration of solutions for a quasilinear choquard equation*, Journal of Mathematical Physics **55** (6): 061502, 2014.

[3] C. O. ALVES AND Y. H. DING, *Multiplicity of positive solutions to a p-laplacian equation involving critical nonlinearity*, Journal of Mathematical Analysis and Applications **279** (2): 508–521, 2003.

[4] ANTONIO AMBROSETTI, HAÏM BREZIS, AND GIOVANNA CERAMI, *Combined effects of concave and convex nonlinearities in some elliptic problems*, Journal of Functional Analysis **122** (2): 519–543, 1994.

[5] VIERI BENCI AND GIOVANNA CERAMI, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Archive for Rational Mechanics and Analysis **114** (1): 79–93, 1991.

[6] SAMIRA BENMOULOUD, RACHID ECHARGHAOUI, AND SI MOHAMMED SBAÏ, *Multiplicity of positive solutions for a critical quasilinear elliptic system with concave and convex nonlinearities*, Journal of Mathematical Analysis and Applications **396** (1): 375–385, 2012.

[7] HAÏM BRÉZIS AND ELLIOTT LIEB, *A relation between pointwise convergence of functions and convergence of functionals*, Proceedings of the American Mathematical Society **88** (3): 486–490, 1983.

[8] FASHUN GAO AND MINBO YANG, *On nonlocal choquard equations with Hardy-Littlewood-Sobolev criticalexponents*, Journal of Mathematical Analysis and Applications **448** (2): 1006–1041, 2017.

[9] FASHUN GAO AND MINBO YANG, *On the Brezis-Nirenberg type critical problem for nonlinear choquard equation*, Sci China Math, doi:10.1007(/s):10.1007/s11425-016-9067-5, 2017.

[10] TSING-SAN HSU, *Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities*, Nonlinear Analysis: Theory, Methods and Applications **71** (7): 2688–2698, 2009.

[11] QIN LI AND ZUODONG YANG, *Multiplicity of positive solutions for a $p-q$-Laplacian system with concaveand critical nonlinearities*, Journal of Mathematical Analysis and Applications **423** (1): 660–680, 2015.

[12] ELLIOTT H. LIEB AND MICHAEL LOSS, *Analysis, volume 14 of graduate studies in mathematics*, American Mathematical Society, Providence, RI, 4, 2001.

[13] ELLIOTT H. LIEB AND BARRY SIMON, *The hartree-fock theory for coulomb systems*, Communications in Mathematical Physics **53** (3): 185–194, 1977.

[14] VITALY MOROZ AND JEAN VAN SCHAFTINGEN, *A guide to the choquard equation*, preprint arXiv:1606.02158, 2016.

[15] VITALY MOROZ AND JEAN VAN SCHAFTINGEN, *A guide to the choquard equation*, Journal of Fixed Point Theory and Applications, **19** (1): 773–813, 2017.

[16] SOLOMON ISAAKOVICH PEKAR, *Untersuchungen uber die Elektronentheorie der Kristalle*, Akademie-Verlag, 1954.

[17] GABRIELLA TARANTELLO, *On nonhomogeneous elliptic equations involving critical sobolev exponent*, In Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. **9**, pp. 281–304, Elsevier, 2016.

[18] M. WILLEM, *Minimax theorems, progress in nonlinear differential equations and applications 24*, 1996.

(Received July 7, 2017)

R. Echarghaoui
E.G.A.L, Dépt. Maths, Fac. Sciences
Université Ibn Tofail
BP. 133, Kénitra, Morocco
e-mail: rachid.echarghaoui@yahoo.com

M. Khiddi
E.G.A.L, Dépt. Maths, Fac. Sciences
Université Ibn Tofail
BP. 133, Kénitra, Morocco
e-mail: mostapha-2@hotmail.com

S.M. Sbai
E.G.A.L, Dépt. Maths, Fac. Sciences
Université Ibn Tofail
BP. 133, Kénitra, Morocco
e-mail: sbaisimo@netcourrier.com