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A REMARK FOR SPATIAL ANALYTICITY AROUND STRAINING FLOWS

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Abstract. Time-local existence of unique smooth solutions to the Navier-Stokes equations in the whole space with linearly growing initial data has been established, via smoothing properties of Ornstein-Uhlenbeck semigroup. It has also been shown that the solution is real-analytic in spatial variables around rotating flows. This note is devoted to prove the spatial analyticity for cases of straining flows and shear flows. It is estimated the size of radius of convergence of Taylor series, due to estimates for higher order derivatives and Cauchy-Hadamard theorem.

1. Introduction

We consider the Navier-Stokes equations which describe incompressible, viscous fluid flows in the whole space $\mathbb{R}^n$ for $n \in \mathbb{N}, \geqslant 2$:

$$\begin{aligned} \partial_t U - \Delta U + U \cdot \nabla U + \nabla P = F \quad &\text{in} \quad \mathbb{R}^n \times (0,T), \\ \nabla \cdot U = 0 \quad &\text{in} \quad \mathbb{R}^n \times (0,T), \\ U|_{t=0} = U_0 \quad &\text{in} \quad \mathbb{R}^n. \end{aligned} \tag{1.1}$$

Here, $U = (U^1(x,t), \ldots, U^n(x,t))$ and $P = P(x,t)$ stand for the unknown velocity and the unknown pressure at $x \in \mathbb{R}^n$ and $t \in (0,T)$, respectively; $U_0 = (U_0^1(x), \ldots, U_0^n(x))$ is a given initial velocity, and $F = (F^1(x,t), \ldots, F^n(x,t))$ is a given external force. We have used the notation of differentiation; $\partial_t := \partial/\partial t$, $\Delta := \sum_{i=1}^n \partial_i^2$, $\partial_i := \partial/\partial x_i$ for $i = 1, \ldots, n$, $\nabla := (\partial_1, \ldots, \partial_n)$, $\nabla \cdot U := \sum_{i=1}^n \partial_i U^i$. It is always imposed the compatibility condition, that is, $\nabla \cdot U_0 = 0$.

There is huge literature on time-local well-posedness (i.e., existence, uniqueness, smoothness and equi-continuity of solutions) to (1.1); see e.g. [5, 6, 9, 14, 16]. In their results, it is assumed that velocities decay at $|x| \to \infty$. Dealing with nondecaying velocities, in [8] Giga and his collaborators have established time-local well-posedness in BUC as well as L^∞. In fact, one can construct time-local unique smooth mild solutions, when U_0 is bounded uniformly continuous; see also [2, 3, 15]. For the case of linearly growing velocities, in [12] Hieber and the second author of this note constructed time-local smooth solutions for

$$U_0(x) = -Mx + u_0(x)$$

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at $x \in \mathbb{R}^n$ with $M = (m_{ij})_{1 \leqslant i,j \leqslant n} \in \mathbb{R}^{n\times n}$ and $u_0 \in L^q(\mathbb{R}^n)$ for $q \in [n,\infty)$ satisfying the trace-free condition $\mathrm{tr}M = 0$ and the divergence-free condition $\nabla \cdot u_0 = 0$. It might be supposed that $U = -Mx$ is a stationary solution to (1.1) with some P and F, thus we are required to investigate time-evolution of disturbance from the initial disturbance u_0. In the cases of more general situation, the reader can find the existence results in [11, 21].

In [12] the spatial real analyticity was also proved, provided M is skew-symmetric. The aim of this note is to show that U is real analytic in x for the cases of general M, including straining flows and shear flows.

In $n = 3$, typical examples of M are as follows:

$$R = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 2b \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $a,b,c \in \mathbb{R}$, and these sum. Note that R, J and S correspond to rotating, straining and shear flows, respectively.

Substituting $u := U + Mx$, (1.1) are rewritten as

$$\begin{aligned} \partial_t u - \Delta u + u\cdot\nabla u - Mx\cdot\nabla u - Mu + \nabla p &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T), \\ \nabla \cdot u &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T), \\ u|_{t=0} &= u_0 \quad \text{in} \quad \mathbb{R}^n. \end{aligned} \tag{1.2}$$

Here, p is a scalar function satisfying $\nabla p = \nabla P - F + M^2 x$. It is rather easy to prove the existence of weak solutions to (1.2); see e.g. [1, 4]. However, for solving (1.1) by the converse transformation $U = -Mx + u$, we are forced to construct classical solutions to (1.2). For this purpose, we select a semigroup approach. Hence, (1.2) is formally equivalent to the integral equation

$$\text{(INT)} \qquad u(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A}\mathbb{P}\{u(s)\cdot\nabla u(s) - 2Mu(s)\}ds$$

with $u_0 \in L^q_\sigma(\mathbb{R}^n)$ for $q \in [n,\infty)$, since $-A$ generates the Ornstein-Uhlenbeck semigroup $\{e^{tA}\}_{t\geqslant 0}$ in L^q_σ. Here, we have used the Helmholtz projection $\mathbb{P} = (\delta_{ij} + R_iR_j)_{1\leqslant i,j\leqslant n}$ onto the solenoidal subspace L^q_σ of the Lebesgue space L^q for $q \in (1,\infty)$ associated with Kronecker's delta δ_{ij} and the Riesz transform R_i defined as $R_i := \partial_i(-\Delta)^{-1/2}$ for $i = 1,\dots,n$ as well as $Av := \Delta v + Mx\cdot\nabla v - Mv$ with domain $D(A) := \{v \in W^{2,q}\cap L^q_\sigma; Mx\cdot\nabla v \in L^q\}$. Note that A and $\mathbb{P}$ commute, since $\nabla\cdot Av = 0$ provided $\nabla \cdot v = 0$. The solution $u \in C([0,T];L^q_\sigma(\mathbb{R}^n))$ to the integral equation (INT) is often called a *mild solution*, so we use the terminology.

The aim of this note is to show the real analyticity of a mild solution u with respect to spatial variables x, whence it exists. We will establish the L^∞-norm estimates of higher order derivatives, and appeal to the Cauchy-Hadamard theorem for estimating the size of radius of convergence of Taylor series. Besides, in [17, 18] Masuda discussed the real analyticity of solutions to (1.1) in t and x from a different approach; his proof is based on the implicit function theory. The reader can find recent improvement of his method in e.g. [7] and references therein. It however looks hard to apply his method into our situation, at least directly. The difficulty comes from the fact that the semigroup $\{e^{tA}\}_{t\geqslant 0}$ is not analytic, this means that arguments of the maximal regularity do not work, so it is not clear how to obtain certain a priori estimates.

This note is organized as follows. In section 2 we state the main results. Section 3 is to recall smoothing properties of the Ornstein-Uhlenbeck semigroup and Kahane's lemma for bilinear estimates. We will give a complete proof of our main results in section 4.

Throughout this note, we denote positive constants by C the value of which may differ from one occasion to another.

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2. Main Results

This section is devoted to state the main results of this note. We first recall the time-local existence and uniqueness results for mild solutions.

THEOREM 1. ([12]) *Let* $n \geqslant 2$, $k \in \mathbb{N}$, $q \in [n,\infty)$. *If* $M \in \mathbb{R}^{n\times n}$, $\mathrm{tr} M = 0$ *and* $u_0 \in L^q_\sigma(\mathbb{R}^n)$, *then there exist* $T_k > 0$ *and a unique mild solution* $u \in C([0,T_k];L^q_\sigma(\mathbb{R}^n))$ *such that*

$$t^{k/2+(1/q-1/r)n/2}\nabla^k u \in C([0,T_k];L^r(\mathbb{R}^n)) \quad \textit{for} \quad r \in [q,\infty].$$

To prove this theorem, one may argue by successive approximation as in [9, 14]. When $n = 2$, it is easy to obtain a time-global unique mild solution. It seems to be hard to gain a time-global solution even for small initial $u_0 \in L^n_\sigma(\mathbb{R}^n)$. For the case when $q = \infty$, because there is a lack of boundedness of $\mathbb{P}$ in L^∞, we need some restriction, for example, $u_0 \in \dot{B}^0_{\infty,1} \subset L^\infty$ for dealing with nondecaying data; see [21]. Remark that T_k must be chosen small for large $k \in \mathbb{N}$. Indeed, when $q > n$, by the iteration scheme, we deduce $T_k \geqslant Ck^{-k}/\|u_0\|_q^{2-2n/q}$ with some C depending only on n, q and M. Nevertheless, the mild solution u is unique as long as it exists, one can extend the existence time of the mild solution up to T_1 having bounds for k-th derivatives. We hence confirm that $u(t) \in C^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and $t \in (0,T_1]$, which means that $u(t) \in C^\infty(\mathbb{R}^n)$, whence the mild solution exists.

Because the semigroup $\{e^{tA}\}_{t\geqslant 0}$ is not analytic, it is impossible to control L^r-norm of $\partial_t u$, at least directly. In fact, $u \notin C^1(0,T_1;L^q)$, that is, u can not be a strong solution. However, it might be shown that u is smooth in t, using the notion of weak solutions; see [21]. We finally reach to $u \in C^\infty(\mathbb{R}^n \times (0,T_1])$. Therefore, u is a classical solution to (1.2) associated with

$$p = \sum_{i,j=1}^n R_iR_ju^iu^j - 2(-\Delta)^{-1}\partial_i m_{ij}u^j.$$

We thus construct a time-local classical solution to (1.1) by $U = -Mx + u$ and suitable choice of P.

In [12], the real analyticity of mild solutions in x is also shown for skew-symmetric M. In this note, we relax the condition on M to obtain the spatial analyticity. For making short of this note, we deal with the case when $u_0 \in L^n$, and estimate for $\|\nabla^k u(t)\|_\infty$, only. Note that the same assertion holds when $u_0 \in L^q$ for $q \in (n,\infty)$ or $u_0 \in \dot{B}^0_{\infty,1}$.

THEOREM 2. *Let $n \geqslant 2$, $M \in \mathbb{R}^{n\times n}$, $\mathrm{tr}M = 0$, $u_0 \in L^n_\sigma(\mathbb{R}^n)$, $T > 0$, and let u be a mild solution in $[0,T]$. Assume further that there exist constants L_1 and L_2 such that*

$$\sup_{0\leqslant t\leqslant T} \|u(t)\|_n \leqslant L_1 \quad \text{and} \quad \sup_{0<t\leqslant T} t^{1/2}\|u(t)\|_\infty \leqslant L_2. \tag{2.1}$$

Then there exist constants K_1 and K_2 depending only on n, M, T, L_1 and L_2 such that

$$\|\nabla^k u(t)\|_\infty \leqslant K_1(K_2k)^k t^{-k/2-1/2} \quad \text{for} \quad t\in(0,T], \quad k\in\mathbb{N}. \tag{2.2}$$

When $M = 0$, the same assertion was proved in [10] with $u_0 \in L^q(\mathbb{R}^n)$ for $q \in [n,\infty]$; with $u_0 \in BMO^{-1}$ see [20]. One may take L_1 and L_2 in (2.1) as finite quantities, whence the mild solution exists up to T. Consequently, it follows from (2.2) that the mild solution $u(t)$ is real analytic in x as long as it exists. More precisely, the size of radius of convergence of Taylor series ($=: \rho$) is estimated from below by

$$\rho = \rho(t) = \liminf_{k\to\infty}\left(\frac{\|\nabla^k u(t)\|_\infty}{k!}\right)^{-1/k} \geqslant \frac{e}{K_2}\sqrt{t}, \quad t\in(0,T].$$

Here, we use Stirling's formula (3.3) in section 3 below and the Cauchy-Hadamard theorem. This assertion implies that the propagation speed is infinite as well as the heat equation, that is, even if the support of initial data is compact, the support of solutions coincides the whole space, instantaneously. It is open whether u is real analytic in t.

By (2.1), it is easy to see that

$$\sup_{0<t\leqslant T} t^{1/2}\|\nabla u(t)\|_n \leqslant L_3 \tag{2.3}$$

with some constant L_3 depending only on n, M, L_1 and L_2.

Similarly, $\sup_{0<t\leqslant T} t^{k/2}\|\nabla^k u(t)\|_n$ is bounded for each finite $k \in \mathbb{N}$.

The assumption on L_2 may be relaxed slightly. In fact, instead of assuming on L_2, we are allowed to suppose

$$\sup_{0<t\leqslant T} t^{(1/q-1/s)n/2}\|u(t)\|_s \leqslant L_4 \quad \text{for some } s\in(q,\infty],$$

since bounds of L_2 and L_3 are ensured by those of L_1 and L_4. Notice that the uniform bound of L_2 (or L_3, L_4) in t, up to $T = \infty$ in particular, is still open for $n \geqslant 3$, even if $M = 0$ and $\|u_0\|_n$ is small.

3. Ornstein-Uhlenbeck Semigroup

Let $n \in \mathbb{N}$. For $q \in [1,\infty]$, $L^q = L^q(\mathbb{R}^n)$ denote by the usual Lebesgue spaces in $\mathbb{R}^n$ with norm $\|f\|_q := (\int_{\mathbb{R}^n} |f(x)|^q\, dx)^{1/q}$ for $q < \infty$, and $\|f\|_\infty := \mathrm{ess.sup}_{x\in\mathbb{R}^n}|f(x)|$. We often omit the notation $(\mathbb{R}^n)$, if no confusion occurs likely; we sometimes do not distinguish the vector valued function and scalar as well as function spaces. The Sobolev space stands for $W^{m,q}$ for $q \in [1,\infty]$ and $m \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The solenoidal subspace of L^q denotes by L^q_σ for $q \in (1,\infty)$.

Secondly, we recall properties of the Ornstein-Uhlenbeck semigroup.

PROPOSTION 1. ([12, 19]) (a) Let $n \in \mathbb{N}$, $\geqslant 2$, $q \in (1,\infty)$, $M \in \mathbb{R}^{n\times n}$, $\mathrm{tr} M = 0$. Put $A := \Delta + Mx\cdot\nabla - M$ with domain $D(A) := \{v \in W^{2,q} \cap L^q_\sigma; Mx\cdot\nabla v \in L^q\}$, $-A$ generates a non-analytic (C_0)-semigroup $\{e^{tA}\}_{t\geqslant 0}$ on L^q_σ. Further, it has the following explicit formula

$$e^{tA}v(x) := \frac{e^{-tM}}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} v(e^{tM}x - y)e^{-Q_t^{-1}y\cdot y/4}dy$$

for $x \in \mathbb{R}^n$, $t > 0$ and $v \in L^q_\sigma$, where $Q_t := \int_0^t e^{sM}e^{sM^T}\,ds$.
(b) Let $T > 0$, $r \in [q,\infty]$. Thus, there exist constants $C > 0$ depending only on n, q, r, M and $\omega \geqslant 0$ depending only on M such that

$$\|\nabla^k e^{tA} v\|_r \leqslant C e^{\omega k t} t^{-(1/q-1/r)n/2} \|\nabla^k v\|_q \tag{3.1}$$

for $t > 0$, $k \in \mathbb{N}_0$ and $v \in W^{k,q}$ as well as

$$\|\nabla^k e^{tA} v\|_r \leqslant C(Ck)^{k/2} e^{\omega k t} t^{-k/2-(1/q-1/r)n/2} \|v\|_q \tag{3.2}$$

for $t > 0$, $k \in \mathbb{N}_0$ and $v \in L^q$.

This proposition is based on the results by Metafune and his collaborators in [19]. The proofs of above estimates are precisely shown in [12], so we omit them in this note. Remark that the semigroup $\{e^{tA}\}_{t\geqslant 0}$ is neither analytic nor commutative to ∇. Indeed, it holds that

$$\nabla e^{tA} v = e^{tM} e^{tA} \nabla v \quad \text{for} \quad v \in (W^{1,q})^n.$$

From above, it is clear that ω essentially depends only on the maximum of absolute value of real part of eigenvalues of M. If M is skew-symmetric, then $\omega = 0$, since e^{tM} is unitary and $Q_t = t\mathbb{I}$.

We recall Kahane's lemma for control the bilinear terms.

LEMMA 1. ([13]) *Let $n \in \mathbb{N}$. Put $|\alpha| := \sum_{i=1}^n \alpha_i$ and $\alpha! := \Pi_{i=1}^n(\alpha_i!)$ for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$. Denote $\beta \leqslant \alpha$ by $\beta_i \leqslant \alpha_i$ for all i, and $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$. Assume that multi-sequences $\{S_\alpha\}$, $\{T_\alpha\}$ satisfy*

$$|S_0| \leqslant \sigma, \quad |T_0| \leqslant \theta, \quad |S_\alpha| \leqslant \sigma|\alpha|^{|\alpha|-\delta} \quad \textit{and} \quad |T_\alpha| \leqslant \theta|\alpha|^{|\alpha|-\delta'}$$

for $\alpha \in \mathbb{N}_0^n$, $\neq 0$, *where* $\delta, \delta' \in \mathbb{R}$, $\sigma, \theta \geqslant 0$ *are constants. If* $\delta, \delta' > 1/2$, *then there exists* $\gamma > 0$ *depending only on* n, δ *and* δ' *such that*

$$\left| \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} S_\beta T_{\alpha-\beta} \right| \leqslant \gamma\sigma\theta |\alpha|^{|\alpha|-\min\{\delta,\delta'\}} \quad \textit{for} \quad \alpha \in \mathbb{N}_0^n, \neq 0.$$

This lemma follows from Stirling's formula

$$k! \sim \sqrt{2\pi k}\,(k/e)^k \quad \text{for large } k \in \mathbb{N}. \tag{3.3}$$

4. Proof

We give the proof of Theorem 2. For $M = 0$, it was obtained by [10] that mild solution is real analytic in x. We modify their proof. It suffices to show the following assertion essentially equivalent to (2.2).

PROPOSTION 2. Suppose that the assumptions of Theorem 2 are satisfied. Let $\delta \in (1/2, 1]$. Then there exist positive constants K_1 and K_2 depending only on n, M, L_1, L_2, T and δ such that

$$\|\nabla^k u(t)\|_\infty \leqslant K_1 (K_2 k)^{k-\delta} t^{-k/2-1/2}, \; t \in (0,T], \; k \in \mathbb{N}. \tag{4.1}$$

Proof. We may assume that u is smooth, i.e., $u \in C^\infty(\mathbb{R}^n \times (0,T])$ by construction. It is used an induction argument with respect to $k \in \mathbb{N}$. For $k = 0$ and 1, we see that $t^{k/2+1/2-n/2r}\nabla^k u(t)$ is uniformly bounded in $[0,T]$ with valued in $L^r(\mathbb{R}^n)$ for $r \in [n,\infty]$, using L_1 and L_2 as (2.3). This means that (4.1) holds for $k = 1$, taking $K_1 \geqslant C_1'$ with some large $C_1' > 0$ and $K_2 = 1$. Similarly, assuming that for $k_* \geqslant 2$ determined later, (4.1) holds for $k \leqslant k_*$ with some large $K_1 \geqslant C_1 \geqslant C_1'$ and $K_2 = 1$.

Let $k \geqslant k_* + 1$. Suppose that (4.1) hold from 1 to $k-1$. We now claim that (4.1) holds for k with suitable K_1 and K_2. Put $\varepsilon \in (0,1)$, we divide the integral in $(0,t)$ of (INT) into two parts to have

$$\begin{aligned}
\|\nabla^k u(t)\|_\infty \leqslant{}& \|\nabla^k e^{tA} u_0\|_\infty \\
&+ \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k e^{(t-s)A} \mathbb{P} u \cdot \nabla u(s)\|_\infty ds \\
&+ \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k e^{(t-s)A} \mathbb{P} 2Mu(s)\|_\infty ds \\
=:{}& I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

We shall estimate $I_1, \ldots, I_5$ above, separately.

In what follows, for the sake of simplicity, let $T \leqslant 1$, and then $t \leqslant 1$. For estimating I_1, by the smoothing estimate (3.2) it holds that

$$I_1 \leqslant C_\spadesuit (C_\heartsuit k)^{k/2} e^{\omega k t} t^{-k/2-1/2} \|u_0\|_n \quad \text{for} \quad t \in (0,T]$$

with some $C_\spadesuit$ and $C_\heartsuit$ independent of k. By Stirling's formula (3.3),

$$I_1 \leqslant C_2'(C_3'k)^{k-\delta}t^{-k/2-1/2}$$

for all $k \geqslant 2$ with constants chosen as $C_3' := C_\heartsuit e^{3\omega}$ and $C_2' := 2C_\spadesuit L_1 C_3'$.

It is also easy to derive the estimate to I_2. Indeed, for $t \in (0,T]$

$$\begin{aligned} I_2 &\leqslant \int_0^{(1-\varepsilon)t} C(Ck)^{k/2}e^{k\omega(t-s)}(t-s)^{-k/2-1}\|u(s)\cdot\nabla u(s)\|_{n/2}ds \\ &\leqslant C(Ck)^{k/2}e^{k\omega t}\int_0^{(1-\varepsilon)t}(t-s)^{-k/2-1}\|u(s)\|_n\|\nabla u(s)\|_n ds \\ &\leqslant C(Ck)^{k/2}e^{k\omega t}\varepsilon^{-k/2-1}(1-\varepsilon)^{1/2}t^{-k/2-1/2}. \end{aligned}$$

Here, we have used the Hölder inequality and (2.3). Similarly as I_2,

$$\begin{aligned} I_4 &\leqslant \int_0^{(1-\varepsilon)t} C(Ck)^{k/2}e^{k\omega(t-s)}(t-s)^{-k/2-1/2}\|u(s)\|_n ds \\ &\leqslant C(Ck)^{k/2}e^{k\omega t}\varepsilon^{-k/2-1}(1-\varepsilon)t^{-k/2+1/2} \end{aligned}$$

hold. Hereafter, we choose $\varepsilon := \eta/k$ for $\eta \geqslant 1$ and $k \geqslant \eta+1$. So,

$$I_2 + I_4 \leqslant C_\diamondsuit C_\clubsuit^{k/2}k^{k+1}e^{k\omega}\eta^{-k/2-1}t^{-k/2-1/2} \leqslant C_2''(C_3''k)^{k-\delta}t^{-k/2-1/2}$$

are satisfied with constants $C_\diamondsuit$ and $C_\clubsuit$; we have chosen $C_2'' := C_\diamondsuit$, $C_3'' := \max\{1, C_\clubsuit\}$ and $\eta \geqslant e^{2\omega+2}$, since $k/2 \leqslant k-\delta$ for $k \geqslant 2$ and $\delta \leqslant 1$ as well as $k^2 \leqslant e^k$. Hence, one can see

$$I_1 + I_2 + I_4 \leqslant C_2(C_3k)^{k-\delta}t^{-k/2-1/2}$$

for $k \geqslant \eta+1$ with $\eta := e^{2\omega+2}$, $C_2 := C_2' + C_2''$ and $C_3 := \max\{C_3', C_3''\}$.

The estimate for I_5 is as follows. By (3.1) we shift ∇^k to u, then

$$I_5 \leqslant \int_{(1-\varepsilon)t}^t Ce^{k\omega(t-s)}\|\nabla^k u(s)\|_\infty ds \leqslant Ce^{k\omega\varepsilon t}\int_{(1-\varepsilon)t}^t \|\nabla^k u(s)\|_\infty ds.$$

This term remains for applying the Gronwall inequality, later.

For I_3, we shift ∇^{k-1} to the bilinear terms. It leads us to

$$\begin{aligned} I_3 &\leqslant \int_{(1-\varepsilon)t}^t Ce^{(k-1)\omega(t-s)}(t-s)^{-1/2}\left\|\sum_{j=0}^k \binom{k}{j}\{\nabla^j u(s)\}\nabla^{k-j}u(s)\right\|_\infty ds \\ &\leqslant Ce^{k\omega\varepsilon t}\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}\|u(s)\|_\infty\|\nabla^k u(s)\|_\infty ds \\ &\quad + Ce^{k\omega\varepsilon t}\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}\sum_{j=1}^{k-1}\binom{k}{j}\|\nabla^j u(s)\|_\infty\|\nabla^{k-j}u(s)\|_\infty ds \\ &=: I_3^\natural + I_3^*. \end{aligned}$$

For $t \leqslant 1$ and $\varepsilon := \eta/k$, combining $I_3^\natural$ and I_5, we derive

$$I_3^\natural + I_5 \leqslant Ce^{\omega\eta} \int_{(1-\eta/k)t}^t (t-s)^{-1/2} s^{-1/2} \|\nabla^k u(s)\|_\infty ds.$$

Put $\psi(t) := \sup_{0<\tau\leqslant t} \tau^{k/2+1/2} \|\nabla^k u(\tau)\|_\infty$. We thus see that

$$\begin{aligned} I_3^\natural + I_5 &\leqslant Ce^{\omega\eta} \int_{(1-\eta/k)t}^t (t-s)^{-1/2} s^{-k/2-1} \psi(s) ds \\ &\leqslant Ce^{\omega\eta} \{(1-\eta/k)t\}^{-k/2-1/2} \psi(t) \int_{(1-\eta/k)t}^t (t-s)^{-1/2} s^{-1/2} ds \\ &\leqslant t^{-k/2-1/2} \psi(t)/2 \quad \text{for } k \geqslant k_* \end{aligned}$$

with some large $k_* \geqslant \eta + 1$. For the last inequality, the definition of Napier's constant leads us to take a constant C_* independently of k:

$$(1-\eta/k)^{-k/2-1/2} \leqslant 2e^{\eta/2} (1-\eta/k)^{-1/2} \leqslant C_*.$$

By assumption of induction and Lemma 1, the last term I_3^* is estimated as follows:

$$\begin{aligned} I_3^* &\leqslant Ce^{k\omega\varepsilon t} \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} \sum_{j=1}^{k-1} \binom{k}{j} K_1 (K_2 j)^{j-\delta} s^{-j/2-1/2} \\ &\qquad\qquad \cdot K_1 \{K_2(k-j)\}^{k-j-\delta} s^{-k/2+j/2-1/2} ds \\ &\leqslant Ce^{k\omega\varepsilon t} K_1^2 K_2^{k-2\delta} \int_{(1-\varepsilon)t}^t (t-s)^{-1/2} s^{-k/2-1} ds \\ &\qquad\qquad \cdot \sum_{j=0}^{k} \binom{k}{j} j^{j-\delta} (k-j)^{k-j-\delta} \\ &\leqslant Ce^{k\omega\varepsilon t} K_1^2 K_2^{k-2\delta} k^{k-\delta} (1-\eta/k)^{-k/2-1} \varepsilon^{1/2} t^{-k/2-1/2} \\ &\leqslant C_4 K_1^2 K_2^{k-2\delta} k^{k-\delta} t^{-k/2-1/2} \end{aligned}$$

with some C_4 independent of k, since η is fixed. Therefore, we have

$$t^{k/2+1/2} \|\nabla^k u(t)\|_\infty \leqslant \psi(t) \leqslant 2C_2 (C_3 k)^{k-\delta} + 2C_4 K_1^2 K_2^{-\delta} (K_2 k)^{k-\delta}$$

for $k \geqslant k_*$. Finally, we take

$$K_1 := \max\{C_1, 4C_2\} \quad \text{and} \quad K_2 := \max\{C_3, (4C_4 K_1)^{1/\delta}\}$$

to get (4.1) with k. This completes the proof of Proposition 2.

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