Differential
Equations
& Applications
Volume 10, Number 3 (2018), 277–298
doi:10.7153/dea-2018-10-20

PROPERTIES OF SOLUTIONS OF THE SCALAR RICCATI EQUATION WITH COMPLEX COEFFICIENTS AND SOME THEIR APPLICATIONS

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(*Communicated by Tingbin Cao*)

Abstract. The definition of normal and extremal solutions of the scalar Riccati equation with complex coefficients is given. Some properties of normal and extremal solutions to Riccati equation are studied. On the basis of the obtained, some theorems which describe the asymptotic behavior of solutions of the system of two linear first order ordinary differential equations are proved (in particular a minimality theorem of a solution of the system of two linear first order ordinary differential equations is proved).

1. Introduction

Let $a(t)$, $b(t)$ and $c(t)$ be complex valued continuous functions on $[t_0;+\infty)$. Consider the Riccati equation

$$z'(t)+a(t)z^2(t)+b(t)z(t)+c(t)=0,\ \ t\geqslant t_0, \tag{1.1}$$

and associated with it the linear system

$$\begin{cases}\phi'(t)=a(t)\psi(t); \\ \psi'(t)=-c(t)\phi(t)-b(t)\psi(t),\end{cases} \quad t\geqslant t_0. \tag{1.2}$$

The solutions $z(t)$ of Eq. (1.1), existing on the some interval $[t_1;t_2)$ $(t_0\leqslant t_1<t_2\leqslant+\infty)$, are connected with solutions $(\phi(t),\psi(t))$ of the last systems by the relations (see [1], pp. 153–154):

$$\phi(t)=\phi(t_1)\exp\left\{\int\limits_{t_1}^{t}a(\tau)z(\tau)d\tau\right\},\quad \phi(t_1)\neq 0,\quad \psi(t)=z(t)\phi(t). \tag{1.3}$$

In the particular case when $a(t)\equiv 1$, $b(t)\equiv 0$ the system (1.2) is reducible to the following second order linear differential equation

$$\phi''(t)+c(t)\phi(t)=0,\ \ t\geqslant t_0. \tag{1.4}$$

Mathematics subject classification (2010): 34E99.

Keywords and phrases: Riccati equation, regular, normal and extremal solutions, global solvability, regular solutions of the system, normal, extremal, super extremal and irreconcilable systems, minimality property.

Paper DEA-10-20

Any solution $z(t)$ of Eq. (1.1), with the initial condition $z(t_0) = z_{(0)} (\in C)$, is or else continuable on $[t_0;+\infty)$, or else not continuable on $[t_0;+\infty)$ and therefore exists on $[t_0;t_1)$ for some $t_1 > t_0$. In the first case we call the solution $z(t)$ regular. For the case real $c(t)$ I. M. Sobol (see. [2,3]) successfully applied some important properties of real and regular solutions of the equation

$$y'(t) + y^2(t) + c(t) = 0, \ \ t \geqslant t_0, \tag{1.5}$$

to establish asymptotic behavior of solutions of Eq. (1.4). In the work [4] these properties of regular solutions of Eq. (1.5) were spread on the real and regular solutions of Eq. (1.1) to the case of real $a(t)$, $b(t)$ and $c(t)$. These properties have found applications in the questions of global solvability of Riccati equation [5], of stability [6–8] and oscillation [9–11] of systems of two linear first order ordinary differential equations, as well of linear second order differential equations.

In the section 2 we represent two global existence criteria for the scalar Riccati equation. In the section 3 we prove some important properties of solutions of scalar Riccati equation spreading the results of work [4] on the case of complex coefficients of Eq. (1.1). Some of the main results of section 3 as well as a particular version of Theorem 4.1 of section 4 (see below) are presented in [12] without their's proofs. The results of sections 2 and 3 are used in the section 4 to investigate some asymptotic properties of solutions to the systems of two first order linear ordinary differential equations.

2. Two global existence criteria

In this paragraph we represent two global existence criteria for the scalar Riccati equation. They will be used with the results of paragraph 3 to investigate in the paragraph 4 some asymptotic properties of solutions to the systems of two first order linear ordinary differential equations.

Denote $a_1(t) \equiv \operatorname{Re} a(t)$, $a_2(t) \equiv \operatorname{Im} a(t)$, $b_1(t) \equiv \operatorname{Re} b(t)$, $b_2(t) \equiv \operatorname{Im} b(t)$, $c_1(t) \equiv \operatorname{Re} c(t)$, $c_2(t) \equiv \operatorname{Im} c(t)$. In this paragraph we will assume that

I) $a_k(t) \geqslant 0$, $t \geqslant t_0$, $k = 1,2$.

The cases: II) $\begin{cases} a_1(t) \geqslant 0; \\ a_2(t) \leqslant 0, \ t \geqslant t_0, \end{cases}$ III) $\begin{cases} a_1(t) \leqslant 0; \\ a_2(t) \geqslant 0, \ t \geqslant t_0, \end{cases}$ IV) $\begin{cases} a_1(t) \leqslant 0; \\ a_2(t) \leqslant 0, \ t \geqslant t_0, \end{cases}$
can be reduced to the case I) by replacing $z(t) \to \overline{z}(t)$, $z(t) \to -\overline{z}(t)$, $z(t) \to -z(t)$ in (1.1) respectively.

Set

$$D_k(t) \equiv b_2^2(t) - 4(-1)^k a_k(t) c_k(t), \ \ t \geqslant t_0.$$

THEOREM 1. *Let $a_k(t) \geqslant 0$, $t \geqslant t_0$, and let $D_k(t) \leqslant 0$ if $a_k(t) \neq 0$ and $(-1)^k c_k(t) \geqslant 0$ if $a_k(t) = 0$, $k = 1,2$, $t \geqslant t_0$; moreover assume that if $a_1(t)a_2(t) = 0$, then $b_2(t) = 0$,. Then for arbitrary $\gamma_0 \geqslant 0$ and $\delta_0 \geqslant 0$ Eq. (1.1) has a solution $z_0(t)$ on $[t_0;+\infty)$, satisfying the initial condition $z_0(t_0) = \gamma_0 - i\delta_0$, moreover*

$$\operatorname{Re} z_0(t) \geqslant 0 \quad \operatorname{Im} z_0(t) \leqslant 0, \ \ t \geqslant t_0.$$

If $\gamma_0 > 0$ *and* $\delta_0 > 0$, *then*

$$\operatorname{Re} z_0(t) > 0, \quad \operatorname{Im} z_0(t) < 0, \ t \geqslant t_0. \tag{2.1}$$

See proof in [12]. Set

$$r_k^{\pm}(t) \equiv \begin{cases} (-1)^{k+1} b_2(t) \pm \sqrt{D_k(t)}, & \text{if } a_k(t) \neq 0; \\ 0, & \text{if } a_k(t) = 0, \end{cases} \qquad k = 1,2, \ t \geqslant t_0.$$

THEOREM 2. *Let* $a_k(t) \geqslant 0$, $t \geqslant t_0$, *and let* $D_k(t) \geqslant 0$ *if* $a_k(t) \neq 0$ *and* $(-1)^k c_k(t) \geqslant 0$ *if* $a_k(t) = 0$, $k = 1,2$, $t \geqslant t_0$*; moreover, assume that if* $a_1(t)a_2(t) = 0$, *then* $b_2(t) = 0$. *In addition assume that either* $r_1^+(t) \leqslant 0$ *and* $r_2^-(t) \geqslant \varepsilon$ *or* $r_1^-(t) \geqslant \varepsilon$ *and* $r_2^+(t) \leqslant 0$, $t \geqslant t_0$ *for some* $\varepsilon > 0$. *Then for arbitrary* $\gamma_0 > 0$, $\delta_0 > 0$ *Eq. (1.1) has a solution* $z_0(t)$ *on* $[t_0; +\infty)$, *satisfying the initial condition* $z_0(t_0) = \gamma_0 - i\delta_0$, *and inequalities (2.1).*

See proof in [13].

3. Some properties of solutions of the scalar Riccati equation

Let $t_1 \geqslant t_0$.

DEFINITION 1. A solution of Eq. (1.1) is said to be t_1-regular, if it exists on $[t_1; +\infty)$.

DEFINITION 2. A t_1-regular solution $z(t)$ of Eq. (1.1) is said to be t_1-normal, if there exists a neighborhood U of the point $z(t_1)$ such that each solution $\widetilde{z}(t)$ of Eq. (1.1) with $\widetilde{z}(t_1) \in U$ is t_1-regular. Otherwise $z(t)$ is said to be t_1-extremal.

DEFINITION 3. Eq. (1.1) is said to be regular, if it has a t_1-regular solution for some $t_1 \geqslant t_0$.

REMARK 1. For $a(t) > 0$, $\operatorname{Im} b(t) = \operatorname{Im} c(t) \equiv 0$ the regularity of Eq. (1.1) is equivalent to the non oscillation of the equation

$$\left(\frac{\phi'(t)}{a(t)}\right)' + \frac{b(t)}{a(t)}\phi'(t) + c(t)\phi(t) = 0, \ t \geqslant t_0.$$

Regularity criteria of Eq. (1.1) in the real-valued case of $a(t)$, $b(t)$ and $c(t)$ are proved in the works [5] and [15]. A non oscillatory criterion for the last equation is proved in [16].

In what follows a t_0-regular (t_0-normal, t_0-extremal) solution we will just call a regular (normal, extremal) solution. Since the solutions of Eq. (1.1) are continuously depending on their initial values, each t_1-regular solution of Eq. (1.1) is t_1-normal (t_1-extremal) if and only if it is t_2-normal (t_2)-extremal for $t_2 > t_1$. Note that a t_2-regular solution for $t_2 > t_1$ in general is not t_1-regular.

For arbitrary continuous on $[t_0;+\infty)$ function $u(t)$ denote:

$$e_u(t_1;t) \equiv \exp\left\{-\int\limits_{t_1}^{t}[2a(s)u(s)+b(s)]ds\right\},\ \ \mu_u(t_1;t) \equiv \int\limits_{t_1}^{t} a(\tau)e_u(t_1;\tau)d\tau,\ \ t_1,t \geqslant t_0.$$

Let $Z(t;t_1;\lambda)$ be the general solution of Eq. (1.1) in the region $G_{t_1} \equiv \{(t;z) : t \in I_{t_1}(\lambda),\ z \in C\}$, where $I_{t_1}(\lambda)$ is the maximal existence interval for the solution $\widetilde{z}_\lambda(t)$ of Eq. (1.1) whit $\widetilde{z}_\lambda(t_1) = \lambda\ \ (\in C)$.

LEMMA 1. *If $z_0(t)$ is a regular solution of Eq. (1.1), then on the $G_{t_1} \cap [t_0;+\infty) \times C$ the general solution $Z(t;t_1;\lambda)$ of Eq. (1.1) is given by*

$$Z(t;t_1;\lambda) = z_0(t) + \frac{\lambda e_{z_0}(t_1;t)}{1+\lambda\mu_{z_0}(t_1;t)},\ \ t,t_1 \geqslant t_0,\ \ \lambda \in C. \tag{3.1}$$

The proof of this lemma is elementary and we omit it (see [4]).

EXAMPLE 3.1. Consider the equation

$$z'(t) + a(t)z^2(t) = 0,\ \ t \geqslant -1. \tag{3.2}$$

The general solution of this equation in the region $G_0 \cap [-1;+\infty) \times C$ is given by formula

$$Z(t;0;\lambda) = \frac{\lambda}{1+\lambda\int\limits_0^t a(\tau)d\tau},\ \ \lambda \in C,\ \ 1+\lambda\int\limits_0^t a(\tau)d\tau \neq 0,\ \ t \geqslant -1. \tag{3.3}$$

Suppose $\int\limits_0^t a(\tau)d\tau = \text{arctg}\, t(\cos t + i\sin t)$, $t \geqslant -1$. Then from (3.3) is seen that all solutions $z(t)$ of Eq. (3.2) with $|z(0)| = \frac{\pi}{2}$ are 0-extremal, and with $|z(0)| < \frac{\pi}{2}$ they are 0-normal (for $|z(0)| > \frac{\pi}{2}$ the solution $z(t)$ is not 0-regular). Suppose in (3.2) the function $a(t)$ has bounded support. Then from (3.3) is seen that (3.2) has no extremal solution. For $u_0 \in C$ and $0 < r < R \leqslant +\infty$ denote $K_{r,R}(u_0) \equiv \{z \in C : r < |z-u_0| < R\}$ a ring with its center in u_0 and radiuses r and R. Let $\varepsilon > 0$ and let $K_{\varepsilon,r,R}(u_0) (\subset K_{r,R}(u_0))$ denotes a ε - net for $K_{r,R}(u_0)$, i. e. a finite set $\xi_1,\dots,\xi_{m_\varepsilon} \in K_{r,R}(u_0)$ such that for each $u \in K_{r,R}(u_0)$ there exists $v \in K_{\varepsilon,r,R}(u_0)$ with $|u-v| < \varepsilon$. Consider the sequence of sets $\{K_{\frac{1}{2n},\frac{1}{n},n}(u_0)\}_{n=1}^{+\infty}$. Let the function $f(t) \equiv \int\limits_0^t a(\tau)d\tau$, $t \geqslant 0$, has the following properties. $f(t) \neq u_0$, $t \in [0;1]$ when t varies from n to $n+1$ the curve $f(t)$ crosses all points of $K_{\frac{1}{2n},\frac{1}{n},n}(u_0)$ (i. e. for each $v \in K_{\frac{1}{2n},\frac{1}{n},n}(u_0)$ there exists $\xi_v \in [n;n+1]$ such that $f(\xi_v) = v$) and values of $f(t)$ remain in $K_{\frac{1}{2n},+\infty}(u_0)$. Obviously for all $T \geqslant 0$ the set of values of $f(t)$ on $[T;+\infty)$ is everywhere dense in C and $f(t) \neq u_0$, $t \geqslant t_0$. From here and from (3.3) it follows that Eq. (3.2) has no t_1-normal solutions for all $t_1 \geqslant t_0$ and has at least two extremal solutions $z_0(t) \equiv 0$ and $z_1(t)$ with $z_1(0) = -\frac{1}{u_0}$. By analogy using $1/2n$ - nets $K_{\frac{1}{2n},\frac{1}{n},n}(u_0;u_1;\dots,u_m)$ of intersections $\cap_{k=0}^m K_{\frac{1}{n},n}(u_k)$ in place of $K_{\frac{1}{2n},\frac{1}{n},n}(u_0)$, $n=1,2,\dots$ one can show that there exists a Riccati equation which has no t_1-normal solutions for all $t_1 \geqslant t_0$ and has at least $m+2$ extremal solutions.

REMARK 2. Let $z_0(t)$ be a t_1-regular solution of Eq. (1.1). Then since the function $\mu_{z_0}(t_1;t)$ is continuously differentiable by t, there exists $\lambda_0 \neq 0$ such that $1+\lambda_0\mu_{z_0}(t_1;t) \neq 0$, $t \geqslant t_1$ (the curve $\{\mu_{z_0}(t_1;t) : t \geqslant t_1\}$ is not space filling). Then by virtue of Lemma 1 the function $z_1(t) \equiv z_0(t) + \frac{\lambda_0 e_{z_0}(t_1;t)}{1=\lambda_0\mu_{z_0}(t_1;t)}$, $t \geqslant t_1$, is a t_1-regular solution of Eq. (1.1), different from $z_0(t)$.

From Lemma 1 we immediately get

THEOREM 3. *A t_1-regular solution $z_0(t)$ of Eq. (1.1) is t_1-normal if and only if $\mu_{z_0}(t_1;t)$ is bounded with respect to $t \in [t_1;+\infty)$.*

Let $z_1(t)$ and $z_2(t)$ be regular solutions of Eq. (1.1). Then by virtue of Lemma 1 for each $t_1,t \geqslant t_0$ the following equalities hold

$$z_j(t) = z_k(t) + \frac{\lambda_{jk}(t_1)e_{z_k}(t_1;t)}{1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)}, \quad j,k=1,2, \tag{3.4}$$

where $\lambda_{jk}(t_1) \equiv z_j(t_1) - z_k(t_1)$, $j,k=1,2$. From here we have:

$$z_j(t) - z_k(t) = \frac{\lambda_{jk}(t_1)e_{z_k}(t_1;t)}{1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)}, \quad t_1,t \geqslant t_0, \quad j,k=1,2, \tag{3.5}$$

The left hand sides of these equalities are independent of t_1. Consequently, so are their right hand sides. In view of this following [4] denote:

$$\nu_{z_j,z_k}(t) \equiv -\frac{1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)}{\lambda_{jk}(t_1)e_{z_k}(t_1;t)}, \quad t \geqslant t_0, \quad j,k=1,2 \quad (j \neq k).$$

Then the equalities (3.4) can be rewritten in the form

$$z_j(t) = z_k(t) - \frac{1}{\nu_{z_j,z_k}(t)}, \quad t \geqslant t_0, \quad j,k=1,2 \quad (j \neq k).$$

From here is seen that

$$\nu_{z_1,z_2}(t) = -\nu_{z_2,z_1}(t), \quad t \geqslant t_0. \tag{3.6}$$

THEOREM 4. *Regular solutions $z_1(t)$ and $z_2(t)$ of Eq. (1.1) satisfy the relations*

$$a(t)[z_1(t)+z_2(t)] = \frac{\nu'_{z_1,z_2}(t)}{\nu_{z_1,z_2}(t)} - b(t), \quad t \geqslant t_0. \tag{3.7}$$

$$a(t)z_1(t)z_2(t) = z_1'(t) + z_1(t)\frac{\nu'_{z_1,z_2}(t)}{\nu_{z_1,z_2}(t)} + c(t), \quad t \geqslant t_0. \tag{3.8}$$

The proof of this theorem in the case of real functions $a(t)$, $b(t)$, $c(t)$, $z_1(t)$ and $z_2(t)$ can be found in [4]. The same proof remains valid in the general case.

From (3.4) we deduce:

$$-2a(t)z_j(t)-b(t)=-2a(t)z_k(t)-b(t)-\frac{2\lambda_{jk}(t_1)a(t)e_{z_k}(t_1;t)}{1+\lambda_{jk}(t_1;t)},\ \ t\geqslant t_0,\ j,k=1,2.$$

Integrating these equalities from t_1 to t we obtain:

$$\int\limits_{t_1}^{t}[2a(\tau)z_j(\tau)+b(\tau)]d\tau=\int\limits_{t_1}^{t}[2a(\tau)z_k(\tau)+b(\tau)]d\tau-\ln[1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)]^2,\ t_1,t\geqslant t_0,$$

$j,k=1,2.$ From here it follows

$$e_{z_j}(t_1;t)=\frac{e_{z_k}(t_1;t)}{[1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)]^2},\ \ t_1,t\geqslant t_0,\ \ j,k=1,2,$$

so for all $t_1,t\geqslant t_0$

$$e_{z_1}(t_1;t)e_{z_2}(t_1;t)=\frac{e_{z_1}(t_1;t)e_{z_2}(t_1;t)}{[1+\lambda_{12}(t_1)\mu_{z_2}(t_1;t)]^2[1+\lambda_{21}(t_1)\mu_{z_1}(t_1;t)]^2},\ \ j,k=1,2. \tag{3.9}$$

Since $1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t_1)=1$, $j,k=1,2,$ and the solutions $z_j(t)$ $(j=1,2)$ are regular, from (3.4) it follows that $1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)\neq 0$, $t_1,t\geqslant t_0$, $j,k=1,2.$ Therefore by virtue of (3.9) we will have:

$$[1+\lambda_{12}(t_1)\mu_{z_2}(t_1;t)][1+\lambda_{21}(t_1)\mu_{z_1}(t_1;t)]\equiv 1.$$

From here it is easy to derive the equalities

$$\mu_{z_j}(t_1;t)=\frac{\mu_{z_k}(t_1;t)}{1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)},\ \ t_1,t\geqslant t_0,\ \ j,k=1,2. \tag{3.10}$$

Multiplying (3.5) on $a(t)$ and integrating from t_1 to t we will have:

$$I_{z_j,z_k}(t_1;t)\equiv\int\limits_{t_1}^{t}a(\tau)[z_j(\tau)-z_k(\tau)]d\tau=\ln[1+\lambda_{jk}(t_1)\mu_{z_k}(t_1;t)],\ \ t_1,t\geqslant t_0, \tag{3.11}$$

$j,k=1,2.$ If $z_2(t)\equiv z_{ext}(t)$ is an extremal solution then

$$\liminf_{t\to+\infty}|1+\lambda_{1,2}(t_0)\mu_{z_1}(t_0;t)|=0.$$

Hence by (3.4)

$$\liminf_{t\to+\infty}\mathrm{Re}\int\limits_{t_0}^{t}a(\tau)(z_{ezt}(\tau)-z_0(\tau))d\tau=-\infty. \tag{3.12}$$

The last equality particularly implies that if $z_1^*(t)$ and $z_2^*(t)$ are extremal solutions then

$$\liminf_{t\to+\infty} \operatorname{Re} \int_{t_0}^{t} a(\tau)(z_1^*(\tau)-z_2^*(\tau))d\tau = -\infty. \tag{3.13}$$

$$\limsup_{t\to+\infty} \operatorname{Re} \int_{t_0}^{t} a(\tau)(z_1^*(\tau)-z_2^*(\tau))d\tau = +\infty. \tag{3.14}$$

From (3.4), from Theorem 3 and from (3.11) we conclude

THEOREM 5. *For t_1-regular solutions $z_1(t)$ and $z_2(t)$ the function $I_{z_1,z_2}(t_1;t)$ is bounded with respect to $t\in[t_1;+\infty)$ if and only if these solutions are t_1-normal.*

In what follows, we assume that Eq. (1.1) has at least one regular solution. For an arbitrary continuous function $u(t)$ on $[t_1;+\infty)$ we introduce the notation

$$\nu_u(t) \equiv \int_t^{+\infty} a(\tau)\exp\Big\{-\int_t^{\tau}[2a(s)u(s)+b(s)]ds\Big\}d\tau, \ \ t\geqslant t_0.$$

THEOREM 6. *Let the integral $\nu_{z_0}(t_0)$ be convergent for some regular solution $z_0(t)$ of Eq. (1.1). Then the following assertions hold.*

A) For all normal solutions $z(t)$ of Eq. (1.1) and only for them, the integrals $\nu_z(t)$ are convergent for all $t\geqslant t_0$;

B) The integral $\int\limits_{t_0}^{+\infty} a(\tau)[z_1(\tau)-z_2(\tau)]d\tau$ is convergent for arbitrary normal solutions $z_1(t)$ and $z_2(t)$;

C) Eq. (1.1) has an extremal solution if and only if $\nu_{z_0}(t)\neq 0$, $t\geqslant t_0$. Under this condition the unique extremal solution $z_*(t)$ of Eq. (1.1) has the form

$$z_*(t) = z_0(t) - \frac{1}{\nu_{z_0}(t)}, \ \ t\geqslant t_0, \tag{3.15}$$

in addition

$$\nu_{z_*}(t) = \infty, \ \ t\geqslant t_0, \tag{3.16}$$

$$\operatorname{Re}\int_{t_0}^{+\infty} a(\tau)[z_*(\tau)-z_0(\tau)]d\tau = -\infty, \tag{3.17}$$

$$\int_t^{+\infty} a(\tau)[z_1(\tau)-z_2(\tau)]d\tau = \ln\left[\frac{z_*(t)-z_1(t)}{z_*(t)-z_2(t)}\right], \ \ t\geqslant t_0, \tag{3.18}$$

for arbitrary normal solutions $z_1(t)$ and $z_2(t)$ of Eq. (1.1).

Proof. Note that $\nu_{z_0}(t_0) = \lim\limits_{t\to+\infty} \mu_{z_0}(t_0;t)$. Therefore $\mu_{z_0}(t_0;t)$ is bounded. By virtue of Theorem 3 from here it follows, that $z_0(t)$ is a normal solution. Let $z(t)$ be a normal solution of Eq. (1.1), different from $z_0(t)$. By (3.10) we have

$$\nu_z(t_0) = \lim_{t\to+\infty} \mu_z(t_0;t) = \lim_{t\to+\infty} \frac{\mu_{z_0}(t_0;t)}{1+\lambda(t_0)\mu_{z_0}(t_0;t)}, \tag{3.19}$$

where $\lambda(t_0) \equiv z(t_0) - z_0(t_0)$. On the strength of Theorem 5 the function $I_{z,z_0}(t_0;t)$ is bounded with respect to $t \in [t_0;+\infty)$. From here and from (3.11) it follows $\lim\limits_{t\to+\infty}[1+\lambda(t_0)\mu_{z_0}(t_0;t)] \neq 0$. From here and from (3.19) it follows the convergence of $\nu_z(t_0)$. From the equality $\nu_z(t_0) = \mu_z(t_0;t) + e_z(t_0;t)\nu_z(t)$, $t \geqslant t_0$, and from the convergence of $\nu_z(t_0)$ it follows the convergence of $\nu_z(t)$ for all $t \geqslant t_0$. If $\widetilde{z}(t)$ is an extremal solution of Eq. (1.1), then due to Theorem 3 $\mu_{\widetilde{z}}(t_1;t)$ is not bounded with respect to $t \in [t_1;+\infty)$ for each $t_1 \geqslant t_0$. Therefore $\nu_{\widetilde{z}}(t_1)$ diverges for all $t_1 \geqslant t_0$. The assertion A) is proved. Let us prove B). Denote $\lambda_0 \equiv -\frac{1}{\nu_{z_0}(t_0)}$. Then

$$1+\lambda_0\mu_{z_0}(t_0;t) = -\frac{\nu_{z_0}(t_0)-\mu_{z_0}(t_0;t)}{\nu_{z_0}(t_0)} = -\frac{\nu_{z_0}(t)e_{z_0}(t_0;t)}{\nu_{z_0}(t_0)} \neq 0, \quad t \geqslant t_0.$$

On the strength of Lemma 1 from here it follows that

$$z_*(t) \equiv z_0(t) + \frac{\lambda_0 e_{z_0}(t_0;t)}{1+\lambda_0\mu_{z_0}(t_0;t)}, \quad t \geqslant t_0. \tag{3.20}$$

is a regular solution of Eq. (1.1). Since $\lim\limits_{t\to+\infty}[1+\mu_{z_0}(t_0;t)] = 1+\lambda_0\nu_{z_0}(t_0) = 0$, from (3.20) is seen that $z_*(t)$ is an extremal solution of Eq. (1.1). Moreover by virtue of Lemma 1 from convergence of $\nu_{z_0}(t_0)$ and from (3.20) it follows that $z_*(t)$ is the unique extremal solution of Eq. (1.1). From (3.20) also it follows (3.16). Show that from the existence of an extremal solution $z_{ext}(t)$ of Eq. (1.1) it follows inequality $\nu_{z_0}(t) \neq 0$, $t \geqslant t_0$. Due to Lemma 1 we have

$$z_{ext}(t) = z_0(t) + \frac{\lambda_{ext} e_{z_0}(t_0;t)}{1+\lambda_{ext}\mu_{z_0}(t_0;t)}, \quad t \geqslant t_0, \tag{3.21}$$

for some $\lambda_{ext} \neq 0$. Since $\nu_{z_0}(t_0) = \lim\limits_{t\to+\infty} \mu_{z_o}(t_0;t)$, and $z_{ext}(t)$ is an extremal solution, it is necessary $\lim\limits_{t\to+\infty}[1+\lambda_{ext}\mu_{z_0}(t_0;t)] = 0$. Therefore $\nu_{z_0}(t_0) = -\frac{1}{\lambda_{ext}} \neq 0$. Then

$$1+\lambda_{ext}\mu_{z_0}(t_0;t) = -\frac{e_{z_0}(t_0;t)\nu_{z_0}(t)}{\nu_{z_0}(t_0)}, \quad t \geqslant t_0. \tag{3.22}$$

Since $z_{ext}(t)$ is regular, from (3.18) it follows that $1+\lambda_{ext}\mu_{z_0}(t_0;t) \neq 0$, $t \geqslant t_0$. From here and from (3.22) it follows $\nu_{z_0}(t) \neq 0$, $t \geqslant t_0$. By (3.10) we have

$$\mu_{z_*}(t_1;t) = \frac{\mu_{z_0}(t_1;t)}{1+\lambda_*(t_1)\mu_{z_0}(t_1;t)}, \quad t \geqslant t_1 \geqslant t_0, \tag{3.23}$$

where $\lambda_*(t_1) = z_*(t_1) - z_0(t_1)$, $t_1 \geqslant t_0$. Since according to Theorem 3 $\mu_{z_*}(t_1;t)$ is unbounded with respect to $t \in [t_1;+\infty)$, and there exists a finite limit $\lim\limits_{t\to+\infty} \mu_{z_0}(t_1;t) = \nu_{z_0}(t_1) \neq 0$, from (3.23) it follows that

$$\lim_{t\to+\infty} [1 + \lambda_*(t_1)\mu_{z_0}(t_1;t)] = 0. \tag{3.24}$$

Then passing to the limit in (3.23) when $t \to +\infty$ we arrive at the equality $\nu_{z_*}(t_1) = \infty$, $t \geqslant t_0$. The equality (3.16) is proved. By virtue of Lemma 1 we have

$$z_*(t) - z_0(t) = \frac{\lambda_*(t_0)e_{z_0}(t_0;t)}{1 + \lambda_*(t_0)\mu_{z_0}(t_0;t)}, \quad t \geqslant t_0.$$

Multiplying both sides of this equality on $a(t)$ and integrating from t_0 to t we will get

$$\int_{t_0}^{t} a(\tau)[z_*(\tau) - z_0(\tau)]d\tau = \ln[1 + \lambda_*(t_0)\mu_{z_0}(t_0;t)], \quad t \geqslant t_0.$$

Then

$$\operatorname{Re} \int_{t_0}^{t} a(\tau)[z_*(\tau) - z_0(\tau)]d\tau = \ln|1 + \lambda_*(t_0)\mu_{z_0}(t_0;t)|, \quad t \geqslant t_0.$$

Passing to the limit in this equality when $t \to +\infty$ and taking into account (3.24) we come to (3.17). Let $z_1(t)$ and $z_2(t)$ be normal solutions of Eq. (1.1), and let

$$\phi_1(t) \equiv \exp\left\{ \int_{t_1}^{t} a(\tau)z_1(\tau)d\tau \right\}, \quad \psi_1(t) \equiv z_1(t)\phi_1(t);$$

$$\phi_2(t) \equiv \exp\left\{ \int_{t_1}^{t} a(\tau)z_*(\tau)d\tau \right\}, \quad \psi_2(t) \equiv z_*(t)\phi_2(t), \quad t_1, t \geqslant t_0.$$

By virtue of (1.3) $(\phi_k(t), \psi_k(t))$ $(k = 1,2)$ is a solution of the system (1.1) on $[t_1;+\infty)$. Then by virtue of (1.3) for each $\lambda \in C$ the function

$$\widetilde{Z}(t;t_1;\lambda) \equiv \frac{\psi_1(t) + \lambda\psi_2(t)}{\phi_1(t) + \lambda\phi_2(t)}$$

is a solution of Eq. (1.1) in such a domain of variation of t $(\geqslant t_1)$, in which $\phi_1(t) + \lambda\phi_2(t) \neq 0$. Let us divide the numerator and denominator of the fraction $\widetilde{Z}(t;t_1;\lambda)$ on $\phi_1(t)$. Taking into account (1.3) we will get

$$\widetilde{Z}(t;t_1;\lambda) = \frac{z_1(t) + \lambda z_*(t_1)\exp\left\{ \int_{t_1}^{t} a(\tau)[z_*(\tau) - z_1(\tau)]d\tau \right\}}{1 + \lambda\exp\left\{ \int_{t_1}^{t} a(\tau)[z_*(\tau) - z_1(\tau)]d\tau \right\}}, \quad t \geqslant t_1 \geqslant t_0.$$

In view of the equality $\widetilde{Z}(t_1;t_1;\lambda) = \frac{z_1(t_1)+\lambda z_*(t_1)}{1+\lambda}$ from here we will have

$$z_2(t) = \frac{z_1(t) + \lambda_2 z_*(t_1)\exp\left\{\int_{t_1}^{t} a(\tau)[z_*(\tau) - z_1(\tau)]d\tau\right\}}{1 + \lambda_2\exp\left\{\int_{t_1}^{t} a(\tau)[z_*(\tau) - z_1(\tau)]d\tau\right\}}, \quad t \geqslant t_1 \geqslant t_0.$$

where $\lambda_2 \equiv \frac{z_2(t_1)-z_1(t_1)}{z_*(t_1)-z_1(t_1)}$. Then

$$a(t)[z_2(t) - z_1(t)] = \frac{\lambda_2 a(t)[z_*(t) - z_1(t)]\exp\left\{\int_{t_1}^{t} a(\tau)[z_*(\tau) - z_1(\tau)]d\tau\right\}}{1 + \lambda_2\exp\left\{\int_{t_1}^{t} a(\tau)[z_*(\tau) - z_1(\tau)]d\tau\right\}}, \quad t \geqslant t_1 \geqslant t_0.$$

Integrating this equality from t_1 to t we will get

$$\int_{t_1}^{t} a(\tau)[z_2(\tau) - z_1(\tau)]d\tau = \ln\left[\frac{z_*(t_1) - z_2(t_1)}{z_*(t_1) - z_1(t_1)}\right]$$
$$+ \ln\left[1 + \lambda_2\exp\left\{\int_{t_1}^{t} a(\tau)[z_*(\tau) - z_1(\tau)]d\tau\right\}\right], \quad t \geqslant t_1 \geqslant t_0.$$

Passing to the limit in this equality when $t \to +\infty$ and taking into account (3.14) we will get

$$\int_{t_1}^{+\infty} a(\tau)[z_2(\tau) - z_1(\tau)]d\tau = \ln\left[\frac{z_*(t_1) - z_2(t_1)}{z_*(t_1) - z_1(t_1)}\right], \quad t_1 \geqslant t_0,$$

which proves (3.18). The theorem is proved. □

COROLLARY 1. *If $\nu_{z_*}(t_0) = \infty$ for some regular solution $z_*(t)$ of Eq. (1.1) then $z_*(t)$ is the unique extremal solution of Eq. (1.1). Eq. (1.1) has normal solutions and for each normal solution of Eq. (1.1) and for all $t \geqslant t_0$ the integrals $\nu_z(t)$ are convergent and for arbitrary normal solutions $z_0(t)$, $z_1(t)$, $z_2(t)$ of Eq. (1.1) and for $z_*(t)$ the relations (3.15)–(3.18) are valid.*

Proof. From the condition of the corollary it follows that

$$|\mu_{z_0}(t_0;t)| > 1, \quad t \geqslant T, \tag{3.25}$$

for some $T > t_0$. As a continuously differentiable function the curve $\mu_{z_*}(t)$, $t \in [t_0;T]$ is rectifiable (has finite length). So its plane measure is null. Therefore the set of values of $\mu_{z_*}(t_0;t)$ on $[t_0;T]$ cannot cover the disk $|z| < 1$. From here and from (3.25) it follows that for some $u_0 \neq 0$ with $|u_0| < 1$ the inequality $\mu_{z_*}(t_0;t) \neq 0$, $t \geqslant t_0$, holds.

By (3.4) from here it follows that the solution $z_0(t)$ of Eq. (1.1) with $z_0(t_0) = z_*(t_0) - \frac{1}{u_0}$ is regular. Then by (3.10) we have:

$$\mu_{z_0}(t_1;t) = \frac{\mu_{z_*}(t_1;t)}{1+\lambda_*(t_1)\mu_{z_*}(t_1;t)}, \quad t_1, t \geqslant t_0,$$

where $\lambda_*(t_1) \equiv z_0(t_1) - z_*(t_1) \neq 0$. From here and from the condition of the corollary we have that the integral $\nu_{z_0}(t_1)$ is convergent and $\nu_{z_0}(t_1) = \frac{1}{\lambda_*(t_1)} \neq 0$, $t_1 \geqslant t_0$. So for $z_0(t)$ all conditions of Theorem 6 are fulfilled. Therefore for all normal solutions $z(t)$ and for all $t \geqslant t_0$ the integrals $\nu_z(t)$ are convergent and $\nu_z(t) \neq 0$, $t \geqslant t_0$ and for arbitrary solutions $z_0(t)$, $z_1(t)$, $z_2(t)$ of Eq. (1.1) and for $z_*(t)$ the relations (3.15)–(3.18) are satisfied. The corollary is proved. □

In the sequel we will assume that for some regular solution $z_0(t)$ of Eq. (1.1) the integral $\nu_{z_0}(t_0)$ converges and $\nu_{z_0}(t) \neq 0$, $t \geqslant t_0$. By (3.16) for normal solutions $z_1(t)$, $z_2(t)$ and an extremal solution $z_*(t)$ of Eq. (1.1) the following equality holds

$$\frac{z_*(t) - z_2(t)}{z_*(t) - z_1(t)} = \frac{\nu_{z_2}(t)}{\nu_{z_1}(t)}, \quad t \geqslant t_0. \tag{3.26}$$

Tending in this equality t to $+\infty$ and taking into account (3.18) we will get

$$\lim_{t\to+\infty} \frac{\nu_{z_2}(t)}{\nu_{z_1}(t)} = 1. \tag{3.27}$$

By (3.15) we have

$$\frac{z_*(t) - z_2(t)}{z_*(t) - z_1(t)} = \nu_{z_1}(t)[z_1(t) - z_2(t)], \quad t \geqslant t_0. \tag{3.28}$$

By (3.18) the following equality takes place

$$\lim_{t\to+\infty} \frac{z_*(t) - z_2(t)}{z_*(t) - z_1(t)} = 1.$$

Then

$$\lim_{t\to+\infty} \frac{z_1(t) - z_2(t)}{z_*(t) - z_1(t)} = \lim_{t\to+\infty} \frac{z_1 - z_*(t) + z_*(t) - z_2(t)}{z_*(t) - z_1(t)} = -1 + \lim_{t\to+\infty} \frac{z_*(t) - z_2(t)}{z_*(t) - z_1(t)} = 0.$$

From here and from (3.28) it follows

$$\lim_{t\to+\infty} \nu_{z_1}(t)[z_1(t) - z_2(t)] = 0.$$

From (3.27) it follows that in the last equality the function $\nu_{z_1}(t)$ can be replaced by $\nu_{z_0}(t)$, where $z_0(t)$ is an arbitrary normal solution of Eq. (1.1). Therefore if for some normal solution $z_0(t)$ of Eq. (1.1) the inequality $|\nu_{z_0}(t)| \geqslant \varepsilon > 0$, $t \geqslant t_0$ holds, then all normal solutions of Eq. (1.1) are asymptotically close, i. e. for arbitrary normal solutions $z_1(t)$ and $z_2(t)$ of Eq. (1.1) the following equality takes place $\lim\limits_{t\to+\infty}[z_1(t)-z_2(t)]=0$. By (3.6) and (3.8) for normal solutions $z_1(t)$ and $z_2(t)$ of Eq. (1.1) the following equality is valid

$$\frac{[z_1(t)-z_2(t)]'}{z_1(t)-z_2(t)} = -\frac{\nu'_{z_1,z_2}(t)}{\nu_{z_1,z_2}(t)},\ \ t \geqslant t_0.$$

Integrating this equality from t_0 to t we will get

$$\ln\left[\frac{z_1(t)-z_2(t)}{z_1(t_0)-z_2(t_0)}\right] = -\ln\left[\frac{\nu_{z_1,z_2}(t)}{\nu_{z_1,z_2}(t_0)}\right],\ \ t \geqslant t_0.$$

From here we have

$$\frac{z_1(t)-z_2(t)}{z_1(t_0)-z_2(t_0)} \cdot \frac{\nu_{z_1,z_2}(t)}{\nu_{z_1,z_2}(t_0)} = 1,\ \ t \geqslant t_0. \tag{3.29}$$

Since $z_1(t)$ and $z_2(t)$ are normal, by (3.11) from the assertion B) of Theorem 6 it follows that $\inf\limits_{t\geqslant t_0}|1+\lambda_{1,2}\mu_{z_2}(t_0;t)|>0$. Therefore $\nu_{z_1,z_2}(t)\to\infty$ for $t\to\infty$ if and only if $e^{-1}_{z_2}(t_0;t)\to\infty$ for $t\to+\infty$. By virtue of (3.29) from here we come to the following criterion of asymptotic closeness of normal solutions of Eq. (1.1).

THEOREM 7. *All solutions of Eq. (1.1) are asymptotically close if and only if the relation*

$$\mathrm{Re}\int\limits_{t_0}^{+\infty}[2a(\tau)z_0(\tau)+b(\tau)]d\tau = +\infty.$$

holds for some normal solution $z_0(t)$ of Eq. (1.1).

EXAMPLE 3.2. Assume that $a(t)\equiv a_0\neq 0$, $b(t)\equiv b_0$, $c(t)\equiv c_0$, $t\geqslant t_0$, $\mathrm{Re}\sqrt{b_0^2-4a_0c_0}>0$. Then $z_\pm \equiv \frac{-b_0\pm\sqrt{b_0^2-4a_0c_0}}{2a_0}$ are regular solutions of Eq. (1.1). One can readily see that the integral $\nu_{z_+}(t_0)$ is convergent, $\nu_{z_+}(t)\neq 0$, $t\geqslant t_0$, and the integral $\nu_{z_-}(t_0)$ is divergent. Then by Theorem 6, z_+ is a normal solution and z_- is the unique extremal solution of Eq. (1.1). Note that $z'_+\equiv 0$, $\nu'_{z_+}(t)\equiv 0$. Then using the relations (3.7) and (3.8) we obtain the well-known Viete formulae

$$a_0(z_-+z_+) = -b_0;$$

$$a_0z_-z_+ = c_0.$$

Since $2a_0z_++b_0=\sqrt{b_0^2-4a_0c_0}$, and $\mathrm{Re}\sqrt{b_0^2-4a_0c_0}>0$, we have $\mathrm{Re}\int_{t_0}^{+\infty}[2a_0z_++b_0]d\tau=+\infty$. Therefore by Theorem 7, all normal solutions of Eq. (1.1) are asymptotically close under the above assumptions.

4. The asymptotic behavior of solutions of the system of two first order linear ordinary differential equations

Let $a_{jk}(t)$ $(j,k=1,2)$ be complex-valued continuous functions on $[t_0;+\infty)$. Consider the system

$$\begin{cases} \phi'(t) = a_{11}(t)\phi(t) + a_{12}(t)\psi(t); \\ \psi'(t) = a_{21}(t)\phi(t) + a_{22}(t)\psi(t), \end{cases} \tag{4.1}$$

$t \geqslant t_0$. Let $t_1 \geqslant t_0$.

DEFINITION 4. A solution $(\phi(t),\psi(t))$ of the system (4.1) is said to be t_1-regular, if $\phi(t) \neq 0$, $t \geqslant t_1$.

REMARK 3. From Remark 2 and (4.1)–(4.3) is seen that if the system (4.1) has a t_1-regular solution $(\phi_0(t),\psi_0(t))$, then it has at least one t_1-regular solution, linearly independent of $(\phi_0(t),\psi_0(t))$.

Each T-regular solution of the system (4.1) for any $T \geqslant t_0$ we will just call regular.

DEFINITION 5. The system (4.1) is said to be regular if it has at least one regular solution.

DEFINITION 6. The system (4.1) is said to be strongly regular, if its each nontrivial solution is regular.

The substitution $\psi(t) = z(t)\phi(t)$, $t \geqslant t_0$, in (4.1) leads to the system

$$\begin{cases} \phi'(t) = [a_{11}(t) + a_{12}(t)z(t)]\phi(t); \\ (z'(t) + a_{12}(t)z^2(t) + B(t)z(t) - a_{21}(t))\phi(t) = 0, \end{cases}$$

where $B(t) \equiv a_{11}(t) - a_{22}(t)$, $t \geqslant t_0$. Therefore if $z_0(t)$ is a t_1-regular solution of the Riccati equation

$$z'(t) + a_{12}(t)z^2(t) + B(t)z(t) - a_{21}(t) = 0, \quad t \geqslant t_0. \tag{4.2}$$

then the functions

$$\phi_0(t) \equiv \lambda_0 \exp\left\{\int\limits_{t_1}^{t} [a_{11}(\tau) + a_{12}(\tau)z_0(\tau)]d\tau\right\}, \quad \lambda_0 \neq 0, \quad \psi_0(t) \equiv z_0(t)\phi_0(t), \tag{4.3}$$

$t \geqslant t_0$, form the solution $(\phi_0(t),\psi_0(t))$ for the system (4.1) on $[t_1;+\infty)$.

DEFINITION 7. The regular system (4.1) is called irreconcilable if for its arbitrary two linearly independent regular solutions $(\phi_j(t),\psi_j(t))$, $j=1,2$, the functions $\frac{\phi_1(t)}{\phi_2(t)}$ and $\frac{\phi_2(t)}{\phi_1(t)}$ are unbounded on $[T;+\infty)$, where $T \geqslant t_0$ such that $\phi_j(t) \neq 0$, $t \geqslant T$, $j=1,2$.

DEFINITION 8. The regular system (4.1) is called normal if for its arbitrary two linearly independent regular solutions $(\phi_j(t),\psi_j(t))$, $j=1,2$, the functions $\frac{\phi_1(t)}{\phi_2(t)}$ and $\frac{\phi_2(t)}{\phi_1(t)}$ are bounded on $[T;+\infty)$, where $T\geqslant t_0$ such that $\phi_j(t)\neq 0$, $t\geqslant T$, $j=1,2$.

DEFINITION 9. The regular system (4.1) is called extremal if it has a regular solution $(\phi_*(t),\psi_*(t))$ such that for all regular solutions $(\phi(t),\psi(t))$ of the system (4.1) linearly independent of $(\phi_*(t),\psi_*(t))$, the equality $\lim\limits_{t\to+\infty}\phi_*(t)/\phi(t)=0$ is fulfilled.

The unique (up to arbitrary multiplier) solution $(\phi_*(t),\psi_*(t))$, defined above, we will call the minimal solution of the system (4.1).

DEFINITION 10. The regular system (4.1) is called super extremal if it is neither normal, nor extremal, nor irreconcilable.

By (4.1)–(4.3) from (3.12) and from Theorem 5 it follows that the system (4.1) is normal if and only if Eq. (4.2) is regular and has no t_1-extremal solution for all $t_1\geqslant t_0$, and from (3.12)–(3.14) it follows that the system (4.1) is irreconcilable if and only if Eq. (4.2) is regular and all its t_1-regular solutions are t_1-extremal for all $t_1\geqslant t_0$, the system (4.1) is extremal if and only if it has unique t_1-extremal solution for some $t_1\geqslant t_0$. therefore the system (4.1) is super extremal if and only if Eq. (4.2) has at least two different extremal solutions and at least one normal solution On the basis of Example 3.1 from here we conclude that there exist normal, irreconcilable, extremal and super extremal systems and only these kinds of the system (4.1).

REMARK 4. Each extremal system (4.1) is strongly regular. Indeed, let $(\phi_*(t),$ $\psi_*(t))$ be the minimal solution of the system (4.1). According to Remark 3 the system (4.1) has a regular solution $(\phi_1(t),\psi_1(t))$, linearly independent of $(\phi_*(t),\psi_*(t))$. Let $(\phi(t),\psi(t))$ be arbitrary nontrivial solution of the system (4.1) linearly independent of $(\phi_*(t),\psi_*(t))$. Then

$$\phi(t)=\phi_1(t)\left[c_1+c_2\frac{\phi_*(t)}{\phi_1(t)}\right],\quad t\geqslant t_1, \tag{4.4}$$

for some $t_1\geqslant t_0$, where c_1 and c_2 are some constants and $c_1\neq 0$. Since the system (4.1) is extremal we have

$$\lim_{t\to+\infty}\frac{\phi_*(t)}{\phi_1(t)}=0.$$

From here and from (4.7) it follows that $(\phi(t),\psi(t))$ is regular. therefore the system (4.1) is strongly regular.

REMARK 5. If $a_{jk}(t)$, $j,k=1,2$, are real valued and the system (4.1) is oscillatory (definition of oscillatory system (4.1) see in [9]) then Eq (4.2) has no real valued t_1-regular solution for all $t_1\geqslant t_0$. But by virtue of Lemma 2.1 of work [9] it follows that all solutions $z(t)$ of Eq. (4.2) with $\operatorname{Im} z(t_0)\neq 0$ are regular. Therefore all solutions of Eq. (4.2) are normal. By (4.1)–(4.3) from here and from Theorem 5 it follows that each oscillatory system (4.1) with real valued coefficients is normal.

REMARK 6. If $a_{12}(t)$, $B(t)$ and $a_{21}(t)$ are real valued then from Lemma 2.1 of work [9] and from the results of work [4] it follows that the system (4.1) is or else normal or else extremal or else super extremal, in particular the system (4.1) with real valued $a_{jk}(t),\ j,k=1,2,$ always is or else normal or else extremal or else super extremal.

For arbitrary continuous on $[t_0;+\infty)$ function $u(t)$ denote:

$$J_u(t_1;t) \equiv \left\{\int_{t_1}^{t} u(\tau)d\tau\right\},\ \ \widetilde{v}_u(t) \equiv \int_{t}^{+\infty} a_{12}(\tau)\exp\left\{-\int_{t}^{\tau}[2a_{12}(s)u(s)+B(s)]ds\right\}d\tau,$$

$t_1\,,\,t \geqslant t_0\,.$

THEOREM 8. *Let the system (4.1) has a t_1-regular solution $(\phi_0(t),\psi_0(t))$ for some $t_1 \geqslant t_0$ such that the integral $\int_{t_1}^{+\infty}\frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_0^2(\tau)}d\tau$ be convergent (conditionally), where $S(t)\equiv a_{11}(t)+a_{22}(t),\ t\geqslant t_0$, and for all $t\geqslant t_1$, $\int_{t}^{+\infty}\frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_0^2(\tau)}d\tau \neq 0$. Then the system (4.1) is extremal and:*

a) for its minimal solution $(\phi_*(t),\psi_*(t))$ the equality $\int_{t_1}^{+\infty}\frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_*^2(\tau)d\tau} = \infty$; holds;

b) for each nontrivial solution $(\phi(t),\psi(t))$ of the system (4.1), linearly independent of $(\phi_*(t),\psi_*(t))$, the integral $\int_{T}^{+\infty}\frac{a_{12}(\tau)J_S(\tau)}{\phi^2(\tau)d\tau}$ converges (conditionally), where $T=T(\phi;\psi)\geqslant t_0$ such that $\phi(t)\neq 0,\ t\geqslant T$; if in addition $\int_{t_1}^{+\infty}|a_{12}(\tau)J_S(\tau)|d\tau=+\infty$, then

$$\limsup_{t\to+\infty}|\phi(t)| = +\infty; \tag{4.5}$$

c) for arbitrary solutions $(\phi_j(t);\psi_j(t)),\ j=1,2,$ of the system (4.1),linearly independent of $(\phi_*(t);\psi_*(t))$, there exists a finite limit

$$\lim_{t\to+\infty}\frac{\phi_1(t)}{\phi_2(t)}(\neq 0). \tag{4.6}$$

Proof. Denote: $z_0(t)\equiv\frac{\psi_0(t)}{\phi_0(t)}$, $t\geqslant t_1$. By (4.3) we have $\widetilde{v}_{z_0}(t_1) = \int_{t_1}^{+\infty}\frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_0^2(\tau)}d\tau$. From here and from conditions of the theorem it follows that the integral $\widetilde{v}_{z_0}(t_1)$ is convergent. Then on the strength of Theorem 6 Eq. (4.2) has the unique t_1-extremal solution $z_*(t)$. By virtue of (4.3) the functions

$$\phi_*(t)\equiv\exp\left\{\int_{t_1}^{t}[a_{11}(\tau)+a_{12}(\tau)z_*(\tau)]d\tau\right\},\ \ \psi_*(t)\equiv z_*(t)\phi_0(t),\ \ t\geqslant t_1,$$

form the t_1-regular solution $(\phi_*(t), \psi_*(t))$ of the system (4.1) on $[t_1;+\infty)$, which can be continued on $[t_0;+\infty)$ as a solution of the system (4.1). Then by virtue of (3.13) from the equality $\widetilde{\nu}_{z_*}(t_1) = \int\limits_{t_1}^{+\infty} \frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_*^2(\tau)}d\tau$ and from the conditions of the theorem it follows that $(\phi_0(t), \psi_0(t))$ and $(\phi_*(t), \psi_*(t))$ are linearly independent. Therefore for arbitrary solution $(\phi(t), \psi(t))$ of the system (4.1), linearly independent of $(\phi_*(t), \psi_*(t))$ the equality

$$\phi(t) = \phi_0(t)\left[c_0 + c_*\frac{\phi_*(t)}{\phi_0(t)}\right], \quad t \geqslant t_1, \tag{4.7}$$

holds, where c_0 and c_* are some constants and $c_0 \neq 0$. Due to Theorem 6 from the equality $\widetilde{\nu}_{z_0}(t_1) = \int\limits_{t_1}^{+\infty} \frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_0^2(\tau)}d\tau$ and from the conditions of the theorem it follows that $z_0(t)$ is a t_1-normal solution of Eq. (4.2). Then according to (3.14) from the equality

$$\frac{\phi_*(t)}{\phi_0(t)} = \frac{1}{\phi_0(t)}\exp\left\{\int\limits_{t_1}^{t} a_{12}(\tau)\big[z_0(\tau) - z_*(\tau)\big]d\tau\right\}, \quad t \geqslant t_1,$$

it follows that

$$\lim_{t\to+\infty}\frac{\phi_*(t)}{\phi_0(t)} = 0, \tag{4.8}$$

From here and from (4.7) it follows that $\phi(t) \neq 0$, $t \geqslant t_2$, for some $t_2 \geqslant t_1$. Taking into account (4.7) and (4.8) from here we will get:

$$\lim_{t\to+\infty}\frac{\phi_*(t)}{\phi(t)} = \lim_{t\to+\infty}\frac{\phi_*(t)}{\phi_0(t)}\frac{1}{\left[c_0 + c_*\lim\limits_{t\to+\infty}\frac{\phi_*(t)}{\phi_0(t)}\right]} = 0.$$

From here it follows the uniqueness (up to arbitrary constant multiplier) of the solution $(\phi_*(t), \psi_*(t))$. Therefore the system (4.1) is extremal. The assertion a) immediately follows from the equality $\widetilde{\nu}_{z_*}(t_1) = \int\limits_{t_1}^{+\infty} \frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_*^2(\tau)}d\tau$. The assertion b) follows from Theorem 6 and from the equality $\widetilde{\nu}_{z}(T) = \int\limits_{T}^{+\infty} \frac{a_{12}(\tau)J_S(T;\tau)}{\phi^2(\tau)}d\tau$, where $z(t) \equiv \frac{\psi(t)}{\phi(t)}$ is a T-normal solution of Eq. (4.2), and the equality (4.5) immediately follows from the convergence of the integral $\int\limits_{T}^{+\infty} \frac{a_{12}(\tau)J_S(T;\tau)}{\phi^2(\tau)}d\tau$ and from the equality

$$\int\limits_{t_0}^{+\infty}|a_{12}(\tau)J_S(T;\tau)|d\tau = J_{-S}(t_0;T)\int\limits_{t_0}^{+\infty}|a_{12}(\tau)J_S(t_0;\tau)|d\tau = +\infty.$$

Let us prove c). Let $z_j(t) \equiv \frac{\psi_j(t)}{\phi_j(t)}$, $t \geqslant T$, $j = 1,2$, where $T \geqslant t_0$ such that $\phi_j(t) \neq 0$, $t \geqslant T$, $j = 1,2.$. Since $(\phi_j(t), \psi_j(t))$ $(j = 1,2)$ are linearly independent of $(\phi_*(t), \psi_*(t))$, by (4.3) and Theorem 6 the functions $z_j(t)$ $(j = 1,2)$ are T-normal solutions of Eq. (4.2), and

$$\frac{\phi_1(t)}{\phi_2(t)} = \frac{\phi_1(T)}{\phi_2(T)} \exp\left\{\int_T^t a_{12}(\tau)\big[z_1(\tau) - z_2(\tau)\big]d\tau\right\}, \ t \geqslant T.$$

By (3.15) from here it follows (4.6). The theorem is proved. □

REMARK 7. A criterion for existence of a t_1-regular solution to system (4.1) in the case of real $a_{jk}(t)$ $(j,k = 1,2)$ is given in [9] (see [9, Theorem 4.2]).

COROLLARY 2. *If the system (4.1) has a t_1-regular solution $(\phi_*(t), \psi_*(t))$ for some $t_1 \geqslant t_0$ such that $\int\limits_{t_1}^{+\infty} \frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_*^2(\tau)d\tau} = \infty$, then the assertions of Theorem 8 are valid.*

Proof. Denote $z_*(t) \equiv \frac{\psi_*(t)}{\phi_*(t)}$, $t \geqslant t_1$. Then $\widetilde{\nu}_{z_*}(t_1) = \int\limits_{t_1}^{+\infty} \frac{a_{12}(\tau)J_S(t_1;\tau)}{\phi_*^2(\tau)d\tau} = \infty$. By virtue of Corollary 1 from here it follows that Eq. (4.2) has a t_1-normal solution $z_0(t)$ such that $\widetilde{\nu}_{z_0}(t_1)$ converges and $\widetilde{\nu}_{z_0}(t) \neq 0$ for all $t \geqslant t_0$. Then for the solution $(\phi_0(t), \psi_0(t))$ of the system (4.1), where

$$\phi_0(t) \equiv \exp\left\{\int_{t_1}^t [a_{11}(\tau) + a_{12}(\tau)z_0(\tau)]d\tau\right\}, \ \ \psi_0(t) \equiv z_0(t)\phi_0(t), \ \ t \geqslant t_0,$$

the integrals $I_{\phi_0}(t) \equiv \int\limits_t^{+\infty} \frac{a_{12}(\tau)J_S(t;\tau)}{\phi_0^2(\tau)d\tau} = \widetilde{\nu}_{z_0}(t)$ are convergent and do not vanish for all $t \geqslant t_1$. Thus all conditions of Theorem 8 are fulfilled. The corollary is proved. □

EXAMPLE 4.1. Consider the system

$$\begin{cases} \phi'(t) = a(t)\psi(t); \\ \psi'(t) = -a(t)\phi(t), \ \ t \geqslant t_0, \end{cases} \tag{4.9}$$

where $a(t)$ is a continuous function on $[t_0;+\infty)$. One can readily check that one of the solutions of this system is $(\phi(t), \psi(t))$, where

$$\phi(t) \equiv \sin\left(\int_{t_0}^t a(\tau)d\tau\right), \ \ \psi(t) \equiv \cos\left(\int_{t_0}^t a(\tau)d\tau\right), \ \ t \geqslant t_0.$$

Suppose $a(t) = ia_0(t)$, $a_0(t) \geqslant 0$, $t \geqslant t_0$, and $\int\limits_{t_0}^{+\infty} a_0(\tau)d\tau = +\infty$. Then it is not difficult to verify that $(\phi(t), \psi(t))$ is a t_1-regular solution to the system (4.9) for any $t_1 > t_0$, the integrals $I_\phi(t) \equiv \int\limits_t^{+\infty} \frac{a(\tau)}{\phi^2(\tau)} d\tau$ are convergent and $I_\phi(t) \neq 0$, $t \geqslant t_1$. Therefore for the system (4.9) the conditions of Theorem 8 are fulfilled. Suppose $\sin\Big(\int\limits_{t_0}^t a(\tau)d\tau\Big) \neq 0$, $t \geqslant t_1 > t_0$, $\Big|\mathrm{Im}\int\limits_{t_0}^t a(\tau)d\tau\Big| \leqslant M = const$, $t \geqslant t_1$, and $\int\limits_{t_0}^{+\infty} a(\tau)d\tau = \infty$. Then $(\phi(t), \psi(t))$ is a t_1-regular solution to the system (4.9) and since $|\phi(t)| \leqslant M_1$, $t \geqslant t_1$ for some $M_1 = const$ we have $\int\limits_{t_1}^{+\infty} \frac{a(\tau)}{\phi^2(\tau)} d\tau = \infty$. Therefore all conditions of Corollary 2 for the system (4.9) are satisfied.

REMARK 8. Taking into account the remark 1 we conclude that Corollary 2 is a generalization and a supplement of the assertions (i) and (ii) of Theorem 6.4 from the book [17] (see [17], p. 355), as well as a generalization and a supplement of Theorem 3.1 from [18].

EXAMPLE 4.2. One can readily check that the function $z_0(t) \equiv \sin t + i\cos t$, $t \geqslant t_0$, is a regular solution to the equation

$$z'(t) + [\sin t - i\cos t]z^2(t) + iz(t) - \sin t - i\cos t = 0, \ \ t \geqslant t_0,$$

So it is not difficult to verify that for $a_{12}(t) = \sin t - i\cos t$, $a_{21}(t) = \sigma(t) = \sin t + {}$ $+ i\cos t$, $t \geqslant t_0$, $B(t) \equiv i$ the conditions of Corollary 2 are fulfilled.

EXAMPLE 4.3. Let $a_{21}(t) \equiv 0$. Then $z_0(t) \equiv 0$ is a regular solution of Eq. (4.1) and by (4.3) the functions $\phi_*(t) \equiv \exp\Big\{\int\limits_{t_0}^t a_{11}(\tau)d\tau\Big\}$, $\psi_*(t) \equiv 0$ form the regular solution $(\phi_*(t), \psi_*(t))$ of the system (4.1). Therefore by Corollary 2 if

$$\int\limits_{t_0}^{+\infty} a_{12}(\tau)\exp\Big\{\int\limits_{t_0}^t B(\tau)d\tau\Big\} = \infty,$$

then the system (4.1) is extremal and the assertions of Theorem 8 are valid.

Let $a_{12}(t) \neq 0$, $t \geqslant t_0$ and let $A_{12}(t)$ and $B(t) - \frac{a'_{12}(t)}{a_{12}(t)}$ be continuously differentiable. Then we can rewrite Eq. (4.2) in the form

$$[a_{12}(t)z(t)]' + [a_{12}(t)z(t)]^2 + \Big\{B(t) - \frac{a'_{12}(t)}{a_{12}(t)}\Big\}[a_{12}(t)z(t)] - a_{12}(t)a_{21}(t) = 0, \ \ t \geqslant t_0. \tag{4.10}$$

Set:

$$v(t) \equiv a_{12}(t)z(t) + \frac{1}{2}\Big\{B(t) - \frac{a'_{12}(t)}{a_{12}(t)}\Big\}, \ \ t \geqslant t_0. \tag{4.11}$$

Then from (4.10) we will come to the following Riccati equation with respect to $v(t)$

$$v'(t) + v^2(t) + Q(t) = 0, \ \ t \geqslant t_0, \tag{4.12}$$

where $Q(t) \equiv -\frac{1}{2}\Big\{B(t) - \frac{a_{12}'(t)}{a_{12}(t)}\Big\}' - \frac{1}{4}\Big\{B(t) - \frac{a_{12}'(t)}{a_{12}(t)}\Big\}^2 - a_{12}(t)a_{21}(t)$, $t \geqslant t_0$.

Consider the equation

$$\phi''(t) + Q(t)\phi(t) = 0, \tag{4.13}$$

COROLLARY 3. *Let* $\operatorname{Im} Q(t) \equiv 0$. *Then the following assertions are valid:*

1) if Eq. (4.13) is oscillatory then the system (4.1) is normal;

2) if Eq. (4.13) is non oscillatory then the system (4.1) is extremal and the assertions of Theorem 8 are valid

Proof. If Eq. (4.13) is oscillatory then Eq. (4.12) has no real regular solution and its all complex solutions are regular (see for example [9]). Therefore all regular solutions of Eq. (4.12) are normal. By (4.11) from here it follows that the all regular solutions of Eq. (4.2) are normal. Therefore the system (4.1) is normal. Let Eq. (4.13) be non oscillatory. Then Eq. (4.12) has unique real valued extremal solution (see [8, Lemma 2.1]). Therefore Eq. (4.12) has unique extremal solution (since all complex solutions of Eq. (4.12) are normal). By (4.11) from here it follows that Eq. (4.2) has unique extremal solution. Hence the system (4.1) is extremal and all the conditions of Theorem 8 are fulfilled. The corollary is proved. □

Denote: $a_{12,1}(t) \equiv \operatorname{Re} a_{12}(t)$, $a_{12,2}(t) \equiv \operatorname{Im} a_{12}(t)$, $B_2(t) \equiv \operatorname{Im} B(t)$, $a_{21,1}(t) \equiv \operatorname{Re} a_{21}(t)$, $a_{21,2}(t) \equiv \operatorname{Im} a_{21}(t)$,

$$d_k(t) \equiv B_2(t) - 4(-1)^k a_{12,k}(t)a_{21,k}(t), \;\; k = 1,2m \;\; t \geqslant t_0.$$

$$\mu_u(t) \equiv \int\limits_{t_0}^{t} a_{12}(\tau)\exp\Big\{-\int\limits_{t_0}^{\tau}[2a_{12}(\xi)u(\xi) + B(\xi)]d\xi\Big\}, \;\; t \geqslant t_0,$$

where $u(t)$ is an arbitrary continuous function on $[t_0;+\infty)$.

THEOREM 9. *Let* $a_{12,k}(t) \geqslant 0$, *and let* $d_k(t) \leqslant 0$, *if* $a_{12,k}(t) \neq 0$ *and* $(-1)^k a_{21,k}(t) \leqslant 0$ *if* $a_{12,k}(t) = 0$, $k = 1,2$, $t \geqslant t_0$; *moreover suppose that if* $a_{12,1}(t)a_{12,2}(t) = 0$, *then* $B_2(t) = 0$. *Then for all solutions* $(\phi_j(t), \psi_j(t))$, $j = 1,2$, *of the system (4.1) with* $\operatorname{Re}\frac{\psi_j(t_0)}{\phi_j(t_0)} > 0$, $\operatorname{Im}\frac{\psi_j(t_0)}{\phi_j(t_0)} < 0$, $j = 1,2$, *the functions* $\frac{\phi_1(t)}{\phi_2(t)}$, $I_{\phi_1}(t) \equiv \int\limits_{t_0}^{t}\frac{a_{12}(\tau)J_S(t_0;\tau)}{\phi_1^2(\tau)}d\tau$, $t \geqslant t_0$, *are bounded. The system (4.1) is or else normal or else extremal.*

Proof. Due to (4.3) and Theorem 1 from the conditions of theorem it follows that the functions $z_j(t) \equiv \frac{\psi_j(t)}{\phi_j(t)}$, $t \geqslant t_0$, $j = 1,2$, are t_0-normal solutions of Eq. (4.2). Then by (4.3) we have:

$$\frac{\phi_1(t)}{\phi_2(t)} = \frac{\phi_1(t_0)}{\phi_2(t_0)}\exp\Big\{\int\limits_{t_0}^{t} a_{12}(\tau)\big[z_1(\tau) - z_2(\tau)\big]d\tau\Big\}, \;\; t \geqslant t_0.$$

On the strength of Theorem 5 from here it follows that the function $\frac{\phi_1(t)}{\phi_2(t)}$ is bounded. By virtue of Theorem 3 from t_0-normality of $z_1(t)$ it follows the boundedness of the function $\mu_{z_1}(t)$. By (4.3) we have $\mu_{z_1}(t) = I_{\phi_1}(t)$, $t \geqslant t_0$. Therefore the function $I_{\phi_1}(t)$ is bounded. Let $z_1(t_0) \neq z_2(t_0)$. Then from the boundedness of $\frac{\phi_1(t)}{\phi_2(t)}$ it follows that the system (4.1) is or else normal or else extremal. The theorem is proved. □

Denote:

$$R_k^{\pm}(t) \equiv \begin{cases} (-1)^{k+1}B_2(t) \pm \sqrt{d_k(t)}, & \text{if } a_{12,k}(t) \neq 0; \\ 0, & \text{if } a_{12,k}(t) = 0, \end{cases} \quad t \geqslant t_0, \ \ k = 1,2.$$

Using Theorem 2 in place of Theorem 1 by analogy of Theorem 9 can be proved

THEOREM 10. *Let $a_{12,k}(t) \geqslant 0$, and let $d_k(t) \geqslant 0$ if $a_{12,k}(t) \neq 0$ and $(-1)^k a_{21,k}(t) \leqslant 0$ if $a_{12,k}(t) = 0$, $k = 1,2$, $t \geqslant t_0$ moreover assume that if $a_{12,1}(t)a_{12,2}(t) = 0$ then $B_2(t) = 0$. In addition assume that either $R_1^+(t) \leqslant 0$ and $R_2^-(t) \geqslant \varepsilon$ or $R_1^-(t) \geqslant \varepsilon$ and $R_2^+(t) \leqslant 0$, $t \geqslant t_0$ fir some $\varepsilon > 0$. Then for all solutions $(\phi_j(t), \psi_j(t))$, $j = 1,2$, of the system (4.1) with $\operatorname{Re}\frac{\psi_j(t_0)}{\phi_j(t_0)} > 0$, $\operatorname{Im}\frac{\psi_j(t_0)}{\phi_j(t_0)} < 0$, $j = 1,2$, the functions $\frac{\phi_1(t)}{\phi_2(t)}$ and $I_{\phi_1}(t) \equiv \int\limits_{t_0}^{t} \frac{a_{12}(\tau)J_S(t_0;\tau)}{\phi_1^2(\tau)}d\tau$, $t \geqslant t_0$, are bounded. The system (4.1) is or else normal or else extremal.*

THEOREM 11. *Let the conditions of Theorem 9 (Theorem 10), are fulfilled and let*

$$\int\limits_{t_0}^{+\infty}\left|a_{12}(\tau)\exp\left\{-\int\limits_{t_0}^{\tau} B(\xi)d\xi\right\}\right|d\tau < +\infty. \tag{4.13}$$

Then for each solution $(\phi(t), \psi(t))$ of the system (4.1) with $\operatorname{Re}\frac{\psi(t_0)}{\phi(t_0)} \geqslant 0$, $\operatorname{Im}\frac{\psi(t_0)}{\phi(t_0)} \leqslant 0$, ($\operatorname{Re}\frac{\psi(t_0)}{\phi(t_0)} > 0$, $\operatorname{Im}\frac{\psi(t_0)}{\phi(t_0)} < 0$) the inequality

$$\int\limits_{t_0}^{+\infty}\left|\frac{a_{12}(\tau)J_S(t_0;\tau)}{\phi^2(\tau)}\right|d\tau < +\infty \tag{4.14}$$

holds, and for arbitrary solutions $(\phi_j(t), \psi_j(t))$, $j = 1,2$, of the system (4.1) with $\operatorname{Re}\frac{\psi(t_0)}{\phi(t_0)} \geqslant 0$, $\operatorname{Im}\frac{\psi(t_0)}{\phi(t_0)} \leqslant 0$, ($\operatorname{Re}\frac{\psi(t_0)}{\phi(t_0)} > 0$, $\operatorname{Im}\frac{\psi(t_0)}{\phi(t_0)} < 0$) there exists finite limit $\lim\limits_{t\to+\infty}\frac{\phi_1(t)}{\phi_2(t)}$ $(\neq 0)$. The system (4.1) is or else normal or else extremal. In the last case the assertions A) and B) of Theorem 8 are valid for some $t_1 \geqslant t_0$.

Proof. Let $(\phi(t), \psi(t))$ be a solution of the system (4.1) with $\operatorname{Re}\frac{\psi(t_0)}{\phi(t_0)} \geqslant 0$, $\operatorname{Im}\frac{\psi(t_0)}{\phi(t_0)} \leqslant 0$, ($\operatorname{Re}\frac{\psi(t_0)}{\phi(t_0)} > 0$, $\operatorname{Im}\frac{\psi(t_0)}{\phi(t_0)} < 0$), and let $z(t)$ be a solution of Eq. (4.2) with $z(t_0) = \frac{\psi(t_0)}{\phi(t_0)}$. Then according to Theorem 9 (Theorem 10) $z(t)$ is a t_0-regular solution of

Eq. (4.2), and $\mathrm{Re}\, z(t) \geqslant 0$, $\mathrm{Im}\, z(t) \leqslant 0$ $(\mathrm{Re}\, z(t) > 0,\ \mathrm{Im}\, z(t) < 0)$ $t \geqslant t_0$. Therefore, $\mathrm{Re} \int\limits_{t_0}^{t} a_{12}(\tau) z(\tau) d\tau \geqslant 0$, $t \geqslant t_0$. Then taking into account (4.3) we will have:

$$\int\limits_{t_0}^{+\infty} \left| \frac{a_{12}(\tau) J_S(t_0;\tau)}{\phi^2(\tau)} \right| d\tau = \int\limits_{t_0}^{+\infty} \left| a_{12}(\tau) \exp\left\{ -\int\limits_{t_0}^{\tau} \big[2a_{12}(\xi) z(\xi) + B(\xi)\big] d\xi \right\} \right| d\tau$$

$$\leqslant \int\limits_{t_0}^{+\infty} \left| a_{12}(\tau) \exp\left\{ -\int\limits_{t_0}^{\tau} B(\xi) d\xi \right\} \right| d\tau.$$

From here and from (4.13) it follows (4.14). From the last inequality it follows that $\widetilde{\nu}_z(t)$ converges for all $t \geqslant t_0$. Two cases are possible:

1) $\widetilde{\nu}_z(t) \neq 0$, $t \geqslant T$, for some $T \geqslant t_0$;
2) $\widetilde{\nu}_z(t)$ has arbitrary large zeroes.

In the case 1) we have $\widetilde{\nu}_z(t) = \int\limits_{t}^{+\infty} \frac{a_{12}(\tau) J_S(t_0;\tau)}{\phi^2(\tau)} d\tau \neq 0$, $t \geqslant T$. Then by virtue of Theorem 8 the system (4.1) is extremal and the assertions of Theorem 8 hold for some $t_1 \geqslant t_0$. In the case 2) due to Theorem 8 Eq. (4.2) has no t_1-extremal solution for arbitrary $t_1 \geqslant t_0$. Then by (4.3) for arbitrary two regular solutions of the system (4.1) we have

$$\frac{\phi_1(t)}{\phi_2(t)} = \frac{\phi_1(t_1)}{\phi_2(t_1)} \exp\left\{ \int\limits_{t_1}^{t} a_{12}(\tau)[z_1(\tau) - z_2(\tau)] d\tau \right\}, \ \ \phi_j(t) \neq 0, \ \ t \geqslant t_1,$$

for some $t_1 \geqslant t_0$, where $z_j(t)$, $j = 1,2$, are two t_1-normal solutions of Eq. (4.2). From here and from Theorem 5 it follows that $\frac{\phi_1(t_1)}{\phi_2(t_1)}$ is bounded on $[t_1;+\infty)$. Therefore the system (4.1) is normal. The theorem is proved. □

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(Received December 18, 2017)

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