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ON BOUNDARY VALUE PROBLEM FOR EQUATIONS WITH CUBIC NONLINEARITY AND STEP–WISE COEFFICIENT

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Abstract. The differential equation with cubic nonlinearity $x'' = -ax + bx^3$ is considered together with the boundary conditions $x(-1) = x(1) = 0$. In the autonomous case, $b = const > 0$, the exact number of solutions for the boundary value problem is given. For nonautonomous case, where $b = \beta(t)$ is a step-wise function, the existence of additional solutions is detected. The reasons for such behaviour are revealed. The example considered in this paper is supplemented by a number of visualizations.

1. Introduction

The classic questions studied by the theory of nonlinear boundary value problems are the existence and uniqueness of solutions. Many mathematical models of nonlinear phenomena have multiple solutions, however.

There are still lacking general methods and techniques for studying the problem of multiple solutions. In this paper, we consider problems in which nonlinearity interacts with linearity.

One of the articles that addresses this issue is [8]. When considering superlinear differential equations by variational methods, the authors asked whether this equation can have multiple solutions satisfying the given boundary conditions. They constructed the example of a superlinear equation with three solutions. Their approach was developed in [2], in which the existence of an infinite number of solutions in the same problem was proved.

Developing this approach, we consider a second-order nonlinear differential equation with cubic nonlinearity. First, we consider the autonomous equation and the respective Dirichlet boundary value problem. Equations of this type often arise in applications (for example, in the theory of superconductivity by Ginsburg-Landau ([7]). We prove the result on the exact number of solutions of this boundary value problem. Further, a nonautonomous case is considered. In this case, the equation with a cubic nonlinearity is given on the side intervals, and in the middle interval the equation is linear. How does this affect the number of solutions to this boundary value problem? We consider these questions and prove a series of results.

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In this problem, we consider the effect of three parameters a, b and δ on the number of solutions. There are a significant number of articles devoted to the study of differential equations, combined from several equations defined on nonintersecting subintervals of the main interval. Using constructions of this type, convenient examples of boundary value problems that have multiple solutions, can be provided. In the article by R. Moore and Z. Nehari [8] the equation of the form

$$x'' = -p(t)|x|^{2\varepsilon}x \tag{1.1}$$

was studied. The coefficient $p(t)$ is a step-wise function which is equal to a given positive number in the left and the right subintervals of the interval $[a,b]$ and $p(t)$ is zero in the middle subinterval. It was shown that for appropriate choice of $p(t)$ the equation has three solutions satisfying the boundary conditions $x(a) = 0 = x(b)$. Inspired by this example, in the work [2] the problem

$$x'' = -q(t)x^3, \quad x(-1) = 0, \quad x(1) = 0 \tag{1.2}$$

is considered where q is

$$q(t) = \begin{cases} 2, & t \in [-1, -1+\varepsilon], \\ 0, & t \in (-1+\varepsilon, 1-\varepsilon), \\ 2, & t \in [1-\varepsilon, 1]. \end{cases} \tag{1.3}$$

It was shown that for any $\varepsilon \in (0,1)$ there are infinitely many solutions of the problem. Similar type problems were studied in the papers [1], [6], where the second order differential equations with step-wise coefficients were considered.

We want to compare solutions of the equation (1.6) with solutions of a similar autonomous equation

$$x'' = -ax + bx^3 \tag{1.4}$$

with respect to boundedness of solutions. Next, we consider boundary value problems for both equations (1.6) and (1.4), in which the boundary conditions are

$$x(-1) = 0, \quad x(1) = 0. \tag{1.5}$$

We want to study both problems and compare the number of their solutions. The differential equation (1.4) contains only cubic nonlinearity. The equation (1.6) is also nonlinear with cubic nonlinearity turned off on the middle subinterval.

In this article we study the equation

$$x'' = -ax + \beta(t)x^3, \quad a > 0, \tag{1.6}$$

where $\beta(t)$ is a step-wise function

$$\beta(t) := \begin{cases} b, & t \in [-1, -1+\delta] =: I_1, \\ 0, & t \in (-1+\delta, 1-\delta) =: I_2, \\ b, & t \in [1-\delta, 1] =: I_3, \quad b > 0, \quad 0 < \delta < 1. \end{cases} \tag{1.7}$$

The equation (1.6) includes the equation (1.4), which was previously studied and is often discussed in textbooks. We are not aware, however, of the results on exact estimates of the number of solutions in boundary value problems for this equation. Such an estimate is given in subsection 3.1. Then we study the problem (1.6), (1.5), in which the equation (1.6) is an equation of the type (1.4) in the two side subintervals I_1 and I_3 (see (3.16)) and it is a linear equation on the middle subinterval of I_2.

2. Boundedness of solutions

Consider the autonomous equation (1.4) with positive coefficients a, b. There are three critical points of equation (1.4) at $x_1 = -\sqrt{a/b}$, $x_2 = 0$, $x_3 = \sqrt{a/b}$. The origin is a center and $x_{1,3} = \pm\sqrt{a/b}$ both are saddle points. Two heteroclinic trajectories connect the two saddle points, Fig. 1.

Consider the phase portrait for equation (1.4) depicted in Fig. 1. Introduce the notation G3 for the region bounded by two heteroclinic orbits. It is clear that any trajectory with a points in this region stays in G3 eventually and the respective solutions are bounded. Solutions with a point outside G3 are unbounded.

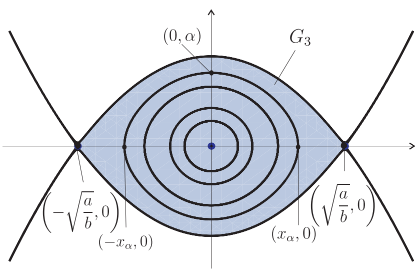

Figure 1: *The phase portrait of equation* (1.4)*, region G3*

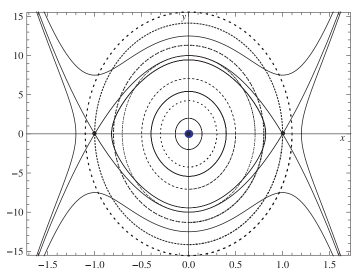

Figure 2: *The phase portrait of equation $x'' = -200x + 200x^3$ and $x'' = -200x$ (dashed)*

Consider now equation (1.6), where $\beta(t)$ is given by (1.7). It appears that for some choices of a, b and δ solutions of the Cauchy problem (1.6), (3.1) escape the region G3. This is possible if after the first stage ($t \in [-1, -1+\delta]$) the respective trajectory switches to a trajectory (dashed) of the equation $x'' = -ax$ (recall that $\beta(t) = 0$ for $t \in (-1+\delta, 1-\delta)$) and leaves the region G3 (Fig. 2). Then on the third stage ($t \geqslant 1-\delta$) the trajectory of equation (1.6) stays outside the region G3 eventually and is therefore unbounded.

The example of such behaviour is given in Fig. 15 and Fig. 16, where $a = 200$, $b = 200$, $\delta = 0.95$.

3. Boundary value problem

Consider now both equations (1.4) and (1.6), given in the interval $[-1,1]$ together with the boundary conditions (1.5). Let us discuss BVPs (1.4), (1.5) and (1.6), (1.5).

3.1. The autonomous equation

Proof. Equation (1.4) has an integral $x'^2(t) = -ax^2(t) + \frac{1}{2}bx^4(t) + C$, where C is an arbitrary constant and solutions $x(t;\alpha)$ of the Cauchy problem (1.4), (3.1)

$$x(-1) = 0, \;\; x'(-1) = \alpha \tag{3.1}$$

satisfy the relation

$$x'^2(t) = -ax^2 + \frac{1}{2}bx^4 + \alpha^2. \tag{3.2}$$

The upper heteroclinic solution satisfies $x'(-1) = a/\sqrt{2b} =: \alpha_{max}$. The time needed for a solution $x(t;\alpha)$ (with $\alpha \in (0,\alpha_{max})$) to pass from $x(-1) = 0$ to the maximal value $x_\alpha = \sqrt{(a-\sqrt{a^2-2b\alpha^2})/b}$ is given by

$$T_\alpha = \int_0^{x_\alpha} \frac{dx}{\sqrt{\frac{1}{2}bx^4 - ax^2 + \alpha^2}}, \quad \text{where} \quad \alpha^2 = ax_\alpha^2 - \frac{1}{2}bx_\alpha^4. \tag{3.3}$$

Therefore

$$x'^2(t) = -ax^2(t) + \frac{1}{2}bx^4(t) + ax_\alpha^2 - \frac{1}{2}bx_\alpha^4, \tag{3.4}$$

$$\frac{dx}{dt} = \pm\sqrt{-ax^2(t) + \frac{1}{2}bx^4(t) + ax_\alpha^2 - \frac{1}{2}bx_\alpha^4}. \tag{3.5}$$

One has, by the variable change $\xi = xx_\alpha^{-1}$, that

$$T_\alpha = \int_0^{x_\alpha} \frac{dx}{\sqrt{-ax^2 + \frac{1}{2}bx^4 - \frac{1}{2}bx_\alpha^4 + ax_\alpha^2}} = \int_0^1 \frac{d\xi}{\sqrt{a\left(1-\xi^2\right) - \frac{1}{2}bx_\alpha^2\left(1-\xi^4\right)}}. \tag{3.6}$$

Compare $T_{\alpha_1} = \int_0^1 d\xi/\sqrt{a\,(1-\xi^2) - 1/2\,bx_{\alpha_1}^2(1-\xi^4)}$ and $T_{\alpha_2} = \int_0^1 d\xi/\sqrt{a(1-\xi^2) - 1/2\,bx_{\alpha_2}^2(1-\xi^4)}$. If $x_{\alpha_1} < x_{\alpha_2}$ then $T_{\alpha_1} < T_{\alpha_2}$. □

LEMMA 1. *The function T_α monotonically increases from $\pi/(2\sqrt{a})$ to $+\infty$ as α changes from zero to α_{max} $(\alpha \in (0,\alpha_{max}))$.*

The number of solutions of the BVP (1.4), (1.5) depends only on a parameter a. The exact number of solutions is given by Theorem 1.

Equation (1.4) written in polar coordinates

$$x(t) = \rho(t)\sin\phi(t), \quad x'(t) = \rho(t)\cos\phi(t) \tag{3.7}$$

turns to a system (3.8):

$$\begin{cases} \phi'(t) = \cos^2\phi(t) + a\sin^2\phi(t) - \rho^2(t)\, b\, \sin^4\phi(t), \\ \rho'(t) = \frac{1}{2}\rho(t)\sin 2\phi(t)\left(1 - a + \rho^2(t)\, b\, \sin^2\phi(t)\right). \end{cases} \tag{3.8}$$

Denote $\phi_\alpha(t)$ the polar function for a solution $x(t,\alpha)$.
The values $(x(1,\alpha),\, x'(1,\alpha))$ form a spiral around the origin and the respective values $\phi_\alpha(1)$ decrease to zero as α changes from zero to α_{max} $(\alpha \in (0,\alpha_{max}))$.

Consider any solution of equation (1.4) with the initial conditions $(x(t_0), x'(t_0)) \in G3$. Let the initial conditions be written as

$$\phi(t_0) = \phi_0, \quad \rho(t_0) = \rho_0, \quad (\phi_0,\rho_0) \in G3, \quad \rho_0 > 0. \tag{3.9}$$

LEMMA 2. *The angular function of any solution of* (1.4), (3.9) *is monotonically increasing.*

Proof. Consider the first equation of system (3.8) multiplied by $\rho^2(t)$

$$\rho^2(t)\phi'(t) = \rho^2(t)\cos^2\phi(t) + a\rho^2(t)\sin^2\phi(t) - b\rho^4(t)\, \sin^4\phi(t). \tag{3.10}$$

Returning to (x,y) coordinates

$$\rho^2(t)\phi'(t) = y^2(t) + ax^2(t) - bx^4(t), \tag{3.11}$$

$$\rho^2(t)\phi'(t) = y^2(t) + x^2(t)(a - bx^2(t)) > 0. \tag{3.12}$$

Since $x^2(t) < \frac{a}{b}$ for any solution of (1.4), (3.9) the angular function $\phi(t)$ is increasing. $\square$

Consider the problem (1.4), (1.5). Due to Lemma 1 the following statement is true.

THEOREM 1. *Let i be a positive integer such that $i\pi/2 < \sqrt{a} < (i+1)\pi/2$. The Dirichlet problem* (1.4), (1.5) *has exactly $2i$ nontrivial solutions such that $x(-1) = 0$, $x'(-1) = \alpha$, $-\alpha_{max} < \alpha < \alpha_{max}$, $\alpha \neq 0$.*

Proof. We consider the Cauchy problem (1.4), (3.1):

$$x'' = -ax + bx^3, \quad x(-1) = 0, \quad x'(-1) = \alpha, \quad 0 < \alpha < \alpha_{max}, \quad \alpha \neq 0. \tag{3.13}$$

It is known [3] that solutions of the equation of variations with respect to the trivial one

$$y'' = -ay \tag{3.14}$$

given with the initial conditions (3.1) approximate solutions of the Cauchy problem (3.13) $x(t,\alpha)$. Solutions of the problem (3.14), (3.1) are given by

$$y(t) = \frac{\alpha}{\sqrt{a}} \sin\sqrt{a}(t+1). \tag{3.15}$$

Due to the assumption $i\,\pi/2 < \sqrt{a} < (i+1)\pi/2$ solutions $y(t)$ along with solutions $x(t,\alpha)$ (for small enough α) have exactly i zeros in the interval $(-1,1)$. These zeros move monotonically to the right as α increases due to Lemma 1. Solutions $x(t,\alpha)$ with $0 < \alpha < \alpha_{max}$ and close enough to α_{max} have not zeros in $(-1,1]$ since the respective trajectories are close to the upper heteroclinic (and the "period" of a heteroclinic solution is infinite). Therefore there are exactly i solutions of the problem (1.4), (1.5). The additional i solutions are obtained considering solutions with $\alpha \in (-\alpha_{max}, 0)$ due to symmetry arguments. Hence the proof. □

REMARK 1. The idea of the proof is to compare the behaviour of solutions with the initial data (3.1) where α varies from 0 to α_{max}. This approach was employed, for example, in the articles [9], [4], [5].

3.2. The nonautonomous equation

We study solutions of boundary value problem (1.6), (1.5) where $\beta(t)$ is a stepwise function given by (1.7). Hence we have the problems

$$\begin{array}{l} x_1'' = -ax_1 + b\,x_1^3, \quad x_1(-1) = 0, \quad x_1'(-1) = \alpha, \quad t \in I_1, \quad \alpha > 0, \\ x_2'' = -a\,x_2, \quad x_2(-1+\delta) = x_1(-1+\delta), \quad x_2(1-\delta) = x_3(1-\delta), \quad t \in I_2, \\ x_3'' = -ax_3 + b\,x_3^3, \quad x_3(1) = 0, \quad x_3'(1) = -\gamma, \quad t \in I_3, \quad \gamma > 0. \end{array} \tag{3.16}$$

To find solutions of the problems we use the Jacobian elliptic functions $\mathrm{sn}\,(u,k)$, $\mathrm{cn}\,(u,k)$, $\mathrm{dn}\,(u,k)$ [10], where u is amplitude and k is elliptic modulus.

The elliptic sine function $\mathrm{sn}\,t$ can be defined as $t = \int\limits_0^{\mathrm{sn}\,t} ds/\sqrt{(1-s^2)(1-k^2s^2)}$ in the interval $\left[0, \int\limits_0^1 ds/\sqrt{(1-s^2)(1-k^2s^2)}\right]$ and extended to infinity by periodicity. The period is $4T$, where $T = \int\limits_0^1 ds/\sqrt{(1-s^2)(1-k^2s^2)}$.

A solution of the Cauchy problem

$$x'' = -ax + bx^3, \quad x(0) = 0, \quad x'(0) = \mu \quad \text{for} \quad a^2 - 2b\,\mu^2 > 0 \tag{3.17}$$

is

$$x(t,\mu) = \mathrm{sign}\,\mu \sqrt{\frac{a - \sqrt{a^2 - 2b\mu^2}}{b}}\ \mathrm{sn}\left(\sqrt{\frac{a + \sqrt{a^2 - 2b\mu^2}}{2}}\,t, \sqrt{\frac{a - \sqrt{a^2 - 2b\mu^2}}{a + \sqrt{a^2 - 2b\mu^2}}}\right). \tag{3.18}$$

Then, using the change of the independent variable, solutions of the problems

$$x'' = -ax + bx^3, \quad x(-1) = 0, \quad x'(-1) = \alpha, \tag{3.19}$$

$$x'' = -ax + bx^3, \quad x(1) = 0, \quad x'(1) = \gamma \tag{3.20}$$

are respectively

$$x_1(t,\alpha) = \operatorname{sign}\alpha \sqrt{\frac{a-\sqrt{a^2-2b\alpha^2}}{b}} \operatorname{sn}\left(\sqrt{\frac{a+\sqrt{a^2-2b\alpha^2}}{2}}\,(t+1), \sqrt{\frac{a-\sqrt{a^2-2b\alpha^2}}{a+\sqrt{a^2-2b\alpha^2}}}\right) \tag{3.21}$$

and

$$x_3(t,\gamma) = \operatorname{sign}\gamma \sqrt{\frac{a-\sqrt{a^2-2b\gamma^2}}{b}} \operatorname{sn}\left(\sqrt{\frac{a+\sqrt{a^2-2b\gamma^2}}{2}}\,(t-1), \sqrt{\frac{a-\sqrt{a^2-2b\gamma^2}}{a+\sqrt{a^2-2b\gamma^2}}}\right), \tag{3.22}$$

where $a^2 - 2b\alpha^2 > 0$ and $a^2 - 2b\gamma^2 > 0$ (this means that trajectories $x_1(t)$ and $x_3(t)$ are located in G3). In equation (3.22) the sign is "$-$" if solutions with even number of zeros in $(-1,1)$ are considered, and, respectively, "$+$" for solutions with odd number of zeros [2].

Denote a solution in the middle interval $x_2(t)$. This solution is a sum of trigonometric functions that connects smoothly solutions $x_1(t)$ and $x_3(t)$.

Introduce

$$\begin{aligned} A(\alpha) &= \sqrt{a-\sqrt{a^2-2b\alpha^2}},\ B(\alpha) = \sqrt{a+\sqrt{a^2-2b\alpha^2}}, \\ A(\gamma) &= \sqrt{a-\sqrt{a^2-2b\gamma^2}},\ \ B(\gamma) = \sqrt{a+\sqrt{a^2-2b\gamma^2}}. \end{aligned} \tag{3.23}$$

Consider solutions of (1.6), (1.5) which have even number of zeros. Then

$$\begin{aligned} x_1(t,\alpha) &= \tfrac{A(\alpha)}{\sqrt{b}} \operatorname{sn}\left(\tfrac{B(\alpha)}{\sqrt{2}}\,(t+1), \tfrac{A(\alpha)}{B(\alpha)}\right), \\ x_3(t,\alpha) &= -\tfrac{A(\gamma)}{\sqrt{b}} \operatorname{sn}\left(\tfrac{B(\gamma)}{\sqrt{2}}\,(t-1), \tfrac{A(\gamma)}{B(\gamma)}\right) \end{aligned} \tag{3.24}$$

and $x_1'(-1) = \alpha$, $x_3'(1) = -\gamma$.

In order $x(t)$ to be C^2-function both solutions x_1 and x_3 are to be smoothly connected by a middle function $x_2(t)$:

$$x_2(t) = C_1 \sin\sqrt{a}t + C_2 \cos\sqrt{a}t. \tag{3.25}$$

The following relations are to be satisfied:

$$\begin{cases} x_1(-1+\delta) = x_2(-1+\delta), \\ x_1'(-1+\delta) = x_2'(-1+\delta), \\ x_3(1-\delta) = x_2(1-\delta), \\ x_3'(1-\delta) = x_2'(1-\delta). \end{cases} \tag{3.26}$$

$$\begin{cases} \frac{A(\alpha)}{\sqrt{b}}\,\mathrm{sn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big) = C_1\sin\sqrt{a}(\delta-1) + C_2\cos\sqrt{a}(\delta-1), \\ \alpha\,\mathrm{cn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big)\,\mathrm{dn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big) \\ \quad = C_1\sqrt{a}\cos\sqrt{a}(\delta-1) - C_2\sqrt{a}\sin\sqrt{a}(\delta-1), \\ \frac{A(\gamma)}{\sqrt{b}}\,\mathrm{sn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big) = -C_1\sin\sqrt{a}(\delta-1) + C_2\cos\sqrt{a}(\delta-1), \\ -\gamma\,\mathrm{cn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big)\,\mathrm{dn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big) \\ \quad = C_1\sqrt{a}\cos\sqrt{a}(\delta-1) + C_2\sqrt{a}\sin\sqrt{a}(\delta-1). \end{cases} \tag{3.27}$$

Solving the system (3.27) with respect to constants C_1 and C_2 we get

$$\begin{cases} \sqrt{a}\cos\sqrt{a}(\delta-1)\big(\frac{A(\alpha)}{\sqrt{b}}\,\mathrm{sn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big) - \frac{A(\gamma)}{\sqrt{b}}\,\mathrm{sn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big)\big) \\ \quad = \sin\sqrt{a}(\delta-1)\big(\alpha\,\mathrm{cn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big)\,\mathrm{dn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big) \\ \qquad -\gamma\,\mathrm{cn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big)\,\mathrm{dn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big)\big), \\ -\sqrt{a}\sin\sqrt{a}(\delta-1)\big(\frac{A(\alpha)}{\sqrt{b}}\,\mathrm{sn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big) + \frac{A(\gamma)}{\sqrt{b}}\,\mathrm{sn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big)\big) \\ \quad = \cos\sqrt{a}(\delta-1)\big(\alpha\,\mathrm{cn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big)\,\mathrm{dn}\,\big(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\big) \\ \qquad +\gamma\,\mathrm{cn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big)\,\mathrm{dn}\,\big(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\big)\big). \end{cases} \tag{3.28}$$

Introduce new notation

$$\begin{aligned} \Phi(\alpha,\gamma) = {} & \sqrt{a}\cos\sqrt{a}(\delta-1)\left(\frac{A(\alpha)}{\sqrt{b}}\,\mathrm{sn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right) - \frac{A(\gamma)}{\sqrt{b}}\right. \\ & \left.\times\,\mathrm{sn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\right) - \sin\sqrt{a}(\delta-1)\left(\alpha\,\mathrm{cn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)\right. \\ & \left.\times\,\mathrm{dn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right) - \gamma\,\mathrm{cn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\mathrm{dn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\right), \end{aligned} \tag{3.29}$$

$$\begin{aligned} \Psi(\alpha,\gamma) = {} & \cos\sqrt{a}(\delta-1)\left(\alpha\,\mathrm{cn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)\mathrm{dn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)\right. \\ & \left.+\gamma\,\mathrm{cn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\mathrm{dn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\right) + \sqrt{a}\sin\sqrt{a}(\delta-1) \\ & \times\left(\frac{A(\alpha)}{\sqrt{b}}\,\mathrm{sn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right) + \frac{A(\gamma)}{\sqrt{b}}\,\mathrm{sn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\right). \end{aligned} \tag{3.30}$$

Then system (3.28) can be rewritten as

$$\begin{cases} \Phi(\alpha,\gamma) = 0, \\ \Psi(\alpha,\gamma) = 0. \end{cases} \tag{3.31}$$

We are interested in the number of solutions of boundary value problem (1.6), (1.5).

PROPOSITION 1. *For a, b and δ given positive solution $(\alpha > 0, \gamma > 0)$ of the system* (3.31) *produces a solution with even number of zeros in $(-1,1)$ of the Dirichlet problem* (1.6), (1.5).

Compatibility of the system (3.31) is shown in the sequel.

Consider solutions of (1.6), (1.5) which have odd number of zeros in $(-1,1)$. Like before we get

$$\begin{aligned}\Phi_1(\alpha,\gamma) = {}& \sqrt{a}\cos\sqrt{a}(\delta-1)\Big(\frac{A(\alpha)}{\sqrt{b}}\,\mathrm{sn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)+\frac{A(\gamma)}{\sqrt{b}}\\ &\times\mathrm{sn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\Big)-\sin\sqrt{a}(\delta-1)\Big(\alpha\,\mathrm{cn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)\\ &\times\mathrm{dn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)+\gamma\,\mathrm{cn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\mathrm{dn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\Big),\end{aligned} \tag{3.32}$$

$$\begin{aligned}\Psi_1(\alpha,\gamma) = {}& -\cos\sqrt{a}(\delta-1)\Big(-\alpha\,\mathrm{cn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)\mathrm{dn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)\\ &+\gamma\,\mathrm{cn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\mathrm{dn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\Big)+\sqrt{a}\sin\sqrt{a}(\delta-1)\\ &\times\Big(\frac{A(\alpha)}{\sqrt{b}}\,\mathrm{sn}\left(\frac{B(\alpha)}{\sqrt{2}}\,\delta,\frac{A(\alpha)}{B(\alpha)}\right)-\frac{A(\gamma)}{\sqrt{b}}\,\mathrm{sn}\left(\frac{B(\gamma)}{\sqrt{2}}\,\delta,\frac{A(\gamma)}{B(\gamma)}\right)\Big).\end{aligned} \tag{3.33}$$

Then we have the system

$$\begin{cases}\Phi_1(\alpha,\gamma)=0,\\ \Psi_1(\alpha,\gamma)=0.\end{cases} \tag{3.34}$$

PROPOSITION 2. *For a, b and δ given positive solution $(\alpha > 0, \gamma > 0)$ of the system* (3.34) *produces a solution with odd number of zeros in $(-1,1)$ of the Dirichlet problem* (1.6), (1.5).

Solvability of the system (3.34) is discussed in the sequel.

If b in the equation (1.4) is replaced by $\beta(t)$ in the equation (1.6) we can state the following assertion.

PROPOSITION 3. *Let i be a positive integer such that $i\,\pi/2 < \sqrt{a} < (i+1)\pi/2$ and $\delta < 1$. For δ close to 1 the Dirichlet problem* (1.6), (1.5) *has at least $2i$ nontrivial solutions such that $x(-1)=0$, $x'(-1)=\alpha$, $-\alpha_{max} < \alpha < \alpha_{max}$, $\alpha \neq 0$. For δ close to 0 the problem* (1.6), (1.5) *has no solutions.*

For $\delta \sim 1$ equation (1.6) is essentially that of (1.4) and the estimate of Theorem 1 is valid for the number of solutions to the problem (1.6), (1.5). If $\delta \sim 0$ equation (1.6) is essentially the linear one $x'' = -ax$ and, since a is not an eigenvalue of the linear problem, there are no solutions of (1.6), (1.5).

Let $\delta \to 0$. At $\delta = 0$ the function $\Phi(\alpha,\gamma)$ and $\Psi(\alpha,\gamma)$ in (3.29), (3.30) are respectively $(\alpha-\gamma)\sin\sqrt{a}$ and $(\alpha+\gamma)\cos\sqrt{a}$. The system (3.31) looks for $\delta = 0$ as

$$\begin{cases} (\alpha-\gamma)\sin\sqrt{a} = 0, \\ (\alpha+\gamma)\cos\sqrt{a} = 0, \end{cases} \tag{3.35}$$

where $\sqrt{a} \neq i\pi/2$, $i-$positive integer. Then the system (3.35) has only the trivial solutions $\alpha = \gamma = 0$ and the BVP has no solutions for δ sufficiently small.

REMARK 2. It was observed numerically that for large enough values of a (for example, $a = 200$) and $\delta \sim 1$ BVP (1.6), (1.5) has more than $2i$ solutions. If $\delta \to 1$ then the equation (1.6) tends to the equation (1.4). By Theorem 1 the number of solution of BVP (1.4), (1.5) is $2i$ for $\sqrt{a}$ between $i\,\pi/2$ and $(i+1)\pi/2$. We have observed however that already for $\delta = 0.95$ this is not the case and the number of nontrivial solutions is greater than $2i$. The reason why the exact number of solutions as in Theorem 1 cannot be guaranteed for the problem (1.6), (1.5) is that the functions $t_1(\alpha) = 2T_\alpha$, $t_2(\alpha) = 4T_\alpha$ and so on, where t_i are the i-th zeros of a solution $x(t;\alpha)$ of the Cauchy problem (1.6), (3.1), can be non-monotone. Equation (1.6) written in polar coordinates (3.7) takes the form:

$$\begin{cases} \phi'(t) = \cos^2\phi(t) + a\sin^2\phi(t) - \rho^2(t)\,\beta(t)\,\sin^4\phi(t), \\ \rho'(t) = \frac{1}{2}\rho(t)\sin 2\phi(t)\big(1 - a + \rho^2(t)\,\beta(t)\,\sin^2\phi(t)\big). \end{cases} \tag{3.36}$$

Denote $\phi_\alpha(t)$ the polar function for a solution $x(t;\alpha)$. The values $(x(1;\alpha),\ x'(1,\alpha))$ form a right spiral around the origin in case of the equation (1.4) and more complicated spiral-like curve in case of equation (1.6). Both curves are depicted for particular values of parameters in Fig. 15 and Fig. 16 respectively. Any point of intersection of these curves with the axis $x = 0$ corresponds to a solution of the BVP (1.6), (1.5).

3.3. Example

Consider equation (1.6) with $a = 200$, $b = 200$:

$$x'' = -200x + \beta(t)x^3, \quad \beta(t) := \begin{cases} 200, & t \in [-1,-1+\delta] =: I_1, \\ 0, & t \in (-1+\delta, 1-\delta) =: I_2, \\ 200, & t \in [1-\delta, 1] =: I_3. \end{cases} \tag{3.37}$$

In what follows we are changing the parameter δ in this way regulating the width of the interval I_2. We are tracing changes in the number of solutions of BVP and discussing reasons for that. We have observed that when $\delta = 0.95$ then the number of solutions exceeds that predicted by Theorem 1 ($i = 9$ in this case because $9\,\pi/2 < \sqrt{200} < 10\pi/2$).

If $\delta = 1$ and initial conditions are $x(-1) = 0$, $x'(-1) = \alpha \neq 0$, $-\alpha_{max} < \alpha < \alpha_{max}$, then equation (3.37) is equation with cubic nonlinearity $x'' = -200x + 200x^3$ and the number of solutions satisfying the boundary conditions (1.5) is 18. For positive initial conditions $0 < \alpha < \alpha_{max}$ the number of solutions is nine.

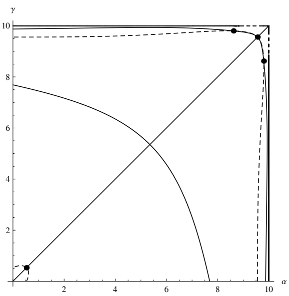

Figure 3: *The trajectory of* $\Phi = 0$ *(solid),* $\Psi = 0$ *(dashed), the points which correspond to solutions of system* (3.31) *and, consequently, to the BVP* (3.37)*,* (1.5)*,* $\delta = 0.7$.

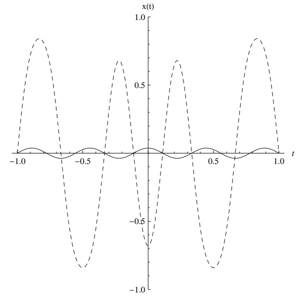

Figure 4: *The symmetric solutions which correspond to the points* $(0.52954, 0.52954)$ *and* $(9.55986, 9.55986)$ *(dashed) in Fig.* 3*,* $\delta = 0.7$.

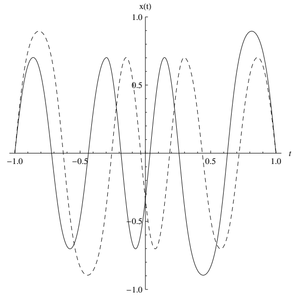

Figure 5: *The asymmetric solutions which correspond to the points* $(8.61987, 9.80176)$ *and* $(9.80176, 8.61987)$ *(dashed) in Fig.* 3*,* $\delta = 0.7$.

Now we look for solutions of the system (3.31) which are intersection points of graphs $\Phi(\alpha,\gamma) = 0$ (solid line) and $\Psi(\alpha,\gamma) = 0$ (dashed line) (Fig. 3).

Let $\delta = 0.7$. There are four solutions of the system (3.31). There are two symmetric solutions of BVP (3.37), (1.5) with even number of zeros on the bisectrix. These solutions are depicted in Fig. 4, but Fig. 5 shows asymmetric solutions that relate to two side points in Fig. 3. Therefore the system (3.31) and problem (3.37), (1.5) have 2 symmetric and 2 asymmetric solutions with even number of zeros in $(-1, 1)$.

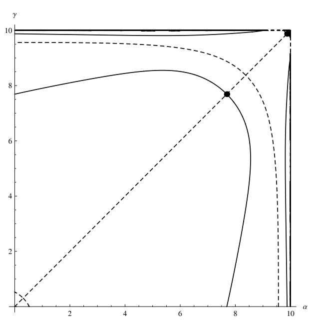

Figure 6: *The trajectory of* $\Phi_1 = 0$ *(solid),* $\Psi_1 = 0$ *(dashed), the points which correspond to solutions of system* (3.34) *and to the problem* (3.37)*,* (1.5)*,* $\delta = 0.7$.

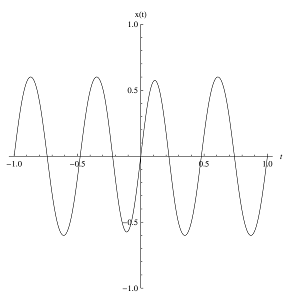

Figure 7: *The symmetric solution which corresponds to the point* $(7.68865, 7.68865)$ *in Fig.* 6*,* $\delta = 0.7$.

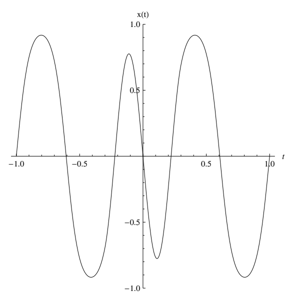

Figure 8: *The symmetric solution which corresponds to the point* $(9.87437, 9.87437)$ *in Fig.* 6*,* $\delta = 0.7$.

In Fig. 6, Fig. 7, Fig. 8 the case of solutions of BVP (3.37), (1.5) with odd number of zeros in $(-1,1)$ is illustrated. Therefore the system (3.31) has 4 solutions, the system (3.34) has 2 solutions. Totally the problem (3.37), (1.5) has 6 solutions (with $\alpha > 0$). Fig. 7 and Fig. 8 show solutions of BVP that relate to points in Fig. 6.

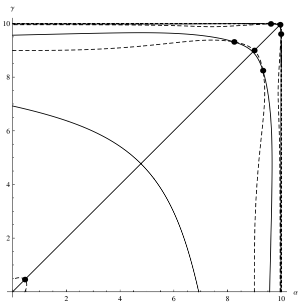

Figure 9: *The trajectory of* $\Phi = 0$ *(solid),* $\Psi = 0$ *(dashed), the points which correspond to solutions of the system* (3.31) *and, respectively, to the BVP* (3.37)*,* (1.5)*,* $\delta = 0.95$.

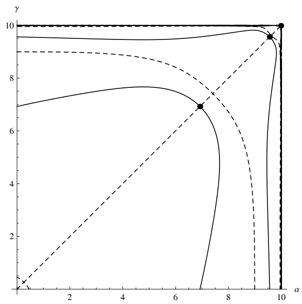

Figure 10: *The trajectory of* $\Phi_1 = 0$ *(solid),* $\Psi_1 = 0$ *(dashed), the points which correspond to solutions of the system* (3.34) *and, respectively, to BVP* (3.37)*,* (1.5)*,* $\delta = 0.95$.

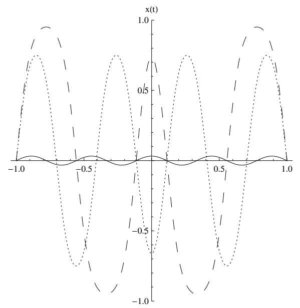

Figure 11: *The symmetric solution which correspond to the points* $(0.455712, 0.455712)$, $(8.99365, 8.99365)$ *(dashing tiny),* $(9.541, 9.541)$ *(dashing large) in Fig.* 9*,* $\delta = 0.95$.

Finally consider the case of $\delta = 0.95$. The system (3.31) and problem (3.37), (1.5) have 3 symmetric (Fig. 11) and four asymmetric solutions with even number of zeros in $(-1,1)$ and these solution are depicted in Fig. 12, Fig. 13 and corresponding points in the Fig. 9 are marked.

The system (3.34) and problem (3.37), (1.5) have 3 symmetric (Fig. 14) solutions with odd number of zeros in the interval (-1,1), corresponding points in the Fig. 10 are depicted. Therefore if $\delta = 0.95$, then the problem (3.37), (1.5) has 10 solutions (with $\alpha > 0$).

In what follows we show how the additional solutions of BVP appear in nonautonomous case comparing with the autonomous one.

The phase trajectories of equations $x'' = -200x + 200x^3$ and $x'' = -200x$ are similar for α small enough but for α large enough trajectories of equation $x'' = -200x$ are stretched along the vertical axis (Fig. 2).

In Fig. 15 and Fig. 16 we compare the behaviours of curves of end-points (at $t = 1$) for autonomous and nonautonomous case. The birth of two additional solutions of BVP is reflected in Fig. 16.

The following figures Fig. 17, Fig. 18 are for the case $a = b = 200$, $\delta = 0.95$.

In Fig. 18 we give the graphs of the polar angle $\phi_\alpha(t)$ for both equations $x'' = -ax + \beta(t)x^3$, (dashed) and $x'' = -ax + bx^3$, $\delta = 0.95$ (solid), $\alpha = 9.985561$. Both curves do not correspond and a non-monotonicity for $\phi_\alpha(t)$ of equation (1.6) is observed (dashed).

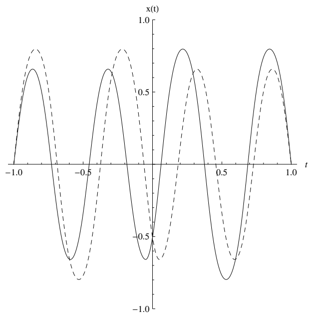

Figure 12: *The asymmetric solutions which correspond to the points* $(8.24265, 9.31334)$ *and* $(9.31334, 8.24265)$ *(dashed) in Fig.* 9*,* $\delta = 0.95$.

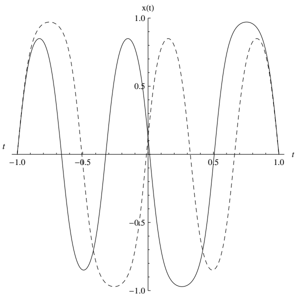

Figure 13: *The asymmetric solutions which correspond to the points* $(9.60487, 9.98353)$ *and* $(9.98353, 9.60487)$ *(dashed) in Fig.* 9*,* $\delta = 0.95$.

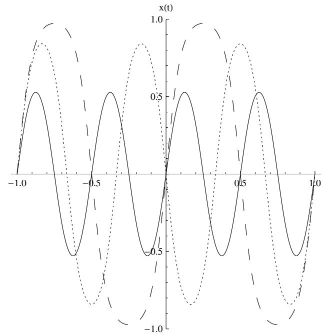

Figure 14: *The symmetric solutions which correspond to the points* $(6.92452, 6.92452)$, $(9.56297, 9.56297)$ *(dashing tiny) and* $(9.98556, 9.98556)$ *(dashing large) in Fig.* 10*,* $\delta = 0.95$.

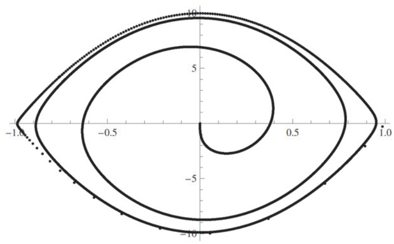

Figure 15: *Curve* $(x(1;\alpha), x'(1;\alpha))$ *for equation* $x'' = -200x + 200x^3$, $0 < \alpha < 10$.

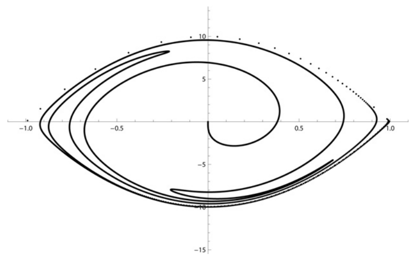

Figure 16: *Curve* $(x(1;\alpha), x'(1;\alpha))$ *for equation* $x'' = -200x + \beta(t)x^3$, $\delta = 0.95$, $0 < \alpha < 10$.

4. Conclusions

The number of solutions of the autonomous problem (1.4), (1.5) is known precisely (Theorem 1) and this number depends on the parameter a only.

For nonautonomous case (1.6), (1.5) additional solutions of BVP are detected for some values of parameters.

It is possible that some solutions of nonautonomous equation (1.6) can escape the region G3 (between the two heteroclinic trajectories). In our example the coefficient $a = 200$ is large enough for a solution to escape $G3$ during small $(\delta \sim 1)$ middle interval $(-1+\delta, 1-\delta)$ where the linear equation $x'' = -ax$ acts.

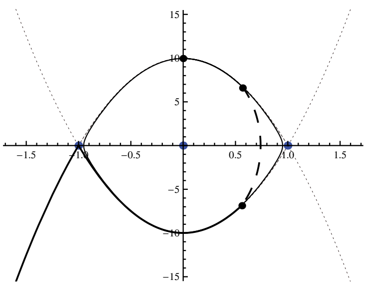

Figure 17: *Phase trajectory escaping the region G3. Equation* $x'' = -ax + \beta(t)x^3$*,* $\alpha = 9.985561$

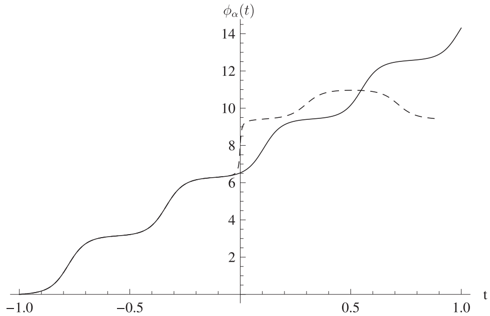

Figure 18: *Graphs of* $\phi_\alpha(t)$ *for equations* $x'' = -ax + \beta(t)x^3$ *(dashed) and* $x'' = -ax + bx^3$ *(solid),* $\alpha = 9.985561$

For nonautonomous case it is possible to use the explicit formulas for solutions and therefore it is possible to evaluate the number of solutions of BVP considering the auxiliary system (3.31) and (3.34).

Finally the non-autonomity may cause the non-monotonicity of the angular functions (Fig. 18) and the zeros $t_i(\alpha)$ and this leads to additional solutions of boundary value problem (1.6), (1.5) comparing with the autonomous (1.4), (1.5).

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