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SOLUTIONS FOR A SECOND–ORDER DELAY DIFFERENTIAL INCLUSION ON THE HALF–LINE WITH BOUNDARY VALUES

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Abstract. In [15], Wei solved a delay differential equation on the half-line. The current paper is an extension of these results to the set-valued case. The results involve measurable selections and the contraction mapping theorem for set-valued functions.

1. Introduction

In [15], Wei solved a delay differential equation on the half-line. It is always a useful challenge to extend single valued results to the set-valued case and that is what we do here. There have been a number of papers involving the generalization of single-valued results to the differential inclusion case. See for example, [4] and [13]. We will need a fixed point theorem and some results concerning selections of set-valued maps and will determine the existence of solutions to a particular differential inclusion with boundary values.

The use of fixed point theorems and measurable selections for such problems with boundary values other than those here is quite common. For example in [7] and [10] the Covitz-Nadler theorem (see below) is used to obtain existence of solutions for a second-order differential inclusion while in [6] a measurable selection theorem is employed for such a problem. In [17] a fixed point result for upper semicontinuous maps is applied and in [14] fixed point index theory is applied. General results for set-valued analysis can be obtained in many places. See for example [1] and [2].

In this article the following *delay boundary value inclusion, (DBVI)*, will be considered:

$$[p(t)x'(t)]' \in F(t, x_t, p(t)x'(t))$$

a.e. where $x_t(s) = x(t+s)$ for $s \in [-r,0]$, $t \in [0,\infty)$. and $x(t) = \varphi(t)$, $t \in [-r,0]$, x is continuously differentiable on $(0,\infty)$ as is $p(t)x'(t)$, and $\lim_{t\to\infty} p(t)x'(t) = L$.

Further we assume the following conditions which are similar to those found in [15]:

1. Let $\varphi(t) \in C[-r,0]$ *and* L *be given with* $\varphi(0) = 0$.

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2. Let $p(t) > 0$ *be continuous on* $[0,\infty)$ *and satisfy* $\int_0^\infty \frac{dt}{p(t)} < \infty$ *and let* $P(t) = \int_0^t \frac{dt}{p(t)}$ *for* $t > 0$.

3. Let F *map* $(0,\infty)\times C[-r,0]\times R$ *to nonempty, closed and convex valued subsets of* R.

4. There exists a nonnegative real-valued function h *defined on* $(0,\infty)\times C([-r,0], [0,\infty))\times[0,\infty)$ *such that* $\sup_{y\in F(t,u,v)} \mid y \mid \leqslant h(t, \mid u \mid, \mid v \mid)$ *for almost all* t *whenever* $u \in C[-r,0]$ *and* $v \in R$. *Furthermore* h *is increasing in its second and third variables in the sense that if* u_1 *and* u_2 *are elements of* $C([-r,0],[0,\infty))$ *with* $u_1(w) \leqslant u_2(w)$ *for all* $w \in [-r,0]$ *and* $0 \leqslant y_1 \leqslant y_2$, *then for almost all* $t \in (0,\infty)$, $h(t,u_1,y_1) \leqslant h(t,u_2,y_2)$.

5. Assume that for all $(u,v) \in C[-r,0]\times C^1(0,\infty)$, *the functions defined by* $F1^{(u,v)}(s) \equiv \inf F(s,u_s,v(s))$ *and* $F2^{(u,v)}(s) \equiv \sup F(s,u_s,v(s))$ *for almost all* s *are measurable functions and finite a.e. Note that conditions 3 and 4 imply that* F *is compact valued a.e. and thus* $F1^{(u,v)}(s)$ *and* $F2^{(u,v)}(s)$ *are selections of* $F(s,u_s,v(s))$ *for almost all* s.

6. There exists $c > \mid L \mid$ *such that if we define* $\eta(t)$ *by:* $\eta(t) = \{\mid \varphi(t) \mid,\ -r \leqslant t \leqslant 0,\ cP(t),\ t > 0\}$, *then* $\int_0^\infty h(t,\eta_t,c)dt \leqslant c - \mid L \mid$.

7. There exists $k(s) : [0,\infty) \to [0,\infty)$ *measurable such that for all* $x_1,x_2 \in C[-r,0]$ *and* $v_1,v_2 \in R$ *we have* $H_d(F(s,x_1,v_1),F(s,x_2,v_2)) \leqslant k(s)\times\max\{\| x_1 - x_2 \|_{[-r,0]}, \mid v_1 - v_2 \mid\}$ *a.e. and* $\int_0^\infty k(s)\times\max\{1,P(s)\}ds = L_1 < 1$ *where* H_d *is the Hausdorff metric.*

NOTE. Many of these conditions are set-valued versions of conditions found in [15], though in that paper all functions were single-valued and f which takes the place of our F was continuous there. f had a single-valued integral boundedness condition which has been generalized in condition 4 here. Condition 5 specifies the existence of a particular measurable selection. A more general measurable selection would be guaranteed when F is £ × ß$(C[-r,0]\times R)$ measurable. See [5], [8], and [9] for details. Condition 7 is a generalized set-valued Lipschitz condition.

By a solution to the above boundary value inclusion we mean that there exists $x \in C[-r,\infty)\cap AC^1(0,\infty)$ and $p(t)x'(t) \in AC(0,\infty)$ where $AC^1(0,\infty)$ is the set of functions which have an absolutely continuous derivative on $(0,\infty)$, and $[p(t)x'(t)]' \in F(t,x_t,p(t)x'(t))$ a.e. on $(0,\infty)$, where $x(t) = \varphi(t)$, $t \in [-r,0]$, and $\lim_{t\to\infty} p(t)x'(t) = L$.

In order to demonstrate the existence of a solution to the above DBVI we will use a set-valued version of the contraction mapping theorem. The definition and theorem below can be found in a number of places. See [3].

DEFINITION. $M : X \to P_{cl}(X)$ is a contraction if and only if $\exists k_1$ such that $0 \leqslant k_1 < 1$ and M is Lipschitz with constant k_1 with respect to the Hausdorff metric. By $P_{cl}(X)$ we mean the closed subsets of X.

COVITZ-NADLER CONTRACTION MAPPING THEOREM. [3] *Let* (X,d) *be a complete metric space. If* $M : X \to P_{cl}(X)$ *is a contraction, then* $Fix(M) \neq \phi$ *where* $Fix(M) = \{x \in X \mid x \in Mx\}$.

We will also use the fact that $L^1(0,\infty)$ is a Banach Lattice. See [12] for details. As in [15] we define the spaces E and Ω by $E = \{x \in C[-r,\infty) \mid x$ is continuously differentiable on $(0,\infty)$ and $p(t)x'(t)$ is bounded on $(0,\infty)\}$ and $\Omega=\{x\in E \mid x(t)=\varphi(t)$ for $t\in[-r,0]$ and $p(t)\mid x'(t)\mid\leqslant c$ for $t>0\}$. E is a Banach space with norm $\|\cdot\|_E$ defined by $\| u \|_E=\max\{\max_{-r\leqslant t\leqslant 0}\mid u(t)\mid, \sup_{t>0} p(t)\mid u'(t)\mid\}$ for $u\in E$.

NOTE. In what follows for all $x\in\Omega$ we define Mx by $Mx=\{m(t)\in C[-r,\infty)\cap AC(0,\infty)\mid \exists\ f$ measurable on $(0,\infty)$ such that $f(s)\in F(s,x_s,p(s)x'(s))$ a.e. with $m(t)=\varphi(t)$ for $t\in[-r,0]$ and $m(t)=\int_0^t\frac{1}{p(\tau)}\int_\tau^\infty f(s)ds\,d\tau+LP(t)$, for $t>0\}$. Condition 5 guarantees that there are indeed some selections of $F(s,x_s,p(s)x'(s))$, namely $F1^{(x,px')}(s)$ and $F2^{(x,px')}(s)$.

2. Preliminary results

We will require a number of lemmas and theorems in order to establish the main result.

LEMMA 1. *$Mx\subseteq\Omega$ for all $x\in\Omega$.*

Proof. Let $x\in\Omega$ and let $m\in Mx$ with associated measurable function $f(s)$ which is a selection of $F(s,x_s,p(s)x'(s))$ a.e. Clearly $m(t)=\varphi(t)$ for $t\in[-r,0]$.

Now let $t>0$. It can be easily shown that $m'(t)=\frac{1}{p(t)}\int_t^\infty f(s)ds+\frac{L}{p(t)}$. Thus $m'\in C(0,\infty)$ and $m\in C[-r,\infty)$.

Following the argument in [14] we note that for $x\in\Omega$ and $s\in(t,\infty)$ it is clear that $p(s)\mid x'(s)\mid\leqslant c$ and for $t_1>0$ since $x(0)=\varphi(0)=0$ we have:

$$\begin{aligned}\mid x(t_1)\mid &= \left|\int_0^{t_1} x'(\tau)d\tau\right| = \left|\int_0^{t_1}\frac{p(\tau)}{p(\tau)}x'(\tau)d\tau\right|\\ &\leqslant \int_0^{t_1}\frac{p(\tau)}{p(\tau)}\mid x'(\tau)\mid d\tau\leqslant cP(t_1)=\eta(t_1).\end{aligned}$$

Thus $\mid x_s(t)\mid\leqslant\eta_s(t)$ if $s+t>0$ and if $s+t\leqslant 0$, then $\mid x_s(t)\mid=\eta_s(t)=\varphi(s+t)$, so we have that $\mid x_s\mid\leqslant\eta_s$ for all $s\geqslant 0$.

Thus we have

$$\begin{aligned}\mid p(t)m'(t)\mid &= \left|\int_t^\infty f(s)ds+L\right|\\ &\leqslant \mid L\mid+\int_t^\infty\mid f(s)\mid ds\\ &\leqslant \mid L\mid+\int_t^\infty h(s,\mid x_s\mid,p(s)\mid x'(s)\mid)ds\\ &\leqslant \mid L\mid+\int_t^\infty h(s,\eta_s,c)ds\\ &\leqslant \mid L\mid+c-\mid L\mid=c\end{aligned}$$

since $h(s,\eta_s,c)\in L^1(0,\infty)$ and h is increasing in its second and third arguments.

Thus $m\in\Omega$ so the lemma is proven.

LEMMA 2. *Let $x \in \Omega$ and $h_n \in Mx$ $\forall n \in N$. If $h_n \to \widetilde{h}$ in E, then $\lim_{b\to\infty} p(b)\widetilde{h}'(b) = L$.*

Proof. Since $\widetilde{h}$ is in E, $p(t)\widetilde{h}'(t)$ is bounded for $t \in (0,\infty)$. Since $h_n \to \widetilde{h}$ in E we have that ph_n' converges to $p\widetilde{h}'$ uniformly on $(0,\infty)$. By the definition of Mx there are selections, $f_n(s)$ of $F(s,x_s,p(s)x'(s))$ a.e. such that $\forall n \in N$, $h_n(t) = \int_0^t \frac{1}{p(\tau)} \int_\tau^\infty f_n(s)ds\,d\tau$ $+LP(t)$. Thus: $p(t)h_n'(t) = p(t)[\frac{1}{p(t)}\int_t^\infty f_n(s)ds + \frac{L}{p(t)}] = \int_t^\infty f_n(s)ds + L$. As before: $|f_n(s)| \leqslant h(s,\eta_s,c) \in L^1(0,\infty)$ which implies that $f_n \in L^1(0,\infty)$ and so $\lim_{t\to\infty}\int_t^\infty f_n(s)ds$ $= 0$. This means that: $\lim_{t\to\infty} p(t)h_n'(t) = L$ $\forall n \in N$.

Now consider $|p(t)\widetilde{h}'(t) - L|$. Let $\varepsilon > 0$. Using the uniform convergence of ph_n' select N' such that $n > N'$ implies that $\forall t > 0$, $|p(t)h_n'(t) - p(t)\widetilde{h}'(t)| < \frac{\varepsilon}{2}$. Fix $n_0 > N'$ and choose $t_0 \in (0,\infty)$ such that $t > t_0$ implies that $|p(t)\widetilde{h_{n_0}}'(t) - L| < \frac{\varepsilon}{2}$.

Now for $t > t_0$ we have that $|p(t)\widetilde{h}'(t) - L| \leqslant |p(t)h_{n_0}'(t) - p(t)\widetilde{h}'(t)| + |p(t)h_{n_0}'(t)$ $-L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

This proves the lemma. □

The next lemma is similar to many found in basic measure theory texts. See for example [11] and [16].

LEMMA 3. *Let $f,g \in L^1(0,\infty)$. If $\forall a,b \in (0,\infty)$, $\int_a^b f(\tau)d\tau \geqslant \int_a^b g(\tau)d\tau$ then $f \geqslant g$ a.e. on $(0,\infty)$.*

Proof. First assume that $z \in L^1(0,\infty)$ and $\forall a,b \in (0,\infty)$, $\int_a^b z(\tau)d\tau \geqslant 0$. We will prove by contradiction that $z(\tau) \geqslant 0$ a.e. The lemma follows by applying this result to the function $z(\tau) = f(\tau) - g(\tau)$.

$B = \{s \in (0,\infty) \mid z(s) < 0\}$ and assume B has positive measure. Choose $a,b \in (0,\infty)$ with $a < b$ such that $B^* \equiv B \cap (a,b)$ has positive measure. $\forall n \in N$ one can find an open set $O_n \supseteq B^*$ such that $O_n \subseteq [a,b]$ and $meas(O_n \setminus B^*) < \frac{1}{n}$. The sequence of sets $\{O_n\}_{n\in N}$ may be chosen to be decreasing.

For each $n \in N$, O_n is a countable union of open intervals which are its components and since by hypothesis the integral of z over such intervals is nonnegative we know that $\int_{O_n} z \geqslant 0$.

Also $0 \leqslant \int_{O_n} z = \int_{B^*} z + \int_{(O_n\setminus B^*)} z$. Thus $\int_{(O_n\setminus B^*)} z \geqslant -\int_{B^*} z$. Note that $z < 0$ on B^*, z is integrable, and B^* has positive measure so we know that $-\int_{B^*} z > 0$. Since the integral is an absolutely continuous set function $\lim_{n\to\infty} meas(O_n \setminus B^*) \leqslant \lim_{n\to\infty}\frac{1}{n} = 0$ implies that $\lim_{n\to\infty}\int_{(O_n\setminus B^*)} z = 0$. This is impossible since $\forall n \in N$, $\int_{(O_n\setminus B^*)} z \geqslant -\int_{B^*} z$, which is a fixed positive value. This is the contradiction that we seek so the lemma is proven. □

In order to apply the contraction mapping theorem we will need to establish certain properties of the operator M.

THEOREM 1. *The operator $\forall x \in \Omega$, Mx is closed, i.e. $M : \Omega \to P_{cl}(\Omega)$.*

Proof. Let $x \in \Omega$ and suppose $h_n \to \widetilde{h}$ in E where $h_n \in Mx$ $\forall n \in N$. We will show that $\widetilde{h} \in Mx$.

$\forall n \in N$ let $f_n(s)$ be the selection of $F(s, x_s, p(s)x'(s))$ for almost all s associated with h_n.

a) $h_n \to \widetilde{h}$ in E and $h_n = \varphi$ on $[-r, 0]$ $\forall n \in N$. Thus clearly $\widetilde{h} = \varphi$ on $[-r, 0]$. In particular note that $\widetilde{h}(0) = 0$.

b) $\forall t > 0$,

$$\lim_{n\to\infty}(p(t)h_n'(t) - L) = p(t)\widetilde{h}(t) - L. \tag{1}$$

and

$$\begin{aligned} p(t)h_n'(t) &= p(t)\frac{d}{dt}\Big[\int_0^t \frac{1}{p(\tau)}\int_\tau^\infty f_n(s)ds\,d\tau + LP(t)\Big] \\ &= p(t)\Big[\frac{1}{p(t)}\int_t^\infty f_n(s)ds + LP'(t)\Big] \\ &= p(t)\Big[\frac{1}{p(t)}\int_t^\infty f_n(s)ds + \frac{L}{p(t)}\Big] \\ &= \int_t^\infty f_n(s)ds + L. \end{aligned}$$

Thus from (1) above we have

$$p(t)\widetilde{h}'(t) = \lim_{n\to\infty} p(t)h_n'(t) = \lim_{n\to\infty}\int_t^\infty f_n(s)ds + L. \tag{2}$$

and this implies that

$$\lim_{n\to\infty}\int_t^\infty f_n(s)ds = p(t)\widetilde{h}'(t) - L.$$

Now let A be the set of full measure in $(0, \infty)$ which satisfies the following conditions.

$\forall s \in A$:

i) $f_n(s) \in F(s, x_s, p(s)x'(s))$ $\forall n \in N$. To accomplish this for each n find a set of full measure which satisfies the condition and then intersect them.

ii) $\sup_{y\in F(s,x_s,p(s)x'(s))} | y | \leqslant h(s, \eta_s, c)$ and

iii) $h(s, \eta_s, c) < \infty$. Note that since $h(s, \eta_s, c) \in L^1(0, \infty)$ it is finite almost everywhere.

Now let $s \in A$.

$$\begin{aligned} &\Longrightarrow \forall n \in N, \quad | f_n(s) | \leqslant h(s, \eta_s, c) < \infty \\ &\Longrightarrow -h(s, \eta_s, c) \leqslant \inf_{n\in N} f_n(s) \leqslant \sup_{n\in N} f_n(s) \leqslant h(s, \eta_s, c). \end{aligned}$$

Recall that $\forall s$, $F(s, x_s, p(s)x'(s))$ is a closed and convex set.

Thus $[\inf_{n\in N} f_n(s),\ \sup_{n\in N} f_n(s)] \subseteq F(s, x_s, p(s)x'(s))$.

For $a,b\in(0,\infty)$ we have

$$\lim_{n\to\infty}\int_a^b f_n(\tau)d\tau=\lim_{n\to\infty}\left[\int_a^\infty f_n(\tau)d\tau-\int_b^\infty f_n(\tau)d\tau\right]$$
$$=p(a)\widetilde{h}'(a)-p(b)\widetilde{h}'(b)\quad\text{by (2).}$$

Now $\forall n\in N$, $\int_a^b \inf_{m\in N} f_m(\tau)d\tau\leqslant\int_a^b f_n(\tau)d\tau\leqslant\int_a^b \sup_{m\in N} f_m(\tau)d\tau$. which implies that

$$\int_a^b \inf_{m\in N} f_m(\tau)d\tau\leqslant\lim_{n\in N}\int_a^b f_n(\tau)d\tau\leqslant\int_a^b \sup_{m\in N} f_m(\tau)d\tau,$$

so

$$\int_a^b \inf_{m\in N} f_m(\tau)d\tau\leqslant p(a)\widetilde{h}'(a)-p(b)\widetilde{h}'(b)\leqslant\int_a^b \sup_{m\in N} f_m(\tau)d\tau.$$

Thus $\int_a^b \inf_{m\in N} f_m(\tau)d\tau\leqslant\int_a^b -[p(\tau)\widetilde{h}'(\tau)]'d\tau\leqslant\int_a^b \sup_{m\in N} f_m(\tau)d\tau$. Since $-h(s,\eta_s,c)$ $\leqslant\inf_{n\in N} f_n(s)\leqslant\sup_{n\in N} f_n(s)\leqslant h(s,\eta_s,c)$ and $h(s,\eta_s,c)\in L^1(0,\infty)$, Lemma 3 implies:

$$\inf_{n\in N} f_n(\tau)\leqslant-[p(\tau)\widetilde{h}'(\tau)]'\leqslant\sup_{n\in N} f_n(\tau)\quad\text{a.e.}\tag{3}$$

Now let A_1 be the intersection of the set A with the set of full measure obtained in (3) and let $s\in A_1$. Since $[\inf_{n\in N} f_n(s),\sup_{n\in N} f_n(s)]\subseteq F(s,x_s,p(s)x'(s))$ we have that $f(s)\equiv[-p(s)\widetilde{h}'(s)]'$ is a selection of $F(s,x_s,p(s)x'(s))$.

As in the note at the end of the introduction we will define $m(t)$ for $t>0$ by

$$\begin{aligned}
m(t)&=\int_0^t \frac{1}{p(\tau)}\int_\tau^\infty[-p(s)\widetilde{h}'(s)]'ds\,d\tau+LP(t)\\
&=\int_0^t \frac{1}{p(\tau)}\lim_{b\to\infty}\int_\tau^b[-p(s)\widetilde{h}'(s)]'ds\,d\tau+LP(t)\\
&=\int_0^t \frac{1}{p(\tau)}\lim_{b\to\infty}-[p(b)\widetilde{h}'(b)-p(\tau)\widetilde{h}'(\tau)]d\tau+LP(t)\\
&=\int_0^t \frac{1}{p(\tau)}\lim_{b\to\infty}[p(\tau)\widetilde{h}'(\tau)-p(b)\widetilde{h}'(b)]d\tau+LP(t)\\
&=\int_0^t \frac{1}{p(\tau)}[p(\tau)\widetilde{h}'(\tau)-L+L]d\tau\\
&=\int_0^t \widetilde{h}'(\tau)d\tau=\widetilde{h}(t)-\widetilde{h}(0)=\widetilde{h}(t).
\end{aligned}$$

The last two equations above come from Lemma 2 and the facts that $P(t)=\int_0^t\frac{1}{p(\tau)}d\tau$ and $\widetilde{h}(0)=0$.

Thus $f(s)\equiv[-p(s)\widetilde{h}'(s)]'$ is the selection of $F(s,x_s,p(s)x'(s))$ associated with $\widetilde{h}$, thus $\widetilde{h}\in Mx$, and Mx is closed. □

THEOREM 2. *M is a contraction.*

Proof. Let $x_1, x_2 \in \Omega$. Recall that $H_d(Mx_1, Mx_2) = \max\{\sup_{h_2 \in Mx_2}\{\inf_{h_1 \in Mx_1} \| h_1 - h_2 \|_E\}, \sup_{h_1 \in Mx_1}\{\inf_{h_2 \in Mx_2} \| h_1 - h_2 \|_E\}\}$.

In what follows the notation $F_i(s)$, $i = 1,2$, will be used to denote $F(s, x_{i_s}, p(s)x_i'(s))$.

Fix $h_1 \in Mx_1$ and let f_1 be the associated a.e. selection of F_1.

For all $h_2 \in Mx_2$ with associated selection f_2 we have $\| h_1 - h_2 \|_E = \sup_{t>0} p(t) \mid h_1'(t) - h_2'(t) \mid$ since $h_1(t) = h_2(t) = \varphi(t)$ for $t \in [-r, 0]$. Also recall that $p(t)h_i'(t) = \int_t^\infty f_i(s)ds + L$ for $i = 1,2$ and that $\mid f_1(s) - f_2(s) \mid \leqslant 2h(s, \eta_s, c)$ a.e. $\in L^1(0,\infty)$.

Thus by the definition of Mx_2 we have:

$$
\begin{aligned}
\inf_{h_2 \in Mx_2} \| h_1 - h_2 \|_E &= \inf_{h_2 \in Mx_2} \{\sup_{t>0} \mid \int_t^\infty (f_1(s) - f_2(s))ds \mid\} \\
&= \inf_{f_2 \in F_2 \text{ a.e.}} \{\sup_{t>0} \mid \int_t^\infty (f_1(s) - f_2(s))ds \mid\} \\
&\leqslant \inf_{f_2 \in F_2 \text{ a.e.}} \{\sup_{t>0} \int_t^\infty \mid f_1(s) - f_2(s) \mid ds\} \\
&= \inf_{f_2 \in F_2 \text{ a.e.}} \int_0^\infty \mid f_1(s) - f_2(s) \mid ds.
\end{aligned}
$$

Claim $\inf_{f \in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid \in L^1(0,\infty)$.

Proof of claim. For all $f \in F_2$ a.e., $\mid f_1(s) - f(s) \mid \leqslant 2h(s, \eta_s, c) \in L^1(0,\infty)$ and since $L^1(0,\infty)$ is a Banach lattice every nonempty order bounded subset of it is order complete. Thus $\inf_{f \in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid$ exists and is an element of the order bounded set $\{g \in L^1(0,\infty) \mid -2h(s,\eta_s,c) \leqslant g(s) \leqslant 2h(s,\eta_s,c) \text{ a.e.}\}$ proving the claim. Again see [12] for details about Banach lattices. Note also that $Y = \{\mid f_1(s) - f(s) \mid : f \in F_2 \text{ a.e.}\} \subseteq \{g \in L^1(0,\infty) \mid -2h(s,\eta_s,c) \leqslant g(s) \leqslant 2h(s,\eta_s,c) \text{ a.e.}\}$ so it is also order bounded and its infimum must exist and be in $L^1(0.\infty)$.

Thus we have $\int_0^\infty \inf_{f \in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid ds \leqslant \int_0^\infty \mid f_1(s) - f_2(s) \mid ds$ for any $f_2 \in F_2$ a.e. which implies that $\int_0^\infty \inf_{f \in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid ds \leqslant \inf_{f \in F_2 \text{ a.e.}} \{\int_0^\infty \mid f_1(s) - f_2(s) \mid ds\}$.

Claim $\inf_{f \in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid \in Y$.

Proof of claim. We know that $\inf_{f \in F_2 \text{ a.e.}} |f_1(s) - f(s)| \in \{g \in L^1(0,\infty) \mid -2h(s,\eta_s,c) \leqslant g(s) \leqslant 2h(s,\eta_s,c) \text{ a.e.}\}$ Let us consider the following measurable function:

$$
\widehat{f}(s) = \begin{cases} f_1(s) & \text{if } F1^{(x_2, px_2')}(s) \leqslant f_1(s) \leqslant F2^{(x_2, px_2')}(s) \\ F2^{(x_2, px_2')}(s) & \text{if } f_1(s) \geqslant F2^{(x_2, px_2')}(s) \\ F1^{(x_2, px_2')}(s) & \text{if } F1^{(x_2, px_2')}(s) \geqslant f_1(s) \end{cases}
$$

which is a measurable function by condition 5 and note that $\widehat{f} \in F_2$ a.e. because $F1^{(x_2, px_2')}(s)$ and $F2^{(x_2, px_2')}(s)$ are measurable selections of F_2 a.e. and the fact that $F_2(s)$ is an interval. In fact $F_2(s) = [F1^{(x_2, px_2')}(s), F2^{(x_2, px_2')}(s)]$ a.e.

Thus:

$$\inf_{f\in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid = \mid f_1(s) - \widehat{f}(s) \mid$$

$$= \begin{cases} 0 & \text{if } F1^{(x_2,px_2')}(s) \leqslant f_1(s) \leqslant F2^{(x_2,px_2')}(s) \\ f_1(s) - F2^{(x_2,px_2')}(s) & \text{if } f_1(s) \geqslant F2^{(x_2,px_2')}(s) \\ F1^{(x_2,px_2')}(s) - f_1(s) & \text{if } F1^{(x_2,px_2')}(s) \geqslant f_1(s) \end{cases}$$

Clearly $\mid f_1(s) - \widehat{f}(s) \mid = \inf Y \in Y$ proving the claim. Note that this implies that $\inf_{f\in F_2 \text{ a.e.}} \{\int_0^\infty \mid f_1(s) - f_2(s) \mid ds\} = \int_0^\infty \inf_{f\in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid ds = \int_0^\infty \mid f_1(s) - \widehat{f}(s) \mid ds$.

The claim above will allow us to relate Y to $H_d(F_1, F_2)$ as follows:

$$\begin{aligned} \inf Y &= \mid f_1(s) - \widehat{f}(s) \mid \text{ a.e.} = \inf_{y_2\in F_2(s)} \mid f_1(s) - y_2 \mid \text{ a.e.} \\ &\leqslant \sup_{y_1\in F_1(s)} \{ \inf_{y_2\in F_2(s)} \mid y_1 - y_2 \mid \} \text{ a.e.} \\ &\leqslant H_d(F_1(s), F_2(s)) \text{ a.e.} \end{aligned}$$

Thus condition 7 implies:

$$\int_0^\infty \inf_{f\in F_2 \text{ a.e.}} \mid f_1(s) - f(s) \mid ds \leqslant \int_0^\infty k(s) \max\{\| x_{1_s} - x_{2_s} \|_{[-r,0]},\ p(s) \mid x_1'(s) - x_2'(s) \mid\} ds.$$

Now let us examine $\| x_{1_s} - x_{2_s} \|_{[-r,0]} = \sup_{t\in[-r,0]} \mid x_1(t+s) - x_2(t+s) \mid$.
If $t+s \leqslant 0$, then $x_1(t+s) = x_2(t+s) = \phi(t+s)$ so $x_1(t+s) - x_2(t+s) = 0$.
If $t+s \geqslant 0$, then

$$\begin{aligned} \sup_{t\in[-r,0],\ s\geqslant -t} \mid x_1(t+s) - x_2(t+s) \mid &= \sup_{t\in[-r,0],\ s\geqslant -t} \left| \int_0^{t+s} (x_1'(u) - x_2'(u)) du \right| \\ &\leqslant \sup_{t\in[-r,0],\ s\geqslant -t} \int_0^{t+s} \mid x_1'(u) - x_2'(u) \mid du \\ &= \sup_{t\in[-r,0],\ s\geqslant -t} \int_0^{t+s} \frac{1}{p(u)} p(u) \mid x_1'(u) - x_2'(u) \mid du \\ &\leqslant \| x_1 - x_2 \|_E \cdot \sup_{t\in[-r,0],\ s\geqslant -t} \int_0^{t+s} \frac{1}{p(u)} du \\ &\leqslant \| x_1 - x_2 \|_E \cdot \sup_{t\in[-r,0],\ s\geqslant -t} P(t+s) \\ &\leqslant \| x_1 - x_2 \|_E\, P(s) \end{aligned}$$

because $t+s \leqslant s$ and $p(s) \geqslant 0$.

Thus the above calculation and the definition of the norm in the space E imply that

$$\begin{aligned}\inf_{h_2\in Mx_2} \parallel h_1-h_2 \parallel_E &\leqslant \int_0^\infty k(s)\max\{P(s)\parallel x_1-x_2\parallel_E,\ p(s)\mid x_1'(s)-x_2'(s)\mid\}ds\\ &\leqslant \int_0^\infty k(s)\max\{P(s)\parallel x_1-x_2\parallel_E,\ \parallel x_1-x_2\parallel_E\}ds\\ &= \parallel x_1-x_2\parallel_E \int_0^\infty k(s)\max\{P(s),1\}ds\\ &= L_1\parallel x_1-x_2\parallel_E\end{aligned}$$

where $L_1 < 1$ by condition 7.

Therefore $\sup_{h_1\in Mx_1}\inf_{h_2\in Mx_2} \parallel h_1-h_2\parallel_E \leqslant L_1 \parallel x_1-x_2\parallel_E$ and by an identical argument $\sup_{h_2\in Mx_2}\inf_{h_1\in Mx_1} \parallel h_1-h_2\parallel_E \leqslant L_1 \parallel x_1-x_2\parallel_E$ which together imply that $H_d(Mx_1,Mx_2)\leqslant L_1\parallel x_1-x_2\parallel_E$ with $L_1<1$. Thus M is a contraction from Ω to $P_{cl}(\Omega)$. □

Now we are ready to establish the existence of solutions to our DBVI.

3. Main result

THEOREM 3. *Let L and φ be given and let F, h, φ, η, k, and p satisfy the conditions 1–7 above. Then our DBVI above has a solution.*

Proof. Theorems 1 and 2 and the Contraction Mapping Theorem together imply that our DBVI has a solution. □

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