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EXISTENCE AND MULTIPLICITY SOLUTIONS FOR A NONLOCAL EQUATION OF KIRCHHOFF TYPE

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Abstract. In this paper, we study the nonlinear Kirchhoff equation

$$-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=g(x,u)\quad \text{in } \mathbb{R}^3,$$

where the potential V and the primitive of g are allowed to be sign-changing and g is local superlinear. Under some assumptions on V and g, we get at least one nontrivial solution and infinitely many nontrivial solutions for this equation. Recent results in the literature are generalized and significantly improved.

1. Introduction and Main Results

In this paper, we study the following equation

$$-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=g(x,u)\quad \text{in } \mathbb{R}^3. \tag{$\mathscr{K}$}$$

Problems related to ($\mathscr{K}$) model several physical and biological systems, where u describes a process, which depends on the average of itself, such as the population density, see e.g. [6] and the references therein.

Let us recall some recent results in the literature on the nonlinear Kirchhoff equation ($\mathscr{K}$). To our best knowledge, when the potential V is positive, Wu [15] was the first one who considering problem ($\mathscr{K}$). Four existence results for nontrivial solutions and a sequence of high energy solutions for problem ($\mathscr{K}$) were obtained by using a symmetric mountain pass theorem. Later the existence of infinitely many high energy solutions for problem ($\mathscr{K}$) with the subcritical nonlinearity which needs not to satisfy the usual Ambrosetti-Rabinowitz-type growth conditions was established by Liu and He [12]. Ye and Tang [16] obtained infinitely many large-energy and small-energy solutions for ($\mathscr{K}$), which unify and sharply improve the results of Wu [15]. The existence of nontrivial solutions for problem ($\mathscr{K}$) when the nonlinearity term is asymptotically

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linear or 4-superlinear at infinity was obtained by Cheng [5]. By some special techniques, Li and Wu [10] proved the existence and multiplicity of nontrivial solutions of problem ($\mathscr{K}$) with a widely class of superlinear nonlinearities, which improves and unites Theorems 1–4 in [15]. In [9], Huang and Liu obtained some existence and nonexistence results by using variational methods and also discussed the "energy doubling" property of nodal solutions. When the potential V may sign-changing, Chen and Liu [4] got at least one solution and infinitely many solutions by using local linking and fountain theorem, respectively. Zhang, Tang and Zhang [17] also considered the sign-changing case. A natural question is whether equation ($\mathscr{K}$) admits nontrivial solutions if the potential V and the primitive of g are both allowed to be sign-changing, and g is *local superlinear*. Our idea is mainly from [11, 3].

Firstly, we consider the potential is positive. The conditions list as follows:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) \geqslant a_0 > 0$, where a_0 is a positive constant.

(V2) There exists a constant $b_0 > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}(\{x \in \mathbb{R}^3 : |x-y| \leqslant b_0, V(x) \leqslant M\}) = 0, \quad \forall M > 0,$$

where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^3$.

(G1) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and there exist $c_1 > 0$, $4 < q < 2^* = 6$ such that

$$|g(x,u)| \leqslant c_1(|u| + |u|^{q-1}), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(G2) There exists an open subset Λ_1 of $\mathbb{R}^3$ with $0 < \operatorname{meas}(\Lambda_1) < \infty$ such that

$$\lim_{|u|\to\infty} \frac{G(x,u)}{|u|^4} = +\infty \text{ uniformly for } x \in \Lambda_1,$$

where $G(x,u) = \int_0^u g(x,s)ds$.

(G3) There exit $\eta \geqslant 4$, $0 \leqslant \varsigma < \frac{\eta-2}{2\eta}$, $k_0 > 1$ and $0 < \delta < 2$ such that

$$\frac{1}{\eta} g(x,u)u - G(x,u) \geqslant -\varsigma V(x)|u|^2 - m_1(x)\frac{|u|^2}{\ln(k_0+|u|)} - m_2(x)|u|^\delta - m_3(x),$$

for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$, where m_1, m_2, $m_3 : \mathbb{R}^3 \to \mathbb{R}$ are positive measurable functions such that $m_1 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $m_2 \in L^{\frac{2}{2-\delta}}(\mathbb{R}^3)$ and $m_3 \in L^1(\mathbb{R}^3)$.

(G4) There exist $0 < k_1 < a_0$ and $\zeta > 0$ such that

$$|g(x,u)| \leqslant k_1|u|, \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, |u| \leqslant \zeta.$$

(G5) $g(x,-u) = -g(x,u)$ for any $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

The main results are the following:

THEOREM 1. *Suppose that (V1),(V2),(G1)–(G4) are satisfied. Then equation ($\mathcal{K}$) has at least one nontrivial solution.*

THEOREM 2. *Suppose that (V1),(V2),(G1)–(G3) and (G5) are satisfied. Then equation ($\mathcal{K}$) has infinitely many nontrivial solutions.*

REMARK 1. Because of the local superlinear condition (G2), we show that our theorem 1 and 2 generalize the results in [5, 10, 12, 15, 16].

REMARK 2. It is worth mentioning that we give much weaker conditions to show the (PS) sequence or Cerami sequence of the corresponding functional is bounded. In [15], the authors use the following conditions.

(WU1) There exist $\mu > 4$ and $r > 0$ such that $\inf_{x\in\mathbb{R}^3,|u|=r} G(x,u) > 0$ and $\mu G(x,u) \leqslant g(x,u)u$ for all $x \in \mathbb{R}^3$ and $|u| \geqslant r$.

Liu and He [12] assume that the following conditions:

(LH) For a.e. $x \in \mathbb{R}^3$, $\forall(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, $s \leqslant t$, there holds $F(x,s) \leqslant F(x,t)$, where $F(x,s) := \frac{1}{4}g(x,s)s - G(x,s)$.

Actually, by (LH) and the assumptions of Liu and He, we obtain

$$\frac{1}{4}g(x,u) - G(x,u) \geqslant 0$$

for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$. This condition is also considered by Li and Wu [10]. Later Cheng [5] generalizes this condition to

(C) $4G(x,u) \leqslant g(x,u)u + \alpha u^2$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$, where α is a positive constant.

Very recently, Ye and Tang [16] use the local condition

(YT) There exists $L > 0$ such that $ug(x,u) - 4G(x,u) \geqslant 0$ for all $x \in \mathbb{R}^3$ and $|u| \geqslant L$.

It is easy to check that our condition (G3) is weaker than all the conditions mentioned above (see e.g. [3]).

Now, we consider the case that potential V may be sign-changing, we will use the following conditions instead of (V1), (G2) and (G3), respectively.

(V1') $V \in C(\mathbb{R}^3,\mathbb{R})$ is bounded from below.

(G2') $\lim_{|u|\to\infty} \frac{G(x,u)}{|u|^4} = +\infty$ for a.e. $x \in \mathbb{R}^3$.

(G3') There exist $\mu \geqslant 4$, $\iota_0 > 0$, $0 \leqslant \iota_1 < \frac{\mu-2}{2\mu}$ and $0 < \sigma < 2$ such that

$$\frac{1}{\mu}g(x,u)u - G(x,u) \geqslant -(\iota_0 + \iota_1 V(x))|u|^2 - l_1(x)|u|^\sigma - l_2(x), \forall(x,u) \in \mathbb{R}^3 \times \mathbb{R}, |u| \geqslant \nu,$$

where l_1, $l_2 : \mathbb{R}^3 \to \mathbb{R}$ are positive measurable functions such that $l_1 \in L^{\frac{2}{2-\sigma}}(\mathbb{R}^3)$ and $l_2 \in L^1(\mathbb{R}^3)$.

Our next results are the following:

THEOREM 3. *Suppose that (V1'),(V2),(G1),(G2'),(G3') and (G4) are satisfied. Then equation ($\mathscr{K}$) has at least one nontrivial solution.*

THEOREM 4. *Suppose that (V1'),(V2),(G1),(G2'),(G3') and (G5) are satisfied. Then equation ($\mathscr{K}$) has infinitely many nontrivial solutions.*

REMARK 3. Our theorem 3 and 4 generalize the results in [4, 17] because of the local superlinear condition (G2'),.

REMARK 4. Chen and Liu [4] use the following condition

(CL) There exists $h > 0$ such that $4G(x,u) \leqslant ug(x,u) + hu^2$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$

to show the boundedness of (PS) sequence.
Zhang et al. [17] use

(ZTZ1) there exists $\beta > 0$ such that $G(x,u) \leqslant \frac{1}{4} f(x,u)u + \beta u^2$, for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$;

(ZTZ2) there exists r_1 such that $G(x,u) \leqslant \frac{1}{4} g(x,u)u$ for all $x \in \mathbb{R}^3$ and $|u| \geqslant r_1$;

(ZTZ3) $G(x,u) \geqslant 0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$ and $F(x,s) \leqslant F(x,t)$, whenever $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $s \leqslant t$, where $F(x,u) = \frac{1}{4} g(x,u)u - G(x,u)$;

to show the boundedness of (PS) sequence. It is easy to see that our condition (G3') is weaker than the above conditions. Moreover, after a careful calculus, we notice that in [17], $\tilde{g}(x,u) := g(x,u) + V_0 u$ do not satisfy conditions (ZTZ2) and (ZTZ3).

The remainder of this paper is organized as follows. In Section 2, we derive a variational setting for problem ($\mathscr{K}$) and give some preliminary lemmas. In Section 3, we will prove Theorems 1 and 2. Finally, we will prove Theorems 3 and 4 in Section 4.

2. Preliminaries

We will present some definitions and lemmas that will be used in the proof of our results.

Firstly, by (V1'), there exist constants $d_0 > 0$ and $a_0 > 0$ such that

$$\inf_{x \in \mathbb{R}^3} (V(x) + d_0) \geqslant a_0 > 0. \tag{2.1}$$

Now, define the function space

$$H^1(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3) \right\}$$

with the usual norm

$$\|u\|_{H^1} := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{1/2}.$$

Let

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + (V(x)+d_0)u^2)dx < +\infty \right\}$$

equipped with the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + (V(x)+d_0)u^2)dx \right)^{1/2}, \quad \forall u \in E,$$

and the inner product

$$(u,v) := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + (V(x)+d_0)uv)dx, \quad \forall u,v \in E.$$

Then E is a Hilbert space. Moreover, we write E^* for the dual of E and $\langle \cdot,\cdot \rangle : E^* \times E \to \mathbb{R}$ for the dual pairing. Let $\|\cdot\|_p$ denote the usual norm on $L^p(\mathbb{R}^3)$ for $p \in [1,+\infty)$. Note that E is continuously embedded in $L^p(\mathbb{R}^3)$ for $p \in [2,6]$. Therefore, there exists a constant $D_p > 0$ such that

$$\|u\|_p \leqslant D_p \|u\|, \quad \forall u \in E, \tag{2.2}$$

for any $p \in [2,6]$.

LEMMA 1. ([2]) *Under the assumptions (V1) (or (V1')) and (V2), the embedding from E into $L^p(\mathbb{R}^3)$ is compact for any $p \in [2,6)$.*

Proof. One can proof this lemma as the same way in [2, Lemma 3.1] by using (2.1).We omit the details.
The natural functional of problem ($\mathscr{K}$) is

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{d_0}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} G(x,u)dx. \tag{2.3}$$

The functional is C^1 and its derivative is given by

$$\begin{aligned} \langle \Phi'(u), v \rangle = \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + (V(x)+d_0)uv]dx + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx \\ - \int_{\mathbb{R}^3} (d_0 uv + g(x,u)v)dx, \forall v \in E. \end{aligned} \tag{2.4}$$

As in [15], if $u \in E$ is a critical point of Φ, then u is a solution of equation ($\mathscr{K}$). The following Lemma is already got in [15].

LEMMA 2. *Set $\Psi(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx$. Then Ψ is weakly lower semicontinuous on E.*

LEMMA 3. *If $\{u_n\} \subset E$ is a bounded sequence with $\Phi'(u_n) \to 0$, then $\{u_n\} \subset E$ has a convergent subsequence.*

Proof. In view of the boundedness of $\{u_n\}$, there is a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$ in E. Next, we will verify that $\{u_n\}$ strongly converges to $u \in E$. By Lemma 1, $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2,6)$.

We observe that

$$\begin{aligned}
&\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \\
=&\|u_n-u\|^2 + b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla(u_n-u)|^2dx \\
&-b\left(\int_{\mathbb{R}^3}|\nabla u|^2dx - \int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)dx \\
&-d_0\int_{\mathbb{R}^3}|u_n-u|^2dx - \int_{\mathbb{R}^3}[g(x,u_n)-g(x,u)](u_n-u)dx \\
\geqslant&\|u_n-u\|^2 - b\left(\int_{\mathbb{R}^3}|\nabla u|^2dx - \int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)dx \\
&-d_0\int_{\mathbb{R}^3}|u_n-u|^2dx - \int_{\mathbb{R}^3}[g(x,u_n)-g(x,u)](u_n-u)dx.
\end{aligned} \tag{2.5}$$

Then, (2.5) implies that

$$\begin{aligned}
\|u_n-u\|^2 \leqslant& \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \\
&+b\left(\int_{\mathbb{R}^3}|\nabla u|^2dx - \int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)dx \\
&+d_0\|u_n-u\|_2^2 + \int_{\mathbb{R}^3}[g(x,u_n)-g(x,u)](u_n-u)dx.
\end{aligned} \tag{2.6}$$

By Lemma 2 and $u_n \rightharpoonup u$ in E, one has $\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)dx \to 0$ as $n\to\infty$. Consequently, by the boundedness of $\{u_n\}$, we get

$$b\left(\int_{\mathbb{R}^3}|\nabla u|^2dx - \int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla(u_n-u)dx \to 0, \quad n\to+\infty. \tag{2.7}$$

It is clear that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0. \tag{2.8}$$

By virtue of (G1) and the Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^3}(g(x,u_n)-g(x,u))(u_n-u)dx \leqslant& \int_{\mathbb{R}^3}[c(|u_n|+|u|)+c(|u_n|^{q-1}+|u|^{q-1})]|u_n-u|dx \\
\leqslant& c(\|u_n\|_2^2+\|u\|_2^2)\|u_n-u\|_2^2 \\
&+c(\|u_n\|_q^{q-1}+\|u\|_q^{q-1})\|u_n-u\|_q.
\end{aligned}$$

Then by $u_n\to u$ in $L^s(\mathbb{R}^3)$ for any $s\in[2,6)$, we obtain

$$\|u_n-u\|_2^2 \to 0, \quad \int_{\mathbb{R}^3}(g(x,u_n)-g(x,u))(u_n-u)dx \to 0 \tag{2.9}$$

as $n \to \infty$. Consequently, (2.7), (2.8) and (2.9) imply that

$$u_n \to u \text{ in } E, \text{ as } n \to \infty.$$

This completes the proof.

LEMMA 4. ([13]) *Assume that* $\Omega \subset \mathbb{R}^3$ *is an open set. Then, for any closed set* $\Theta \subset \Omega$, *there exists a function* $\phi \in C_0^\infty(\mathbb{R}^3)$ *such that* $\phi(x) = 0$ *for all* $x \in \mathbb{R}^3 \setminus \Omega$, $\phi(x) = 1$ *for all* $x \in \Theta$ *and* $0 \leqslant \phi(x) \leqslant 1$ *for all* $x \in \Omega \setminus \Theta$.

Let $A = \{u \in E : \|u\|_2 = 1\}$ and $\bar{A}$ be the class of symmetric subsets of A. Set $F_n = \{B \in \bar{A} : \text{Index}(B) \geqslant n\}$ and

$$\lambda_n = \inf_{B \in F_n} \sup_{u \in B} \|u\|^2, \quad n \in \mathbb{N}. \tag{2.10}$$

LEMMA 5. ([11]) *Assume that (V1') and (V2) are satisfied. Then* $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$.

By Lemma 5, choose an integer $m \geqslant 1$ such that $\max\left\{\frac{a_0 d_0}{a_0 - k_1}, \lambda_m\right\} < \lambda_{m+1}$, where k_1 is given in (G4). Define

$$C_- = \{u \in E : \|u\|^2 \leqslant \lambda_m \|u\|_2^2\}, \quad C_+ = \{u \in E : \|u\|^2 \geqslant \lambda_{m+1} \|u\|_2^2\}. \tag{2.11}$$

LEMMA 6. ([11]) *Assume that (V1') and (V2) are satisfied. Then* C_- *and* C_+ *are two symmetric closed cones in* E, $C_- \cap C_+ = \{0\}$ *and*

$$\mathit{Index}(C_- \setminus \{0\}) = \mathit{Index}(E \setminus C_+) = m.$$

In order to prove Theorem 3, we shall use the following critical point theorem. First, recall that $\Phi \in C^1(E, \mathbb{R})$ satisfies $(\mathrm{C})_c$-condition if any sequence $\{u_n\} \subset E$ such that

$$\Phi(u_n) \to c, \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0$$

has a convergent subsequence.

LEMMA 7. ([7, 11]) *Let* E *be a real Banach space and* C_-, C_+ *be two symmetric cones in* E *such that* C_+ *is closed,* $C_- \cap C_+ = \{0\}$ *and*

$$\mathit{Index}(C_- \setminus \{0\}) = \mathit{Index}(E \setminus C_+) < \infty,$$

where Index is the $\mathbb{Z}_2$*-cohomological index of* [8]. *Let* $r_- > r_+ > 0$ *and* $e_1 \in E \setminus C_-$ *with* $\|e_1\| = 1$. *Define the following four sets*

$$\begin{aligned}
S_1 &= \{u \in C_- : \|u\| \leqslant r_-\},\\
K_+ &= \{u \in C_+ : \|u\| = r_+\},\\
S &= \{u + se_1 : u \in C_-, s \geqslant 0, \|u + se_1\| \leqslant r_-\},\\
S_2 &= \{u + se_1 : u \in C_-, s \geqslant 0, \|u + se_1\| = r_-\}.
\end{aligned}$$

Then $(S, S_1 \cup S_2)$ *links* K_+ *cohomologically in dimension* $m+1$ *over* $\mathbb{Z}_2$.

Hence, suppose $\Phi \in C^1(E,\mathbb{R})$ *satisfies* $(C)_c$*-condition for all* $c>0$, *and* $\inf_{x\in K_+}\Phi(x) > \sup_{x\in S_1\cup S_2}\Phi(x)$, $\sup_{x\in S}\Phi(x)<\infty$, *then* Φ *has critical point with value* $c_0 \geqslant \inf_{K_+}\Phi > 0$.

At last, we use the assumptions on g to deduce a useful estimate. By (G1), we have

$$|g(x,u)| \leqslant c_1|u| + c_1|u|^{q-1} \leqslant \left(\frac{c_1}{\zeta^{q-2}} + c_1\right)|u|^{q-1}, \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, |u| \geqslant \zeta. \quad (2.12)$$

Combining (G4) with (2.12), we obtain

$$|G(x,u)| \leqslant \frac{k_1}{2}|u|^2 + c_2|u|^q, \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R} \quad (2.13)$$

where $c_2 = \left(\frac{c_1}{\zeta^{q-2}} + c_1\right)/q$.

3. Positive potential case

We will prove Theorem 1 by using Mountain Pass Theorem [14]. Note that in this case, $d_0 = 0$ in (2.3). Firstly, we give the following three useful lemmas.

LEMMA 8. *Assume that (V1), (V2), (G1) and (G3) hold. Then* Φ *satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence, that is, $\{\Phi(u_n)\}$ is bounded, and $\Phi'(u_n) \to 0$ as $n \to \infty$. We only need to show that $\{u_n\}$ is bounded in E, because of Lemma 3. By (2.2), (G3) and the Hölder's inequality, there holds

$$\begin{aligned}
c_3 + \frac{1}{\eta}c_2\|u_n\| \geqslant & \Phi(u_n) - \frac{1}{\eta}\langle \Phi'(u_n), u_n\rangle \quad (3.1)\\
= & \left(\frac{1}{2} - \frac{1}{\eta}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{\eta}\right)\left(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\right)^2 \\
& + \int_{\mathbb{R}^3}\left[\frac{1}{\eta}g(x,u_n)u_n - G(x,u_n)\right]dx \\
\geqslant & \frac{\eta-2}{2\eta}\|u_n\|^2 - \int_{\mathbb{R}^3}\left(\varsigma V(x)|u_n|^2 + m_2(x)|u_n|^\delta + m_3(x)\right)dx \\
& - \int_{\mathbb{R}^3} m_1(x)\frac{|u_n|^2}{\ln(k_0+|u_n|)}dx \\
\geqslant & \left(\frac{\eta-2}{2\eta} - \varsigma\right)\|u_n\|^2 - \|m_2\|_{\frac{2}{2-\delta}}\|u_n\|_2^\delta - \|m_3\|_1 \\
& - \int_{\{x\in\mathbb{R}^3 : |u_n(x)| \geqslant \sqrt{\|u_n\|}\}} m_1(x)\frac{|u_n|^2}{\ln(k_0+|u_n|)}dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\{x\in\mathbb{R}^3:|u_n(x)|\leqslant\sqrt{\|u_n\|}\}} m_1(x)\frac{|u_n|^2}{\ln(k_0+|u_n|)}dx\\
\geqslant&\left(\frac{\eta-2}{2\eta}-\varsigma\right)\|u_n\|^2-D_2^\delta\|m_2\|_{\frac{2}{2-\delta}}\|u_n\|^\delta-\|m_3\|_1\\
&-\frac{\|m_1\|_2\|u_n\|_4^2}{\ln(k_0+\sqrt{\|u_n\|})}-\frac{\|m_1\|_1}{\ln k_0}\|u_n\|\\
\geqslant&\left(\frac{\eta-2}{2\eta}-\varsigma\right)\|u_n\|^2-D_2^\delta\|m_2\|_{\frac{2}{2-\delta}}\|u_n\|^\delta-\|m_3\|_1\\
&-\frac{D_4^2\|m_1\|_2\|u_n\|^2}{\ln(k_0+\sqrt{\|u_n\|})}-\frac{\|m_1\|_1}{\ln k_0}\|u_n\|,
\end{aligned}
$$

for some $c_3>0$. We obtain that $\{u_n\}$ is bounded in E, by $0<\delta<2$ and $0\leqslant\varsigma<\frac{\eta-2}{2\eta}$. Due to Lemma 3, $\{u_n\}$ has a convergent subsequence in E. Hence, Φ satisfies the (PS) condition.

LEMMA 9. *Assume that (V1), (V2), (G1) and (G4) are satisfied. Then there exist constants* $\rho,\ \alpha_1>0$ *such that* $\Phi|_{\partial B_\rho(0)}\geqslant\alpha_1$.

Proof. By (2.3), (V1), (G4), (2.13) and (2.2), we have

$$
\begin{aligned}
\Phi(u)=&\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2-\int_{\mathbb{R}^3}G(x,u)dx\\
\geqslant&\frac{1}{2}\|u\|^2-\left(\frac{k_1}{2}\|u\|_2^2+c_2\|u\|_q^q\right)\\
\geqslant&\frac{1}{2}\left(1-\frac{k_1}{a_0}\right)\|u\|^2-c_2D_q^q\|u\|^q.
\end{aligned}
$$

The assertion is true follows by $1-\frac{k_1}{a_0}>0$ and $q>4$.

LEMMA 10. *Suppose that (V1), (V2) and (G2) are satisfied. Then, there exists* $e\in E\setminus\bar{B}_\rho(0)$ *such that* $\Phi(e)\leqslant 0$.

Proof. By (G2), for any $c_4>0$ there exists $c_5=c_5(c_4)>0$ such that

$$G(x,u)\geqslant c_4|u|^4,\forall x\in\Lambda_1,|u|\geqslant c_5. \tag{3.2}$$

For any $\varepsilon>0$, there exist a closed set Λ_2 and an open set Λ_3 such that $\Lambda_2\subset\Lambda_1\subset\Lambda_3$ and

$$\mathrm{meas}(\Lambda_2)>0,\ \mathrm{meas}(\Lambda_3\setminus\Lambda_1)<\varepsilon,\ \mathrm{meas}(\Lambda_1\setminus\Lambda_2)<\varepsilon. \tag{3.3}$$

It follows from Lemma 4 that there exists a function $\phi\in C_0^\infty(\mathbb{R}^3)$ such that $\phi(x)=0$ for all $x\in\mathbb{R}^3\setminus\Lambda_3$, $\phi(x)=1$ for all $x\in\Lambda_2$ and $0\leqslant\phi(x)\leqslant 1$ for all $x\in\Lambda_3\setminus\Lambda_2$. Thus,

$\phi \in E$. In view of (G1), (G4) and (3.2), there exists $R_1 = R_1(c_4) > 0$ such that for all $x \in \Lambda_1$, $0 < |u| < c_5$, we have

$$\frac{|g(x,u)u|}{|u|^2} \leqslant R_1. \tag{3.4}$$

Combining (3.4) and the equality $G(x,u) = \int_0^1 g(x,su)uds$, we obtain

$$|G(x,u)| \leqslant \frac{R_1}{2}|u|^2 \tag{3.5}$$

for all $x \in \Lambda_1$, $|u| < c_5$. In view of (3.2) and (3.5), we have

$$G(x,u) \geqslant c_4|u|^4 - R_2|u|^2, \forall (x,u) \in \Lambda_1 \times \mathbb{R}, \tag{3.6}$$

where $R_2 = c_4c_5^2 + \frac{R_1}{2}$. Taking $e(x) = \bar{\delta}\phi(x)$ and $\varepsilon = \frac{1}{\bar{\delta}^{q-2}}$, where $\bar{\delta} > 0$. By (2.2), (2.13) and (3.6), one has

$$\begin{aligned}
\Phi(e) &= \frac{1}{2}\|e\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla e|^2 dx\right)^2 - \int_{\mathbb{R}^3} G(x,e)dx \\
&\leqslant \frac{1}{2}\|e\|^2 + \frac{b}{4}\|e\|^4 - \int_{\Lambda_2} G(x,e)dx - \int_{\Lambda_3\setminus\Lambda_2} G(x,e)dx \\
&\leqslant \frac{1}{2}\|e\|^2 + \frac{b}{4}\|e\|^4 - \int_{\Lambda_2} (c_4|e|^4 - R_2|e|^2)dx + \int_{\Lambda_3\setminus\Lambda_2}\left(\frac{k_1}{2}|e|^2 + c_2|e|^q\right)dx \\
&\leqslant \left(\frac{1}{2} + R_2D_2^2 + \frac{k_1}{2}D_2^2\right)\|e\|^2 + \frac{b}{4}\|e\|^4 - c_4\int_{\Lambda_2}|e|^4dx + c_2\int_{\Lambda_3\setminus\Lambda_2}|e|^q dx \\
&\leqslant \left(\frac{1}{2} + R_2D_2^2 + \frac{k_1}{2}D_2^2\right)\bar{\delta}^2\|\phi\|^2 + \frac{b}{4}\bar{\delta}^4\|\phi\|^4 - c_4\bar{\delta}^4\text{meas}(\Lambda_2) + 2c_2\bar{\delta}^q\varepsilon \\
&= \left(\frac{1}{2} + R_2D_2^2 + \frac{k_1}{2}D_2^2\right)\bar{\delta}^2\|\phi\|^2 + \frac{b}{4}\bar{\delta}^4\|\phi\|^4 - c_4\bar{\delta}^4\text{meas}(\Lambda_2) + 2c_2\bar{\delta}^2.
\end{aligned} \tag{3.7}$$

Choosing c_4 sufficiently large such that

$$\frac{b}{4}\|\phi\|^4 - c_4\text{meas}(\Lambda_2) < 0.$$

Then, when c_4 is fixed, we can choose a large enough δ such that $\|e\| > \rho$ and $\Phi(e) \leqslant 0$.

Proof. [Proof of Theorem 1] Φ satisfies all conditions of Mountain Pass Theorem [14] by Lemmas 8-10. Therefore, equation ($\mathscr{K}$) has at least one nontrivial solution.

Proof. [Proof of Theorem 2] By (G5) and Lemma 8, $\Phi \in C^1(E,\mathbb{R})$ is even and satisfies (PS) condition. Now, we just need to show Φ satisfies other conditions of Fountain Theorem [1].

For any $k \in \mathbb{N}$, we can choose $k+1$ disjoint open sets $\{\Upsilon_i : i = 0,1,...k\}$ such that

$$\sum_{i=0}^{k}\Upsilon_i \subset \Lambda_1, \text{ and } \text{meas}(\Upsilon_i) > 0, i = 0,1,...,k.$$

For any $\varepsilon > 0$ and Υ_i, there exists a closed set A_i such that $A_i \subset \Upsilon_i$ and

$$\operatorname{meas}(A_i) > 0, \ \operatorname{meas}(\Upsilon_i \setminus A_i) < \varepsilon. \tag{3.8}$$

For every Υ_i $(i = 0, 1, ..., k)$, it follows from Lemma 4 that there exists a function $\phi_i \in C_0^\infty(\mathbb{R}^3)$ such that $\phi_i(x) = 0$ for all $x \in \mathbb{R}^3 \setminus \Upsilon_i$, $\phi(x) = 1$ for all $x \in A_i$ and $0 \leqslant \phi_i(x) \leqslant 1$ for all $x \in \Upsilon_i \setminus A_i$. Let $v_i = \frac{\phi_i}{\|\phi_i\|}$, then $v_i \in E$. Because E is a Hilbert space, v_0, $v_1, ..., v_k, ...$ can expended to be an orthonormal basis $\{v_n\}$ of E. Define $X_j = \mathbb{R}v_j$, and set

$$Y_k = \oplus_{j=0}^{k} X_j, \quad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}, \quad k \in \mathbb{N}.$$

For $2 \leqslant t < 6$, we define

$$\lambda_k(t) = \sup_{u \in Z_k, \|u\|=1} \|u\|_t, \quad k = 1, 2, \cdots.$$

Since E is compactly embedded into $L^s(\mathbb{R}^3)$ for $[2,6)$, we have $\lambda_k(t) \to 0$ as $k \to \infty$ ([16]).

Step 1. By (2.3), (G1) and Lemma 5, we obtain

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} G(x,u)dx \\ &\geqslant \frac{1}{2}\|u\|^2 - c_1(\|u\|_2^2 + \|u\|_q^q) \\ &\geqslant \frac{1}{2}\|u\|^2 - c_1\lambda_k^2(2)\|u\|^2 - c_1\lambda_k^q(q)\|u\|^q. \end{aligned} \tag{3.9}$$

Because $\lambda_k(2) \to 0$ as $k \to \infty$, there exists a positive constant c_6 such that

$$c_1\lambda_k^2(2) \leqslant \frac{1}{4}, \quad \forall k \geqslant c_6. \tag{3.10}$$

By (3.9) and (3.10), we have

$$\Phi(u) \geqslant \frac{1}{4}\|u\|^2 - c_1\lambda_k^q(q)\|u\|^q, \quad \forall k \geqslant c_6. \tag{3.11}$$

Choosing $r_k = \left(8c_1\lambda_k^q(q)\right)^{1/(2-q)}$, then

$$b_k = \inf_{u \in Z_k, \|u\|=r_k} \Phi(u) \geqslant \frac{1}{8}r_k^2, \quad \forall k \geqslant c_6. \tag{3.12}$$

Because $\lambda_k(q) \to 0$ as $k \to \infty$ and $q > 2$, we obtain

$$b_k \to \infty \text{ as } k \to \infty.$$

Step 2. All norms are equivalent in the finite-dimensional space, then there exists a constant $c_7 > 0$ such that

$$c_7\|u\| \leqslant \|u\|_4, \quad \forall u \in Y_k. \tag{3.13}$$

By (G1) and (3.2), there exists $R_3 = R_3(c_4) > 0$ such that

$$G(x,u) \geqslant c_4|u|^4 - R_3|u|^2, \quad \forall (x,u) \in \Lambda_1 \times \mathbb{R}. \tag{3.14}$$

By (2.2), (G1), (3.13) and (3.14), one has

$$\begin{aligned}
\Phi(u) =& \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} G(x,u)dx \\
\leqslant& \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{\Lambda_1} G(x,u)dx \\
\leqslant& \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{\Lambda_1} \left(c_4|u|^4 - R_3|u|^2\right) dx \\
\leqslant& \left(\frac{1}{2} + R_3 D_2^2\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - c_4\|u\|_4^4 \\
\leqslant& \left(\frac{1}{2} + R_3 D_2^2\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - c_4 c_7^4\|u\|^4
\end{aligned} \tag{3.15}$$

for any $u \in Y_k$. Choosing c_4 sufficiently large such that

$$\frac{b}{4} - c_4 c_7^4 < 0.$$

Thus, we can choose ρ_k large enough $(\rho_k > r_k)$ such that

$$a_k = \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leqslant 0.$$

Now, from Fountain Theorem [1], Φ has a sequence of critical points $\{u_k\} \subset E$ such that $\Phi(u_k) \to \infty$ as $k \to \infty$. Hence equation $(\mathscr{K})$ has infinitely many high-energy solutions.

4. Sign Changing potentional case

Firstly, we give the following lemmas.

LEMMA 11. *Assume that (V1'), (V2), (G1), (G2') and (G3') hold. Then Φ satisfies $(C)_c$-condition for all $c > 0$.*

Proof. Let $\{u_n\}$ be a sequence in E satisfying

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{4.1}$$

We claim that $\{u_n\}$ is bounded in E. Otherwise, if $\|u_n\| \to \infty$ as $n \to \infty$, set $v_n := \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and there is $v_0 \in E$ such that, up to a subsequence

$$\begin{aligned} &v_n \rightharpoonup v_0 \text{ in } E, \\ &v_n \to v_0 \text{ a.e. in } \mathbb{R}^3, \\ &v_n \to v_0 \text{ in } L^s(\mathbb{R}^3), 2 \leqslant s < 6 \end{aligned} \tag{4.2}$$

as $n \to \infty$.

Case 1. $v_0 = 0$. By $\mu > 4$ and (4.1), we have

$$\begin{aligned} c_8 \geqslant &\Phi(u_n) - \frac{1}{\mu}\langle \Phi'(u_n), u_n\rangle \\ = &\frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\mu-2}{2\mu}d_0\|u_n\|_2^2 + b\frac{\mu-4}{4\mu}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 \\ &+ \int_{\mathbb{R}^3}\left[\frac{1}{\mu}g(x,u_n)u_n - G(x,u_n)\right]dx \\ \geqslant &\frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\mu-2}{2\mu}d_0\|u_n\|_2^2 + \int_{\mathbb{R}^3}\left[\frac{1}{\mu}g(x,u_n)u_n - G(x,u_n)\right]dx \end{aligned} \tag{4.3}$$

for some $c_8 > 0$ and n large enough. For $0 \leqslant c_9 < c_{10}$, let

$$\Omega_n(c_9, c_{10}) = \{x \in \mathbb{R}^3 : c_9 \leqslant |u_n(x)| < c_{10}\}. \tag{4.4}$$

By (G2'), there exists $\nu > 0$ such that $G(x,u) \geqslant 0$ for all $|u| \geqslant \nu$ and a.e. $x \in \mathbb{R}^3$. We claim that $\operatorname{meas}(\Omega_n(\nu,\infty)) > 0$. Arguing indirectly, $\operatorname{meas}(\Omega_n(\nu,\infty)) = 0$. By (G1) and (4.2), one has

$$\begin{aligned} \int_{\Omega_n(0,\nu)}\frac{|\hat{G}(x,u_n)|}{\|u_n\|^2}dx \leqslant &\frac{(\mu+1)c_1}{\mu}\int_{\Omega_n(0,\nu)}\frac{|u_n|^2+|u_n|^q}{\|u_n\|^2}dx \\ \leqslant &\frac{(\mu+1)c_1}{\mu}(1+\nu^{q-2})\int_{\mathbb{R}^3}|v_n|^2dx \to 0 \end{aligned} \tag{4.5}$$

as $n \to \infty$, where $\hat{G}(x,u) = \frac{1}{\mu}g(x,u)u - G(x,u)$. In view of (4.2), (4.3) and (4.5), we obtain

$$0 \geqslant \frac{\mu-2}{2\mu} > 0,$$

which is a contradiction. Hence the claim is true. By (4.3), (G3') and the Hölder's inequality, one has

$$\begin{aligned} c_8 \geqslant &\frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\mu-2}{2\mu}d_0\|u_n\|_2^2 + \int_{\mathbb{R}^3}\hat{G}(x,u_n)dx \\ \geqslant &\frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\mu-2}{2\mu}d_0\|u_n\|_2^2 - \int_{\Omega_n(0,\nu)}|\hat{G}(x,u_n)|dx \\ &- \int_{\Omega_n(\nu,\infty)}(\iota_0|u_n|^2 + \iota_1 V(x)|u_n|^2 + l_1(x)|u_n|^\sigma + l_2(x))dx \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&\geqslant \left(\frac{\mu-2}{2\mu}-\iota_1\right)\|u_n\|^2-\left[\iota_0+\left(\frac{\mu-2}{2\mu}-\iota_1\right)d_0\right]\|u_n\|_2^2\\
&\quad -\|l_1\|_{\frac{2}{2-\sigma}}\|u_n\|_2^{\sigma}-\|l_2\|_1-\int_{\Omega_n(0,\nu)}|\hat{G}(x,u_n)|dx\\
&\geqslant \left(\frac{\mu-2}{2\mu}-\iota_1\right)\|u_n\|^2-\left[\iota_0+\left(\frac{\mu-2}{2\mu}-\iota_1\right)d_0\right]\|u_n\|_2^2\\
&\quad -D_2^{\sigma}\|l_1\|_{\frac{2}{2-\sigma}}\|u_n\|^{\sigma}-\|l_2\|_1-\int_{\Omega_n(0,\nu)}|\hat{G}(x,u_n)|dx.
\end{aligned}
$$

By (4.2), (4.5), (4.6) and (G3'), we obtain

$$
\|v_0\|_2^2 \geqslant \frac{\mu-2}{2\mu c_6}-\frac{\iota_1}{c_6}>0, \tag{4.7}
$$

where $c_6=\iota_0+\left(\frac{\mu-2}{2\mu}-\iota_1\right)d_0$. That implies $v_0\neq 0$, a contradiction.

Case 2. $v_0\neq 0$. Set $\Upsilon:=\{x\in\mathbb{R}^3 : v_0(x)\neq 0\}$, then meas$(\Upsilon)>0$. For a.e. $x\in\Upsilon$, we have $\lim_{n\to\infty}|u_n(x)|=+\infty$. Hence $\Upsilon\subset\Omega_n(\nu,\infty)$ for large $n\in\mathbb{N}$, it follows from (G2') that there

$$
\lim_{|u|\to\infty}\frac{G(x,u)}{|u|^4}=+\infty,\ \text{a.e. } x\in\Upsilon. \tag{4.8}
$$

Because $\{\Phi(u_n)\}$ is bounded, there exists $c_{11}\in\mathbb{R}$ such that

$$
\Phi(u_n)=\frac{1}{2}\|u_n\|^2-\frac{d_0}{2}\|u_n\|_2^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^2-\int_{\mathbb{R}^3}G(x,u_n)dx\geqslant c_{11}. \tag{4.9}
$$

It follows from (4.9) and (2.2) that

$$
\begin{aligned}
\int_{\mathbb{R}^3}\frac{G(x,u_n)}{\|u_n\|^4}dx &\leqslant \frac{1+d_0D_2^2}{2\|u_n\|^2}+\frac{c_{11}}{\|u_n\|^4}+\frac{b}{4\|u_n\|^4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2\\
&\leqslant \frac{1+d_0D_2^2}{2\|u_n\|^2}+\frac{c_{11}}{\|u_n\|^4}+\frac{b}{4}<\infty.
\end{aligned} \tag{4.10}
$$

On the other hand, by (G1), (G2'), (4.8) and Fatou's Lemma, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3}\frac{G(x,u_n)}{\|u_n\|^4}dx &= \int_{\Omega_n(\nu,\infty)}\frac{G(x,u_n)}{\|u_n\|^4}dx+\int_{\Omega_n(0,\nu)}\frac{G(x,u_n)}{\|u_n\|^4}dx\\
&\geqslant \int_{\Omega_n(\nu,\infty)}\frac{G(x,u_n)}{|u_n|^4}|v_n|^4dx-\int_{\Omega_n(0,\nu)}\frac{G(x,u_n)}{\|u_n\|^4}dx\\
&\geqslant \int_{\Omega_n(\nu,\infty)}\frac{G(x,u_n)}{|u_n|^4}|v_n|^4dx-\int_{\Omega_n(0,\nu)}\frac{c_1(|u_n|^2+|u_n|^q)}{\|u_n\|^4}dx\\
&\geqslant \int_{\Omega_n(\nu,\infty)}\frac{G(x,u_n)}{|u_n|^4}|v_n|^4dx-c_1(1+\nu^{q-2})\int_{\Omega_n(0,\nu)}\frac{|u_n|^2}{\|u_n\|^4}dx\\
&\geqslant \int_{H\cap\Upsilon}\frac{G(x,u_n)}{|u_n|^4}|v_n|^4dx-\frac{c_1(1+\nu^{q-2})D_2^2}{\|u_n\|^2}\to\infty,
\end{aligned}
$$

as $n \to \infty$. This contradicts to (4.10). Therefore, $\{u_n\}$ is bounded in E. Due to Lemma 3, there exists a convergence subsequence. Hence, Φ satisfies $(C)_c$-condition for all $c > 0$.

LEMMA 12. *Assume that (V1'), (V2), (G1) and (G4) are satisfied. Then, there exist constants r_+, $\bar{d} > 0$ such that $\Phi(u) \geqslant \bar{d}$ for any $u \in C_+$ with $\|u\| = r_+$.*

Proof. In view of (G4), (2.1), (2.2), (2.13), one has

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \frac{d_0}{2}\int_{\mathbb{R}^3}|u|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 - \int_{\mathbb{R}^3}G(x,u)dx \\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{d_0}{2\lambda_{m+1}}\|u\|^2 - \int_{\mathbb{R}^3}\left(\frac{k_1}{2}|u|^2 + c_2|u|^q\right)dx \\
&\geqslant \frac{\lambda_{m+1}-d_0}{2\lambda_{m+1}}\|u\|^2 - \frac{k_1}{2a_0}\int_{\mathbb{R}^3}(V(x)+d_0)|u|^2dx - c_2D_q^q\|u\|^q \\
&\geqslant \frac{\lambda_{m+1}(a_0-k_1)-a_0d_0}{2a_0\lambda_{m+1}}\|u\|^2 - c_2D_q^q\|u\|^q
\end{aligned}$$

for any $u \in C_+$. Because $[\lambda_{m+1}(a_0-k_1)-a_0d_0]/(2a_0\lambda_{m+1}) > 0$ and $q > 4$, the assertion follows.

LEMMA 13. *Assume that (V1'), (V2), (G1) and (G2') are satisfied. Let $e_1 \in E \setminus C_-$ with $\|e_1\| = 1$. Then there exists a constant $r_- > r_+$ such that $\sup_{x\in S_1\cup S_2}\Phi(x) \leqslant 0$ and $\sup_{x\in S}\Phi(x) < \infty$, where*

$$\begin{aligned}
S_1 &= \{u \in C_- : \|u\| \leqslant r_-\}, \\
S &= \{u + se_1 : u \in C_-, s \geqslant 0, \|u+se_1\| \leqslant r_-\}, \\
S_2 &= \{u + se_1 : u \in C_-, s \geqslant 0, \|u+se_1\| = r_-\}.
\end{aligned}$$

Proof. It is sufficient to show that $\Phi(u) \to -\infty$ as $u \in C_- + \mathbb{R}^+e_1$ with $\|u\| \to \infty$. Arguing indirectly, we assume that for some sequence $\{w_n + s_ne_1\} \subset C_- + \mathbb{R}^+e_1$ with $\|w_n + s_ne_1\| \to \infty$, there exists a constant $T_4 > 0$ such that $\Phi(w_n + s_ne_1) \geqslant -T_4$ for all $n \in \mathbb{N}$. Let $\omega_n = \frac{w_n+s_ne_1}{\|w_n+s_ne_1\|} = z_n + t_ne_1$, then $\|\omega_n\| = 1$. According to Proposition 2.12 of [7], there is a constant $\xi \geqslant 1$ such that

$$\|u\| + \|e_0\| \leqslant \xi\|u+e_0\|, \forall u \in C_-, \tag{4.11}$$

where $e_0 \in E$ with $-e_0 \notin C_-$. By (4.11) and $\|\omega_n\| = 1$, we have

$$\|z_n\| + t_n \leqslant \|z_n\| + t_n\|e_1\| \leqslant \xi\|z_n + t_ne_1\| = \xi. \tag{4.12}$$

Hence, passing to a subsequence if necessary, assume that $t_n \to t_0 \geqslant 0$, $z_n \rightharpoonup z$ in E, $z_n \to z$ a.e. in $\mathbb{R}^3$, and so, by Lemma 1, $z_n \to z$ in $L^s(\mathbb{R}^3)$ for any $s \in [2,6)$. It follows from (2.11) and $\|\omega_n\| = 1$ that

$$1=\|z_n+t_ne_1\|\leqslant\|z_n\|+t_n\leqslant\sqrt{\lambda_m}\|z_n\|_2+t_n\to\sqrt{\lambda_m}\|z\|_2+t_0$$

as $n\to\infty$. This implies that $z+t_0e_1\neq 0$. Similar to the proof of Case 2 in Lemma 11, we can obtain a contradiction.

Proof. [Proof of Theorem 3] It follows from Lemmas 11–13 that Φ satisfies all conditions of Lemma 7. Therefore, equation ($\mathscr{K}$) has at least one nontrivial solution.

Now, we turn to prove Theorem 4.

LEMMA 14. *Assume that (V1'), (V2), (G1) and (G2') hold. Then for any finite dimensional subspace $\tilde{E}\subset E$, there holds*

$$\Phi(u)\to-\infty, as\ \|u\|\to\infty, u\in\tilde{E}.$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\}\subset E$ with $\|u_n\|\to\infty$, there exists $c_{12}>0$ such that $\Phi(u)\geqslant c_{12}$ for all $n\in\mathbb{N}$. Let $v_n=\frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$. Hence, passing to a subsequence if necessary, we can assume that $v_n\rightharpoonup v$ in E. Because $\tilde{E}$ is finite dimensional, then $v_n\to v\in\tilde{E}$ in E. $v_n\to v$ a.e. in $\mathbb{R}^3$, and so $\|v\|=1$. Similar to the proof of Case 2 in Lemma 11, we can obtain a contradiction.

Due to Lemma 14, we have the following Corollary.

COROLLARY 1. *Assume that (V1'), (V2), (G1) and (G2') are satisfied. Then for any finite dimensional subspace $\tilde{E}\subset E$, there is $R=R(\tilde{E})>0$ such that*

$$\Phi(u)\leqslant 0, \forall u\in\tilde{E}\ with\ \|u\|\geqslant R.$$

Let $\{\varphi_j\}$ be a completely orthonormal basis of E and define $E_j:=\mathbb{R}\varphi_j$,

$$\tilde{Y}_k=\oplus_{j=0}^{k}E_j,\quad \tilde{Z}_k=\overline{\oplus_{j=k}^{\infty}E_j},\quad k\in\mathbb{N}. \tag{4.13}$$

LEMMA 15. *Under assumptions (V1') and (V2), for $2\leqslant t<6$,*

$$\tilde{\lambda}_k(t)=\sup_{u\in\tilde{Z}_k,\|u\|=1}\|u\|_t\to 0,\quad k\to\infty.$$

Proof. Because E is compactly embedded into $L^s(\mathbb{R}^3)$ for $s\in[2,6)$, then $\tilde{\lambda}_k(t)\to 0$ as $k\to\infty$ ([16]).

By virtue of Lemma 15, we can choose an integer $N\geqslant 1$ such that

$$\|u\|_2^2\leqslant\frac{1}{2(d_0+c_1)}\|u\|^2,\quad \|u\|_q^q\leqslant\frac{q}{4c_1}\|u\|^q,\quad\forall u\in\tilde{Z}_N. \tag{4.14}$$

LEMMA 16. *Under assumptions (V1'), (V2) and (G1), there exists ρ, $\alpha>0$ such that $\Phi|_{\partial B_\rho\cap\tilde{Z}_N}\geqslant\alpha$.*

Proof. In view of (2.2), (G1) and (4.14), we obtain

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \frac{d_0}{2}\int_{\mathbb{R}^3}|u|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 - \int_{\mathbb{R}^3}G(x,u)dx \\
&\geqslant \frac{1}{2}\|u\|^2 - \frac{d_0+c_1}{2}\|u\|_2^2 - \frac{c_1}{q}\|u\|_q^q \\
&\geqslant \frac{1}{4}\left(\|u\|^2 - \|u\|^q\right) = \frac{4^{q-3}-1}{4^q} := \alpha,
\end{aligned}
$$

for all $u \in \tilde{Z}_N$, $\|u\| = \frac{1}{4} := \rho$.

Proof. [Proof of Theorem 4] Set $X = \tilde{Y}_N$, $Y = \tilde{Z}_N$. Obviously, $\Phi(0) = 0$ and $\Phi(u) \in C^1(E,\mathbb{R})$ are even. It follows from Lemma 11 and 16 and Corollary 1 that Φ satisfies all conditions of Symmetric Mountain Pass Theorem [14]. Hence, equation ($\mathscr{K}$) has infinitely many high-energy solutions.

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