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EXISTENCE THEORY FOR NONLINEAR STURM–LIOUVILLE PROBLEMS WITH NON–LOCAL BOUNDARY CONDITIONS

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(*Communicated by Lingju Kong*)

Abstract. In this work we provide conditions for the existence of solutions to nonlinear Sturm-Liouville problems of the form

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t))$$

subject to non-local boundary conditions

$$ax(0) + bx'(0) = \eta_1(x) \text{ and } cx(1) + dx'(1) = \eta_2(x).$$

Our approach will be topological, utilizing Schaefer's fixed point theorem and the Lyapunov-Schmidt procedure.

1. Introduction

In this paper we provide criteria for the solvability of nonlinear Sturm-Liouville problems of the form,

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = f(x(t)) \qquad t \in [0,1], \tag{1}$$

subject to non-local boundary conditions

$$ax(0) + bx'(0) = \eta_1(x) \text{ and } cx(1) + dx'(1) = \eta_2(x). \tag{2}$$

There are several standard ways in which one may define a solution to problem (1)–(2), and so to maintain completeness, we mention that in this paper we will be interested in proving the existence of classical solutions to (1)–(2). Formally, by a solution to (1)–(2) we mean a function $x : [0,1] \to \mathbb{R}$ such that px' is continuously differentiable and satisfies (1)–(2).

Throughout our analysis, we will assume that $p, q : [0,1] \to \mathbb{R}$ are continuous, $p(t) > 0$ for all $t \in [0,1]$, $a^2 + b^2 > 0$ and $c^2 + d^2 > 0$, λ is an eigenvalue of the associated linear Sturm-Liouville problem, $f : \mathbb{R} \to \mathbb{R}$ is continuous, and for $i = 1,2$, $\eta_i(x) = \int_{[0,1]} g_i(x)d\mu_i$, where $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are continuous and μ_1 and μ_2 are finite Borel measures on $[0,1]$.

The focus of this paper is the analysis of nonlinear Sturm-Liouville problems at resonance subject to non-local boundary conditions, where by resonance we mean that

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the linear homogeneous problem (7)–(8) has nontrivial solutions. Since the pioneering work of Landesman-Lazer, [12], much has been written about resonant nonlinear Sturm-Liouville boundary value problems with linear boundary conditions. Pertinent references from the point of view of this paper are [2, 4, 5, 6, 8, 11, 12, 14, 15, 18, 19]. Less has been said in regard to problems with nonlocal boundary conditions, even for the case of nonresonance; readers ineterested in results in this direction may consult [1, 7, 10, 13, 17, 20, 21, 22, 23].

The novelty of this work is due in large part to the generality of the nonlinear boundary conditions η_1 and η_2. As an important special case we point out that, by taking μ_1 and μ_2 to be point-supported measures, our integral boundary conditions allow for nonlinear multipoint boundary conditions of the form

$$\eta_1(x) = \sum_{k=1}^{n} f_k(x(t_k)), \eta_2(x) = \sum_{j=1}^{m} h_j(x(t_j)),$$

where each f_k, h_j is a continuous function and each t_k, $t_j \in [0,1]$.

Our main result, Theorem 3.1, provides conditions for the existence of solutions to (1)–(2) under a suitable interaction of the eigenspace of the linear Sturm-Liouville problem and the nonlinearities in both the differential equation and the boundary conditions. We would like to remark that the result we obtain in Theorem 3.1 constitutes a significant extension of the work found in [15] by allowing for much more generality in the boundary conditions, (2).

2. Preliminaries

The nonlinear boundary value problem (1)–(2) will be viewed as an operator equation. We let $C := C[0,1]$ denote the space of real-valued continuous functions topologized by the supremum norm, $\|\cdot\|_C$. As usual, $L^2 := L^2[0,1]$ will denote the space of real-valued square-integrable functions defined on $[0,1]$. The topology on L^2 will be that induced by the standard L^2-norm, $\|\cdot\|_{L^2}$. We use H^2 to denote the Sobolev space of functions with two weak derivatives in L^2; that is,

$$H^2 = \{x \in L^2 \mid x' \text{ is absolutely continuous and } x'' \in L^2\}.$$

Unless otherwise stated, the topology on H^2 will be the subspace topology inherited from L^2. However, we will, on several occasions, topologize H^2 with the Sobolev norm,

$$\|x\|_{H^2} = \|x\|_{L^2} + \|x'\|_{L^2} + \|x''\|_{L^2}.$$

On occasion, we may also view H^2 as a subspace of C. We will use $|\cdot|$ to denote the Euclidean norm on $\mathbb{R}^2$ and $\langle\cdot,\cdot\rangle_2$, $\langle\cdot,\cdot\rangle_S$, and $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ will denote the inner products on L^2, H^2, and $\mathbb{R}^2$, respectively. Weak convergence in L^2 will be denoted by $\overset{2}{\rightharpoonup}$ and weak convergence in the Sobolev space H^2 will be denoted by $\overset{S}{\rightharpoonup}$. We make $L^2 \times \mathbb{R}^2$

an inner product space with inner product

$$\left\langle \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} g \\ v_1 \\ v_2 \end{bmatrix} \right\rangle := m\left(\langle h, g\rangle_2 + \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle_{\mathbb{R}} \right), \tag{3}$$

where m is a positive constant which will be chosen later, and we will use $\|\cdot\|_{L^2\times\mathbb{R}^2}$ to denote the norm generated by this inner product. Lastly, we give $C\times\mathbb{R}^2$ the product topology, and we will use $\|\cdot\|_{C\times\mathbb{R}^2}$ to denote the standard product norm on this space.

Linear boundary operators B_1 and B_2 will be defined as follows:

$B_1 : H^2 \to \mathbb{R}$ is given by

$$B_1 x = ax(0) + bx'(0)$$

and $B_2 : H^2 \to \mathbb{R}$ is given by

$$B_2 x = cx(1) + dx'(1).$$

We define $\mathscr{L} : H^2 \to L^2 \times \mathbb{R}^2$

$$\mathscr{L}x = \begin{bmatrix} \mathscr{A}x \\ B_1 x \\ B_2 x \end{bmatrix},$$

where $\mathscr{A} : H^2 \to L^2$ is defined by

$$\mathscr{A}x(t) = (p(t)x'(t))' + (q(t)+\lambda)x(t).$$

Similarly, we define a nonlinear operator $\mathscr{G} : H^2 \to L^2\times\mathbb{R}^2$ by

$$\mathscr{G}(x) = \begin{bmatrix} \mathscr{F}(x) \\ \eta_1(x) \\ \eta_2(x) \end{bmatrix},$$

where $\mathscr{F}(x)(t) = f(x(t))$ and, as before, for $i = 1,2$, $\eta_i(x) = \int_{[0,1]} g_i(x)\,d\mu_i$. Solving the nonlinear boundary value problem (1)–(2) is now equivalent to solving

$$\mathscr{L}x = \mathscr{G}(x). \tag{4}$$

The study of the nonlinear boundary value problem (1)–(2) will be intimately related to the linear nonhomogeneous boundary value problem

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = h(t), \qquad t \in [0,1] \tag{5}$$

$$ax(0) + bx'(0) = w_1 \quad \text{and} \quad cx(1) + dx'(1) = w_2, \tag{6}$$

where h is an element of L^2 and w_1 and w_2 are elements of $\mathbb{R}$. Using our notation from above, we have that solving (5)–(6) is equivalent to solving

$$\mathscr{L}x = \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}.$$

We begin our study of the nonlinear boundary value problem (1)–(2) by analyzing (5)–(6). To aid in this analysis, we first recall some well-known facts regarding the linear homogeneous Sturm-Liouville problem

$$(p(t)x'(t))' + q(t)x(t) + \lambda x(t) = 0 \tag{7}$$

$$ax(0) + bx'(0) = 0 \text{ and } cx(1) + dx'(1) = 0. \tag{8}$$

For those readers interested in a more detailed introduction to linear Sturm-Liouville problems, we suggest [9].

It is well known that λ is a simple eigenvalue; that is, $Ker(\mathscr{L})$ is one-dimensional. We may therefore choose a vector, ψ, which forms a basis for $Ker(\mathscr{L})$. Without loss of generality, we will assume $\|\psi\|_{L^2} = 1$. Since (7) is a second-order linear homogeneous differential equation, we may choose ϕ satisfying (7) so that $\{\psi, \phi\}$ forms a basis for the solution space of this linear homogeneous problem. We will assume $\langle \psi, \phi \rangle_2 = 0$.

For $u, v \in H^2$, let $wr(u,v)$ denote the Wrońskian of u and v; that is, $wr(u,v) = uv' - vu'$. It follows from standard ode theory that if u and v are linearly independent solutions to (7), then $p \cdot wr(u,v)$ is a nonzero constant. We will assume that ϕ has been chosen so that $p \cdot wr(\psi, \phi) = 1$ and define $\omega : [0,1] \times [0,1] \to \mathbb{R}$ by

$$\omega(t,s) = \begin{cases} \psi(t)\phi(s) & \text{if } 0 \leqslant t \leqslant s \leqslant 1 \\ \psi(s)\phi(t) & \text{if } 0 \leqslant s \leqslant t \leqslant 1 \end{cases}. \tag{9}$$

As a reminder to the reader, ω is often referred to as a fundamental solution of (7).

If we define $K : L^2 \to H^2$ by

$$Kh(t) = \int_0^1 \omega(t,s)h(s)ds, \tag{10}$$

then it is easy to verify that K is self-adjoint, compact, and satisfies $\mathscr{A}Kh = h$ for every $h \in L^2$. Differentiating under the integral symbol, one easily establishes that for every $h \in L^2$, $B_1Kh = \langle h, \phi \rangle_2 B_1\psi = 0$ and $B_2Kh = \langle h, \psi \rangle_2 B_2\phi$. Let

$$v_1 = B_1\phi \text{ and } v_2 = B_2\phi.$$

Since ϕ satisfies (7) and is linearly independent of ψ, we must have $B_1\phi \neq 0$ and $B_2\phi \neq 0$; this is a consequence of the uniqueness of solutions to initial value problems and that fact the linear Sturm-Liouville boundary conditions can be thought of as an orthogonality condition.

With the above ideas in hand, we are now in a position characterize the range of $\mathscr{L}$. We have the following result.

PROPOSITION 2.1. *Let $h \in L^2$ and $w_1, w_2 \in \mathbb{R}$. Then $\vec{h} := \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \in Im(\mathscr{L})$ if and only if $\langle \vec{h}, \vec{\psi} \rangle = 0$, where $\vec{\psi} := \begin{bmatrix} \psi \\ v_1^{-1} \\ -v_2^{-1} \end{bmatrix}$. That is, in $L^2 \times \mathbb{R}^2$, $Im(\mathscr{L}) = \{\vec{\psi}\}^{\perp}$.*

Proof. $\mathscr{L}x = \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}$ if and only if $\mathscr{A}x = h$, $B_1 x = w_1$, and $B_2 x = w_2$. However, $\mathscr{A}x = h$ if and only if $x = c_1\psi + c_2\phi + Kh$, for some real numbers c_1, c_2. Applying the boundary map B_1 and recalling $B_1 Kh = 0$, we get $B_1(c_1\psi + c_2\phi + Kh) = c_2 v_1$. Similarly, using $B_2 Kh = \langle h, \psi \rangle_2 B_2 \phi$, we get $B_2(c_1\psi + c_2\phi + Kh) = (c_2 + \langle h, \psi \rangle_2) v_2$.

Now,

$$c_2 v_1 = w_1 \quad \text{and} \quad (c_2 + \langle h, \psi \rangle_2) v_2 = w_2$$

if and only if

$$c_2 = \frac{w_1}{v_1} \text{ and } \langle h, \psi \rangle_2 = \frac{w_2}{v_2} - \frac{w_1}{v_1} = \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} -v_1^{-1} \\ v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}},$$

which happens if and only if $\langle \vec{h}, \vec{\psi} \rangle = 0$. $\square$

With this characterization of the $Im(\mathscr{L})$ in hand, we make the following definitions which will play a crucial role in our ability to analyze the nonlinear Sturm-Liouville problem, (1)–(2), using a projection scheme.

DEFINITION 2.2. Define $P : L^2 \to L^2$ by $Px = \langle x, \psi \rangle_2 \psi$.

It is clear that P is the orthogonal projection onto $Ker(\mathscr{L})$.

Now, choose m, see (3), to be $\dfrac{1}{1 + \left\| \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\|^2}$. With this choice of m, $\vec{\psi}$ is a unit vector in $L^2 \times \mathbb{R}^2$.

DEFINITION 2.3. Define Q: $L^2 \times \mathbb{R}^2 \to L^2 \times \mathbb{R}^2$ by

$$Q\left(\begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix} \right) = \left\langle \begin{bmatrix} h \\ w_1 \\ w_2 \end{bmatrix}, \vec{\psi} \right\rangle \vec{\psi}.$$

From Proposition 2.1, we have that Q is the orthogonal projection of $L^2 \times \mathbb{R}^2$ on $Im(\mathscr{L})^\perp$. Thus, $I - Q$, is a projection onto the $Im(\mathscr{L})$.

In our analysis of the nonlinear Sturm-Liouville problem we will use a projection scheme often referred to as the Lyapunov-Schmidt procedure. The use of the Lyapunov-Schmidt reduction will allow us to write the operator equation (4) as an equivalent equation in which a fixed point argument may be applied to prove the existence of solutions. Interested readers may consult [3, 16] for a more detailed account of these ideas.

PROPOSITION 2.4. *Solving $\mathscr{L}x = \mathscr{G}(x)$ is equivalent to solving the system*

$$\begin{cases} (I - P)x - M(I - Q)\mathscr{G}(x) = 0 \\ \text{and} \\ \left(\langle \mathscr{F}(x), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi = 0 \end{cases},$$

where M denotes $(L|_{H^2\cap Ker(\mathscr{L})^\perp})^{-1}$.

Proof.

$$\mathscr{L}x=\mathscr{G}(x)\Longleftrightarrow\begin{cases}(I-Q)(\mathscr{L}x-\mathscr{G}(x))=0\\ \text{and}\\ Q(\mathscr{L}x-\mathscr{G}(x))=0\end{cases}$$

$$\Longleftrightarrow\begin{cases}\mathscr{L}x-(I-Q)\mathscr{G}(x)=0\\ \text{and}\\ Q\mathscr{G}(x)=0\end{cases}$$

$$\Longleftrightarrow\begin{cases}M\mathscr{L}x-M(I-Q)\mathscr{G}(x)=0\\ \text{and}\\ Q\mathscr{G}(x)=0\end{cases}$$

$$\Longleftrightarrow\begin{cases}(I-P)x-M(I-Q)\mathscr{G}(x)=0\\ \text{and}\\ \left\langle\begin{bmatrix}\mathscr{F}(x)\\ \eta_1(x)\\ \eta_2(x)\end{bmatrix},\begin{bmatrix}\psi\\ v_1^{-1}\\ -v_2^{-1}\end{bmatrix}\right\rangle\vec{\psi}=0\end{cases}$$

$$\Longleftrightarrow\begin{cases}(I-P)x-M(I-Q)\mathscr{G}(x)=0\\ \text{and}\\ \left\langle\begin{bmatrix}\mathscr{F}(x)\\ \eta_1(x)\\ \eta_2(x)\end{bmatrix},\begin{bmatrix}\psi\\ v_1^{-1}\\ -v_2^{-1}\end{bmatrix}\right\rangle\psi=0\end{cases}$$

$$\Longleftrightarrow\begin{cases}(I-P)x-M(I-Q)\mathscr{G}(x)=0\\ \text{and}\\ (\langle\mathscr{F}(x),\psi\rangle_2+\left\langle\begin{bmatrix}\eta_1(x)\\ \eta_2(x)\end{bmatrix},\begin{bmatrix}v_1^{-1}\\ -v_2^{-1}\end{bmatrix}\right\rangle_{\mathbb{R}})\psi=0\end{cases}.\qquad\square$$

3. Main results

We now come to our main result. In what follows, we will assume that the nonlinear integral boundary operators η_1 and η_2 are induced by bounded continuous functions g_1 and g_2.

To simplify the statement of the theorem, we introduce the following notation. For $i=1,2$, we let

$$g_{i,+}(+\infty):=\limsup_{x\to\infty}g_i(x),$$

$$g_{i,-}(+\infty) := \liminf_{x\to\infty} g_i(x),$$

$$g_{i,+}(-\infty) := \limsup_{x\to-\infty} g_i(x),$$

and

$$g_{i,-}(-\infty) := \liminf_{x\to-\infty} g_i(x).$$

We define $\mathscr{O}_0 := \{t \mid \psi(t) = 0\}$, $\mathscr{O}_+ := \{t \mid \psi(t) > 0\}$, and $\mathscr{O}_- := \{t \mid \psi(t) < 0\}$. From Standard Sturm-Liouville theory, we have that $\mathscr{O}_0$ is a finite set consisting of simple zeros. In what follows, this fact will be used several times, possibly without explicit mention. Finally, for $i = 1,2$, we let

$$J_{i,\pm} = g_{i,\pm}(+\infty)\mu_i(\mathscr{O}_+) + g_{i,\pm}(-\infty)\mu_i(\mathscr{O}_-).$$

THEOREM 3.1. *Suppose that the following conditions hold:*

C1. The function f is "sublinear"; that is, there exists real numbers M_1, M_2 and β, with $0 \leqslant \beta < 1$, such that for every $x \in \mathbb{R}$, $|f(x)| \leqslant M_1|x|^\beta + M_2$;

C2. There exist positive real numbers $\hat{z}$ and J such that for all $z > \hat{z}$,

$$f(-z) \leqslant -J < 0 < J \leqslant f(z);$$

C3. For $i = 1,2$, $\mu_i(\mathscr{O}_0) = 0$, where again u_i is the Borel measure in the definition of the boundary operator η_i;

C4. $-J\int_0^1 |\psi| dt < \left\langle \begin{bmatrix} J_{1,sgn(-v_1)} \\ J_{2,sgn(v_2)} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}$, where for a real number, v, $sgn(v) = +$ if $v > 0$ and $sgn(v) = -$ if $v < 0$;

then, there exists a solution to (1)–(2).

Proof. We start by defining $T : L^2 \to H^2$ by

$$T(x) = Px - (\langle \mathscr{F}(x), \psi\rangle_2 + \left\langle \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}})\psi + M(I-Q)\mathscr{G}(x).$$

From Proposition 2.4, we have that the solutions to (1)–(2) are the fixed points of T. Since M is an integral mapping from L^2 into H^2, it is compact, and thus so is T. We will show that

$$FP := \{x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0,1)\}$$

is a priori bounded in L^2. A fixed point will then follow from an application of Schaefer's fixed point theorem.

To this end, suppose that there exist sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ in H^2 and $(0,1)$, respectively, with $\|x_n\|_{L^2} \to \infty$ and $x_n = \gamma_n T(x_n)$. Let $y_n = \dfrac{x_n}{\|x_n\|_{H^2}}$. Since the closed unit ball in the Sobelov space H^2 is weakly compact, by going to a subsequence if necessary, we may assume that $y_n \xrightarrow{S} y$, for some $y \in H^2$. Again, going to a subsequence if necessary, we may assume that γ_n converges to some $\gamma \in [0,1]$.

Now,

$$\begin{aligned} y_n &= \frac{x_n}{\|x_n\|_{H^2}} \\ &= \gamma_n \frac{T(x_n)}{\|x_n\|_{H^2}} \\ &= \gamma_n \frac{Px_n - \left(\langle \mathscr{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi + M(I-Q)\mathscr{G}(x_n)}{\|x_n\|_{H^2}}. \end{aligned}$$

Since f is sublinear (See C1) and g_1 and g_2 are bounded, it follows that

$$\|\mathscr{G}(x)\|_{L^2 \times \mathbb{R}^2} \leqslant K_1 \|x\|_{L^2}^{\beta} + K_2, \tag{11}$$

and

$$\|\mathscr{G}(x)\|_{C \times \mathbb{R}^2} \leqslant K_1 \|x\|_{C}^{\beta} + K_2, \tag{12}$$

for some positive real numbers K_1 and K_2 and every $x \in H^2$. Thus, from (11),

$$\gamma_n \frac{\left(\langle \mathscr{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi + M(I-Q)\mathscr{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} 0,$$

so that

$$\gamma_n \frac{Px_n - \left(\langle \mathscr{F}(x_n), \psi \rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \right) \psi + M(I-Q)\mathscr{G}(x_n)}{\|x_n\|_{H^2}} \xrightarrow{2} \gamma P y.$$

Since $y_n \xrightarrow{S} y$, $y_n \xrightarrow{2} y$, so that we conclude $y = \gamma P y$. Applying P gives

$$Py = \gamma P^2 y = \gamma P y,$$

from which we deduce that $\gamma = 1$ or $Py = 0$. Since $\|y\|_{H^2} = 1$, it follows that $\gamma = 1$. Thus, $Py = y$ and we deduce that $y = \pm \frac{1}{\|\psi\|_{H^2}} \psi$. We will assume that $y = \frac{1}{\|\psi\|_{H^2}} \psi$, as the other case is similar.

Now, by the compact embedding of H^2 in C, we have, since $y_n \xrightarrow{S} y$, that $y_n \to y$ in C. Using the fact that $y_n \xrightarrow{2} \frac{1}{\|\psi\|_{H^2}} \psi$, we have that

$$\langle y_n, \psi \rangle_2 \to \frac{1}{\|\psi\|_{H^2}} \langle \psi, \psi \rangle_2 = \frac{1}{\|\psi\|_{H^2}}. \tag{13}$$

However, $\langle x_n, \psi\rangle_2 = \|x_n\|_{H^2}\langle y_n, \psi\rangle_2$, so that $\langle x_n, \psi\rangle_2 \to \infty$, since $\|x_n\|_{H^2}$ does (recall $\|x_n\|_{L^2} \to \infty$). Without loss of generality, we will assume from now on that $\langle x_n, \psi\rangle_2 > 0$ for each n.

From $x_n = \gamma_n T(x_n)$, it follows that for each n

$$(I-P)x_n = \gamma_n M(I-Q)\mathscr{G}(x_n)$$

and

$$Px_n = \gamma_n Px_n - \gamma_n\left(\langle \mathscr{F}(x_n), \psi\rangle_2 + \left\langle \begin{bmatrix}\eta_1(x_n)\\ \eta_2(x_n)\end{bmatrix}, \begin{bmatrix}v_1^{-1}\\ -v_2^{-1}\end{bmatrix}\right\rangle_{\mathbb{R}}\right)\psi.$$

This is equivalent to

$$(I-P)x_n = \gamma_n M(I-Q)\mathscr{G}(x_n) \tag{14}$$

and

$$(1-\gamma_n)\langle x_n, \psi\rangle_2 + \gamma_n\left(\langle \mathscr{F}(x_n), \psi\rangle_2 + \left\langle \begin{bmatrix}\eta_1(x_n)\\ \eta_2(x_n)\end{bmatrix}, \begin{bmatrix}v_1^{-1}\\ -v_2^{-1}\end{bmatrix}\right\rangle_{\mathbb{R}}\right) = 0. \tag{15}$$

Let v_n denote $(I-P)x_n$. From (12) and (14), we have that

$$\begin{aligned}\|v_n\|_C &\leqslant |\gamma_n|\,\|M(I-Q)\|\,(K_1\,\|x_n\|_C^\beta + K_2)\\ &\leqslant D_1\,\|x_n\|_C^\beta + D_2,\end{aligned}$$

where $\|M(I-Q)\|$ denotes the operator norm of $M(I-Q)$ and for $i=1,2$, $D_i = \|M(I-Q)\|K_i$. Applying the compact embedding theorem again, we may assume, by scaling each D_i, that

$$\|v_n\|_C \leqslant D_1\,\|x_n\|_{H^2}^\beta + D_2.$$

However, from (13) we have that $\dfrac{\langle x_n, \psi\rangle_2}{\|x_n\|_{H^2}} \to \dfrac{1}{\|\psi\|_{H^2}}$, so that by rescaling one more time, we may assume

$$\|v_n\|_C \leqslant D_1\langle x_n, \psi,\rangle_2^\beta + D_2. \tag{16}$$

For the moment, fix $t \in \mathscr{O}_+ \cup \mathscr{O}_-$. Since

$$\begin{aligned}|x_n(t)| &\geqslant \langle x_n, \psi\rangle_2|\psi(t)| - |v_n(t)|\\ &\geqslant \langle x_n, \psi\rangle_2|\psi(t)| - \|v_n\|_C,\end{aligned}$$

we have, using (16), that

$$\lim_{n\to\infty} x_n(t) = \pm\infty, \text{ whenever } t \in \mathscr{O}_\pm. \tag{17}$$

Define

$$E_n = \{t \mid |\psi|(t) \geqslant \varepsilon_n\},$$

where $\varepsilon_n = \dfrac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi \rangle_2}$. If $t \in E_n$, then

$$\begin{aligned} |x_n(t)| &\geqslant \langle x_n, \psi \rangle_2 |\psi(t)| - |v_n(t)| \\ &\geqslant \langle x_n, \psi \rangle_2 |\psi(t)| - \|v_n\|_C, \\ &\geqslant \langle x_n, \psi \rangle_2 \left(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi \rangle_2} \right) - \|v_n\|_C \\ &= \hat{z}. \end{aligned}$$

This gives, using C2, that

$$\begin{aligned} \int_0^1 f(x_n)\psi dt &= \int_{E_n} f(x_n)\psi dt + \int_{E_n^{\mathsf{c}}} f(x_n)\psi dt \\ &\geqslant J \int_{E_n} |\psi| dt + \int_{E_n^{\mathsf{c}}} f(x_n)\psi dt \\ &\geqslant J \int_{E_n} |\psi| dt - \int_{E_n^{\mathsf{c}}} |f(x_n)\psi| dt \end{aligned}$$

We claim that $\int_{E_n^{\mathsf{c}}} |f(x_n)\psi| dt \to 0$, so that by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \liminf_{n\to\infty} \int_0^1 f(x_n)\psi dt &\geqslant \liminf_{n\to\infty} J \int_{E_n} |\psi| dt \\ &= J \int_0^1 |\psi| dt. \end{aligned} \tag{18}$$

To see that $\int_{E_n^{\mathsf{c}}} |f(x_n)\psi| dt \to 0$, first note that for any $t \in E_n^{\mathsf{c}}$

$$\begin{aligned} |x_n(t)| &\leqslant \langle x_n, \psi \rangle_2 \varepsilon_n + \|v_n\|_C \\ &\leqslant \langle x_n, \psi \rangle_2 \left(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi \rangle_2} \right) + \|v_n\|_C \\ &= \hat{z} + 2\|v_n\|_C \\ &\leqslant \hat{z} + 2(D_1 \langle x_n, \psi \rangle_2^{\beta} + D_2) \quad \text{(using (16))}. \end{aligned}$$

It then follows, from C1, that

$$\begin{aligned} |f(x_n)(t)| &\leqslant M_1 |x_n(t)|^{\beta} + M_2 \\ &\leqslant M_1(\hat{z} + 2(D_1 \langle x_n, \psi, \rangle_2^{\beta} + D_2))^{\beta} + M_2, \end{aligned}$$

which gives that

$$\int_{E_n^{\mathsf{c}}} |f(x_n)\psi| dt \leqslant (M_1(\hat{z} + 2(D_1 \langle x_n, \psi, \rangle_2^{\beta} + D_2))^{\beta} + M_2)\varepsilon_n \mu_L(E_n^{\mathsf{c}}),$$

where μ_L denotes Lebesgue measure on $[0,1]$.

Since $\frac{\|v_n\|_C}{\langle x_n, \psi\rangle_2} \to 0$, we have that $E_n^{\mathsf{c}} \to \mathscr{O}_0$. Further, since $\mathscr{O}_0$ consists of finitely many simple zeros, it follows from the Mean Value Theorem that there exists a positive constant, say L, with

$$\mu_L(E_n^{\mathsf{c}}) \leqslant L\varepsilon_n.$$

We then have that

$$\begin{aligned}
\int_{E_n^{\mathsf{c}}} |f(x_n)\psi| dt &\leqslant (M_1(\hat{z} + 2(D_1\langle x_n, \psi, \rangle_2^\beta + D_2))^\beta + M_2)L\varepsilon_n^2 \\
&= (M_1(\hat{z} + 2(D_1\langle x_n, \psi, \rangle_2^\beta + D_2))^\beta + M_2)L\Big(\frac{\hat{z} + \|v_n\|_C}{\langle x_n, \psi\rangle_2}\Big)^2 \\
&\leqslant (M_1(\hat{z} + 2(D_1\langle x_n, \psi, \rangle_2^\beta + D_2))^\beta + M_2)L\Big(\frac{\hat{z} + D_1\langle x_n, \psi\rangle_2^\beta + D_2}{\langle x_n, \psi\rangle_2}\Big)^2,
\end{aligned}$$

so that

$$\int_{E_n^{\mathsf{c}}} |f(x_n)\psi| dt \leqslant R\frac{\langle x_n, \psi\rangle_2^{2\beta^2}}{\langle x_n, \psi\rangle_2^2},$$

for some positive constant R. Letting $n \to \infty$, and using the fact that $\beta < 1$, we conclude that $\int_{E_n^{\mathsf{c}}} |f(x_n)\psi| dt \to 0$.

We now look to analyze $\liminf_{n\to\infty} \int_0^1 g_i(x_n) d\mu_i$ and $\limsup_{n\to\infty} \int_0^1 g_i(x_n) d\mu_i$, for $i = 1,2$. From (17), if $t \in \mathscr{O}_+$, then

$$g_{i,-}(+\infty) \leqslant \liminf_{n\to\infty} g_i(x_n)(t) \text{ and } \limsup_{n\to\infty} g_i(x_n)(t) \leqslant g_{i,+}(+\infty).$$

Similarly, for each $t \in \mathscr{O}_-$ and each i, $i = 1,2$,

$$g_{i,-}(-\infty) \leqslant \liminf_{n\to\infty} g_i(x_n)(t) \text{ and } \limsup_{n\to\infty} g_i(x_n)(t) \leqslant g_{i,+}(-\infty).$$

Since g_1 and g_2 are bounded, we have, by Fatou's lemma, that for each i,

$$\begin{aligned}
J_{i,-} &= g_{i,-}(+\infty)\mu_i(\mathscr{O}_+) + g_{i,-}(-\infty)\mu_i(\mathscr{O}_-) \qquad (19)\\
&= \int_{\mathscr{O}_+} g_{i,-}(+\infty) d\mu_i + \int_{\mathscr{O}_-} g_{i,-}(-\infty) d\mu_i \\
&\leqslant \int_{\mathscr{O}_+ \cup \mathscr{O}_-} \liminf_{n\to\infty} g_i(x_n) d\mu_i \\
&= \int_{[0,1]} \liminf_{n\to\infty} g_i(x_n) d\mu_i \text{ (using C3)} \\
&\leqslant \liminf_{n\to\infty} \int_{[0,1]} g_i(x_n) d\mu_i \\
&\leqslant \limsup_{n\to\infty} \int_{[0,1]} g_i(x_n) d\mu_i
\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_{[0,1]} \limsup_{n\to\infty} g_i(x_n)d\mu_i \\
&\leqslant \int_{\mathscr{O}_+\cup\mathscr{O}_-} \limsup_{n\to\infty} g_i(x_n)d\mu_i \\
&\leqslant \int_{\mathscr{O}_+} g_{i,+}(+\infty)d\mu_i + \int_{\mathscr{O}_-} g_{i,-}(-\infty)d\mu_i \\
&= g_{i,+}(+\infty)\mu_i(\mathscr{O}_+) + g_{i,+}(-\infty)\mu_i(\mathscr{O}_-) \\
&= J_{i,+}.
\end{aligned}$$

Suppose for the moment that $v_1 > 0$ and $-v_2 > 0$ and let s and r be positive real numbers. Using the definitions of limit inferior and limit superior, see (18) and (19), there exists an n_s and an n_r such that if $n \geqslant n_s$, then

$$J\int_0^1 |\psi|dt - s < \langle f(x_n),\psi\rangle_2 = \langle \mathscr{F}(x_n),\psi\rangle_2, \tag{20}$$

and if $n \geqslant n_r$, then

$$J_{i,-} - r < \int_{[0,1]} g_i(x_n)d\mu_i < J_{i,+} + r. \tag{21}$$

Since $v_1 > 0$ and $-v_2 > 0$, it follows that

$$\left\langle \begin{bmatrix} J_{1,-}-r \\ J_{2,-}-r \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \leqslant \left\langle \begin{bmatrix} \int_{[0,1]} g_1(x_n)d\mu_1 \\ \int_{[0,1]} g_2(x_n)d\mu_2 \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} \leqslant \left\langle \begin{bmatrix} J_{1,+}+r \\ J_{2,+}+r \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}. \tag{22}$$

However,

$$\left\langle \begin{bmatrix} J_{1,-} \\ J_{2,-} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} = \left\langle \begin{bmatrix} J_{1,\mathrm{sgn}(-v_1)} \\ J_{2,\mathrm{sgn}(v_2)} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} > -J\int_0^1 |\psi|dt. \tag{23}$$

Thus, it follows, from (20),(21), (22), and (23), that we may choose r and s small enough so that

$$\langle \mathscr{F}(x_n),\psi\rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}} > 0, \tag{24}$$

for large enough n. The other cases for the sign of v_1 and $-v_2$ are similar. In each case, the conclusion in (24) holds. Recalling that $\langle x_n,\psi\rangle_2 \to +\infty$, we have that for large enough n,

$$(1-\gamma_n)\langle x_n,\psi\rangle_2 + \gamma_n\left(\langle \mathscr{F}(x_n),\psi\rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}\right) > 0.$$

However, this contradicts the fact that by (15),

$$(1-\gamma_n)\langle x_n,\psi\rangle_2 + \gamma_n\left(\langle \mathscr{F}(x_n),\psi\rangle_2 + \left\langle \begin{bmatrix} \eta_1(x_n) \\ \eta_2(x_n) \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}}\right) = 0.$$

Thus,

$$FP := \{x \in H^2 \mid x = \gamma T(x) \text{ for some } \gamma \in (0,1)\}$$

must be a priori bounded, and the proof is complete. □

REMARK 3.2. If $\eta_1 = \eta_2 = 0$, then by choosing for each i, $i = 1,2$, $g_i = 0$ and μ_i to be Lebesgue measure on $[0,1]$, we have that $J_{i,\pm} = 0$. Thus, condition C4 of Theorem 3.1 is trivially satisfied. This shows that Theorem 3.1 is a generalization of the result found in [15], where they analyze linear homogeneous boundary conditions.

The following corollary isolates the special case in which the boundary operators η_1 and η_2 are generated by bounded continuous function g_1 and g_2 for which we assume that for $i = 1,2$, $g_i(\pm\infty) := \lim_{x\to\pm\infty} g_i(x)$ exists.

COROLLARY 3.3. *Suppose that the following conditions hold:*

C1. The function f is "sublinear"; that is, there exists real numbers M_1, M_2 and β, with $0 \leqslant \beta < 1$, such that for every $x \in \mathbb{R}$, $|f(x)| \leqslant M_1|x|^\beta + M_2$;*

C2. There exist positive real numbers $\hat{z}$ and J such that for all $z > \hat{z}$,*

$$f(-z) \leqslant -J < 0 < J \leqslant f(z);$$

C3. For $i = 1,2$, $\mu_i(\mathscr{O}_0) = 0$, where again u_i is the Borel measure in the definition of the boundary operator η_i;*

C4. For $i = 1,2$, $g_i(\pm\infty) := \lim_{x\to\pm\infty} g_i(x)$ exists;*

C5.* $$-J\int_0^1 |\psi| dt < \left\langle \begin{bmatrix} J_{1,+} \\ J_{2,+} \end{bmatrix}, \begin{bmatrix} v_1^{-1} \\ -v_2^{-1} \end{bmatrix} \right\rangle_{\mathbb{R}};$$

then, there exists a solution to (1)–(2).

Proof. If for $i = 1,2$, $g_i(\pm\infty) := \lim_{x\to\pm\infty} g_i(x)$ exist, then for each of these i, $J_{i,-} = J_{i,+}$. □

4. Example

In this section we give a concrete example of the application of our main result, Theorem 3.1. We will use an interval of $[0,\pi]$ to simplify calculations.

Consider

$$x'' + m^2 x = f(x(t)) \tag{25}$$

subject to

$$x(0) = \int_{[0,\pi]} g_1(x) du_1 \text{ and } x(\pi) = \int_{[0,\pi]} g_2(x) du_2 \tag{26}$$

where f, g_1, and g_2 are real-valued continuous functions with g_1 and g_2 bounded.

It is well-known that the L^2-normalized eigenfunctions corresponding to the Dirichlet problem

$$x'' + m^2 x = 0$$

subject to boundary conditions

$$x(0) = 0 \text{ and } x(\pi) = 0,$$

are $\pm\frac{2}{\pi}\sin(mt)$. We choose to take $\psi(t) = \frac{2}{\pi}\sin(mt)$. This gives that ϕ, see (9), is $-\frac{\pi}{2}\cos(mt)$. Thus, $v_1 = \phi(0) = -\frac{\pi}{2}$ and $v_2 = \phi(\pi) = \frac{\pi}{2}$. We also have that

$$\mathscr{O}_+ = \begin{cases} \cup_{i=0}^{j}\left(\frac{2i\pi}{m}, \frac{(2i+1)\pi}{m}\right) & \text{if } m = 2j+1 \\ \cup_{i=0}^{j-1}\left(\frac{2i\pi}{m}, \frac{(2i+1)\pi}{m}\right) & \text{if } m = 2j \end{cases}$$

and

$$\mathscr{O}_- = \begin{cases} \cup_{i=0}^{j-1}\left(\frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m}\right) & \text{if } m = 2j+1 \\ \cup_{i=0}^{j-1}\left(\frac{(2i+1)\pi}{m}, \frac{(2i+2)\pi}{m}\right) & \text{if } m = 2j \end{cases}.$$

Suppose for the moment that conditions C1-C3 hold, since these can be trivially satisfied by any number of choices for f and μ_1, μ_2. Condition C4 of Theorem 3.1 in this specific problem becomes

$$-\frac{4}{\pi}J < \left\langle \begin{bmatrix} J_{1,+} \\ J_{2,+} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\pi} \\ -\frac{2}{\pi} \end{bmatrix} \right\rangle_{\mathbb{R}},$$

which is equivalent to $(J_{1,+} + J_{2,+}) < 2J$. It is clear that there are several bounded continuous functions g_1, g_2 and Borel measures μ_1, μ_2 which make the above inequality valid.

As a concrete example, let $E_m = \{t \mid \sin(mt) = 0\}$ and fix $t_0 \notin E_m$. Take $\mu := \mu_1 = \mu_2$ to be the measure point-supported at t_0; that is, for a subset A of $[0,1]$,

$$\mu(A) = \begin{cases} 1 & \text{if } t_0 \in A \\ 0 & \text{if } t_0 \notin A \end{cases}.$$

Since $t_0 \notin E_m$, we have that t_0 is in $\mathscr{O}_+$ or $\mathscr{O}_-$. If for each i, $i = 1,2$, $g_i(\pm\infty) := \lim_{x\to\pm\infty} g_i(x)$ exists, then when $t \in \mathscr{O}_+$, $J_{i,+} = g_i(+\infty)$. Similarly, when $t \in \mathscr{O}_-$, then $J_{i,+} = g_i(-\infty)$. Thus, if $t_0 \in \mathscr{O}_\pm$, then provided $g_1(\pm\infty) + g_2(\pm\infty) < 2J$, we have, from Corollary 3.3, that the nonlinear boundary value problem (25)–(26) has a solution. It is interesting to note that if $t_0 \notin \cup_m E_m$, and both $g_1(+\infty) + g_2(+\infty) < 2J$ and $g_1(-\infty) + g_2(-\infty) < 2J$, then (25)–(26) has a solution for all eigenvalues m.

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