

HÖLDER CONTINUITY UP TO THE BOUNDARY OF SOLUTIONS TO NONLINEAR FOURTH-ORDER ELLIPTIC EQUATIONS WITH NATURAL GROWTH TERMS

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Abstract. In a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we consider the nonlinear fourth-order partial differential equation $\sum_{|\alpha|=1,2} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, Du, D^2u) + B(x, u, Du, D^2u) = 0$. It is assumed that the principal coefficients $\{A_\alpha\}_{|\alpha|=1,2}$ satisfy the growth and coercivity conditions suitable for the energy space $\dot{W}_{2,p}^{1,q}(\Omega) = \dot{W}^{1,q}(\Omega) \cap \dot{W}^{2,p}(\Omega)$, $1 < p < n/2$, $2p < q < n$. The lower-order term $B(x, u, Du, D^2u)$ behaves as $b(u)\{|Du|^q + |D^2u|^p\} + g(x)$ where $g \in L^\tau(\Omega)$, $\tau > n/q$. We establish the Hölder continuity up to the boundary of any solution $u \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ by using the measure density condition on $\partial\Omega$, an interior local result and a modified Moser method with special test function.

1. Introduction

In this paper, we shall deal with nonlinear fourth-order elliptic equations in the divergent form

$$\sum_{|\alpha|=1,2} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla_2 u) + B(x, u, \nabla_2 u) = 0 \quad \text{in } \Omega, \quad (1.1)$$

where Ω is a bounded open set of \mathbb{R}^n , $n \geq 3$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -dimensional multiindex with nonnegative integer components α_i , $i = 1, \dots, n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ and $\nabla_2 u = \{D^\alpha u : |\alpha| = 1, 2\}$.

The coefficients A_α and B for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$ and $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1, 2\}$ satisfy the following strengthened coercivity condition:

$$\sum_{|\alpha|=1,2} A_\alpha(x, s, \xi) \xi_\alpha \geq C \left\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \right\} - f_1(x) \quad (1.2)$$

and the natural (q, p) -growth condition:

$$|B(x, s, \xi)| \leq b(|s|) \left\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \right\} + f_2(x) \quad (1.3)$$

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where $C > 0$, $p \in (1, n/2)$, $q \in (2p, n)$, $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous nondecreasing function and f_i , $i = 1, 2$ are nonnegative functions from $L^\tau(\Omega)$, $\tau > n/q$.

Note that in the case $n > q > mp$, $m \geq 2$, nonlinear elliptic equations in the divergence form

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \dots, D^m u) = 0 \quad \text{in } \Omega, \tag{1.4}$$

with the condition

$$\begin{aligned} \sum_{1 \leq |\alpha| \leq m} A_\alpha(x, \xi) \xi_\alpha \geq c_1 \left\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=m} |\xi_\alpha|^p \right\} \\ - c_2 \sum_{1 < |\alpha| < m} |\xi_\alpha|^{p\alpha} - c_2 |\xi_0|^q - f(x) \end{aligned} \tag{1.5}$$

where c_1, c_2 are positive constants, $f \in L^{t_*}(\Omega)$ with $t_* > 1$, $f \geq 0$, $\{p_\alpha\}$ is a set of exponents and $\xi = \{\xi_\alpha : |\alpha| \leq m\}$, have been introduced in [36], where the boundedness and Hölder continuity have been established for arbitrary weak solutions from

$$W_{m,p}^{1,q}(\Omega) = W^{1,q}(\Omega) \cap W^{m,p}(\Omega) \quad (\dot{W}_{m,p}^{1,q}(\Omega) = \dot{W}^{1,q}(\Omega) \cap \dot{W}^{m,p}(\Omega)).$$

In particular, for $m = 2$, the structure of equation (1.4) with condition (1.5) is determined by the inequality of the form (1.2).

A regularity condition at the boundary (like the Wiener condition) for solutions of equations in the form (1.4), (1.5) was established in [38]. However, equations with the natural growth condition were not considered in [36], [38].

One of the decisive factors in [36] and in subsequent investigations of equations with strengthened coercivity is the fact that, due to the inequality $q > mp$, the chain rule for weak differentiation in $W_{m,p}^{1,q}(\Omega)$ ($\dot{W}_{m,p}^{1,q}(\Omega)$) is valid (see, e.g., [27, Lemma 2.2], [28, Lemma 3.5], [40, Lemmas 3.1, 3.2], Lemma 3.4). Due to this fact, in the case $n > q > mp$, there is an analogy with the known methods in the theory of second-order equations, when the superposition $h(u)$ of the solution u and some specially selected function h is used as a test function. However, the use of this analogy for high order equations encounters a number of significant difficulties. First, the function h must be smoother than a Lipschitz function. Secondly, it is necessary to appropriately take into account the terms associated with the derivatives $h^{(i)}(u)$, $i \geq 2$ in the corresponding integral identities.

The noted mathematical difficulties are further amplified for equations with the natural growth condition, like (1.3). Because in this case, preliminary regularization of the lower-order term is required in order to obtain existence results and the function h should be chosen so as to neutralize the influence of the lower-order term for further considerations. All this is a reason for a separate consideration of equations with natural growing terms. In this regard we refer to [1], [2], [3], [33], [34] for second order equations and, especially, to [39]–[41], where the existence of bounded generalized solutions of the homogeneous Dirichlet problem for equations, like (1.1)–(1.3), is proved. Next, following the approach of [36], appropriately modified, in [42] has been established the interior local result on Hölder continuity of bounded generalized solutions of the same equation. For existence results of unbounded solutions to fourth-order

equations with natural growth condition and L^1 -data see [12]. We also cite papers [11], [14], [15], [16], [17], [18] on the regularity of solutions of second-order equations with L^1 -data.

In this paper, we establish Hölder continuity up to the boundary of any solution $u \in \dot{W}_{m,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ to equation (1.1) under the conditions (1.2), (1.3). The proof follows [42] with additional modifications of test functions near the boundary $\partial\Omega$, which are suggested by the papers [8], [29] and [42]. In particular, due to the chain rule (see below Lemma 3.4), we can use (see below Lemma 4.1)

$$v = \zeta^q(\exp(\lambda|u|) - 1) \operatorname{sign} u, \quad \zeta \in C_0^\infty(B_\rho), \quad B_\rho \cap \partial\Omega \neq \emptyset$$

as a test function to obtain the estimate

$$\begin{aligned} \int_{\Omega \cap B_\rho} \left(\sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p \right) \zeta^q dx \\ \leq c \rho^n \left(1 + \rho^{-q} + \max_{B_\rho} \left\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \right\} \right), \end{aligned}$$

which is the key to the proof. In this case, a suitable choice of λ makes it possible to neutralize the lower order term.

We remark that the Hölder regularity in particular subsets of Ω for solutions of nonlinear fourth-order elliptic equations with the strengthened coercivity condition and L^1 -right-hand sides was studied in [7]. In addition, the integral functionals in the form

$$I(u) = \int_{\Omega} \{A(x, \nabla_2 u) + A_0(x, u)\} dx$$

were considered in [6], [9]. It was assumed that $A(x, \xi)$ is convex respect to ξ and satisfies the following growth condition: for almost every $x \in \Omega$ and for every $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1, 2\}$,

$$\begin{aligned} \tilde{c}_1 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} - \tilde{f}(x) \leq A(x, \xi) \\ \leq \tilde{c}_2 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} + \tilde{f}(x), \end{aligned}$$

where \tilde{c}_1, \tilde{c}_2 are positive constants, \tilde{f} is a nonnegative function belonging to suitable Lebesgue space and ν, μ are positive measurable functions. The boundedness and Hölder regularity of minimizers to such functionals was obtained by using a modified Moser method with a special test function.

Finally, principal results on continuity at boundary points of solutions to nonlinear second-order elliptic equations were established in [22], [23], [26], [32], [37]. For the local interior results see, for example, [10], [13], [30], [31], [33], [34] in the case of second-order equations and systems, as well as [19], [20], [21] for high-order elliptic equations with strengthened coercivity. Moreover, the papers [4], [5], [10], [13], [22], [23], [30], [33], [34] cover the second-order equations with natural growth conditions.

The present paper is organized as follows. In Section 2 we formulate the hypotheses, we state our problem and the main result. Section 3 consists of preliminary Lemmas needed to prove the main result. In Section 4 we prove the main result on the Hölder continuity up to the boundary of solutions of the Dirichlet problem for equation (1.1) assuming that the lower-order term has the natural (q, p) -growth.

2. Main result

Let $n \in \mathbb{N}$, $n > 2$, and let Ω be a bounded open set of \mathbb{R}^n . We shall use the following notation: $\mathbb{R}_+ = [0, +\infty)$; $\partial\Omega$ is the boundary of Ω , $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω ; $B_\rho(y) := \{x \in \mathbb{R}^n : |x - y| < \rho\}$ is the open ball with center y and radius $\rho > 0$; when not important, we shall omit denoting the center as follows: $B_\rho \equiv B_\rho(y)$; Λ is the set of all n -dimensional multi-indices α such that $|\alpha| = 1$ or $|\alpha| = 2$; $\mathbb{R}^{n,2}$ is the space of all mappings $\xi : \Lambda \rightarrow \mathbb{R}$; if $u \in W^{2,1}(\Omega)$, then $\nabla_2 u : \Omega \rightarrow \mathbb{R}^{n,2}$, and for every $x \in \Omega$ and for every $\alpha \in \Lambda$, $(\nabla_2 u(x))_\alpha = D^\alpha u(x)$. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ is the n -dimensional Lebesgue measure of the set E . If $\tau \in [1, +\infty]$, then $\|\cdot\|_{L^\tau(E)}$ is the norm in the usual Lebesgue space $L^\tau(E)$.

Let $p \in (1, n/2)$ and $q \in (2p, n)$. We denote by $W_{2,p}^{1,q}(\Omega)$ the set of all functions in $W^{1,q}(\Omega)$ that have the second-order generalized derivatives in $L^p(\Omega)$. The set $W_{2,p}^{1,q}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \left(\sum_{|\alpha|=2} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}.$$

We denote by $\mathring{W}_{2,p}^{1,q}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W_{2,p}^{1,q}(\Omega)$.

We consider the equation

$$\sum_{\alpha \in \Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla_2 u) + B(x, u, \nabla_2 u) = 0 \quad \text{in } \Omega \tag{2.1}$$

under the following assumptions:

(A1) For every $\alpha \in \Lambda$, $A_\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \rightarrow \mathbb{R}$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \rightarrow \mathbb{R}$ are Carathéodory functions, i.e. for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^{n,2}$, the functions $A_\alpha(\cdot, s, \xi)$ and $B(\cdot, s, \xi)$ are measurable on Ω and, for almost every $x \in \Omega$, the functions $A_\alpha(x, \cdot, \cdot)$ and $B(x, \cdot, \cdot)$ are continuous in $\mathbb{R} \times \mathbb{R}^{n,2}$.

(A2) For almost every $x \in \Omega$ and for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^{n,2}$ the following inequalities hold:

$$\sum_{\alpha \in \Lambda} A_\alpha(x, s, \xi) \xi_\alpha \geq a(|s|) \left\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \right\} - g_0(x), \tag{2.2}$$

$$\sum_{|\alpha|=1} |A_\alpha(x, s, \xi)|^{q/(q-1)} \leq a_1(|s|) \left\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \right\} + g_1(x), \tag{2.3}$$

$$\sum_{|\alpha|=2} |A_\alpha(x, s, \xi)|^{p/(p-1)} \leq a_2(|s|) \left\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \right\} + g_2(x), \tag{2.4}$$

$$|B(x, s, \xi)| \leq b(|s|) \left\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \right\} + g_3(x), \tag{2.5}$$

where $a : \mathbb{R}_+ \rightarrow (0, +\infty)$ is a continuous nonincreasing function, $a_1, a_2, b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous nondecreasing functions, g_0, g_1, g_2, g_3 are nonnegative summable functions on Ω .

DEFINITION 1. A generalized solution of (2.1) is a function $u \in W_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ such that for every function $v \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \left\{ \sum_{\alpha \in \Lambda} A_\alpha(x, u, \nabla_2 u) D^\alpha v + B(x, u, \nabla_2 u) v \right\} dx = 0. \tag{2.6}$$

The existence of a solution $u \in \mathring{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ for equation (2.1) is proved in [39], [40] under additional assumptions on the coefficients $\{A_\alpha\}_{\alpha \in \Lambda}$ and B and on the functions g_0, g_1, g_2, g_3 . In particular, it was assumed that

$$(A3) \quad g_0, g_1, g_2, g_3 \in L^\tau(\Omega), \quad \tau > n/q.$$

The local Hölder continuity in Ω of every solution $u \in W_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ to (2.1) under the assumptions (A1), (A2), (A3) is proved in [42].

Examples illustrating the fulfilment of the assumptions (A1) and (A2) are given in [42, Examples 2.4–2.6].

The main result of the present article is a theorem on the Hölder continuity up to the boundary of any generalized solution $u \in \mathring{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ to equation (2.1) under assumptions (A1), (A2), (A3) and the following assumption on $\partial\Omega$:

$$(A4) \quad \text{there exist } c_*, R_* > 0 \text{ such that for every } y \in \partial\Omega \text{ and } R \in (0, R_*],$$

$$|B_R(y) \setminus \Omega| \geq c_* |B_R(y)|. \tag{2.7}$$

THEOREM 1. Assume that conditions (A1), (A2), (A3) and (A4) are satisfied. Let $u \in \mathring{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ be a generalized solution of equation (2.1) and $M = \|u\|_{L^\infty(\Omega)}$. Then there exists a function $\tilde{u} : \overline{\Omega} \rightarrow \mathbb{R}$ such that $\tilde{u} = u$ a. e. in Ω and for every $x, y \in \overline{\Omega}$,

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C|x - y|^\varepsilon,$$

where the positive constants C and ε depend only on $n, p, q, \tau, R_*, |\Omega|, M, a(M), a_1(M), a_2(M), b(M)$ and $\max_{0 \leq i \leq 3} \|g_i\|_\tau$.

3. Auxiliary results

The following is the well-known Sobolev inequality for functions in $\dot{W}^{1,q}(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^n$; see for example [24, Theorem 7.10].

LEMMA 3.1. *Set $q^* = nq/(n - q)$. Let \mathcal{O} be a bounded open set in \mathbb{R}^n . Then $\dot{W}^{1,q}(\mathcal{O}) \subset L^{q^*}(\mathcal{O})$. Furthermore, there exists a positive constant $c_{n,q}$ depending only on n and q such that, for every function $u \in \dot{W}^{1,q}(\mathcal{O})$,*

$$\left(\int_{\mathcal{O}} |u|^{q^*} dx \right)^{1/q^*} \leq c_{n,q} \left(\sum_{|\alpha|=1} \int_{\mathcal{O}} |D^\alpha u|^q dx \right)^{1/q}. \tag{3.1}$$

The proof of the following lemma is given in [35, Chapter 1, §2, Lemma 4].

LEMMA 3.2. *Let $f \in W^{1,q}(B_\rho)$. Suppose there exists a measurable subset $G \subset B_\rho$ and positive constants C' and C'' such that*

$$|G| \geq C' \rho^n, \quad \max_G |f| \leq C''.$$

Then

$$\int_{B_\rho} |f|^q dx \leq C \rho^q \left(\sum_{|\alpha|=1} \int_{B_\rho} |D^\alpha f|^q dx + \rho^{n-q} \right)$$

where C is a positive constant depending only on n, q, C', C'' .

The following lemma is due to John and Nirenberg [25] (see also [24, Theorem 7.21]).

LEMMA 3.3. *Let $f \in W^{1,1}(\mathcal{O})$ where \mathcal{O} is a convex domain in \mathbb{R}^n . Suppose there exists a positive constant K such that*

$$\sum_{|\alpha|=1} \int_{\mathcal{O} \cap B_\rho} |D^\alpha f| dx \leq K \rho^{n-1} \quad \text{for all balls } B_\rho.$$

Then there exist positive constants σ_0 and C depending only on n such that

$$\int_{\mathcal{O}} \exp \left(\frac{\sigma}{K} |f - (f)_\mathcal{O}| \right) dx \leq C (\text{diam } \mathcal{O})^n$$

where $\sigma = \sigma_0 |\mathcal{O}| (\text{diam } \mathcal{O})^{-n}$, $(f)_\mathcal{O} = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} f dx$.

LEMMA 3.4. *Let h be an odd function on \mathbb{R} such that $h \in C^1(\mathbb{R})$, $h \in C^2(\mathbb{R} \setminus \{0\})$ and h' has a discontinuity of the first kind at the origin. Let $u \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$. Then $h(u) \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:*

(i) *for every n -dimensional multi-index α , $|\alpha| = 1$,*

$$D^\alpha h(u) = h'(u) D^\alpha u \quad \text{a.e. in } \Omega;$$

(ii) for every n -dimensional multi-index α , $|\alpha| = 2$,

$$D^\alpha h(u) = h'(u)D^\alpha u + h''(u)D^\beta u D^\gamma u \mathbb{I}_{\{u \neq 0\}} \quad \text{a.e. in } \Omega,$$

where $\alpha = \beta + \gamma$, $|\beta| = |\gamma| = 1$ and $\mathbb{I}_{\{u \neq 0\}}$ is the indicator of the set $\{u \neq 0\}$.

For the proof of this fact, see [40, Lemma 3.2].

The following result is discussed in [24, Lemma 8.23].

LEMMA 3.5. Let ω be a non-decreasing function on an interval $(0, R_0]$ satisfying, for all $R \leq R_0$, the inequality

$$\omega(\vartheta R) \leq \theta \omega(R) + \varphi(R)$$

where φ is also non-decreasing function and $0 < \vartheta, \theta < 1$. Then, for any $\delta \in (0, 1)$ and $R \leq R_0$, we have

$$\omega(R) \leq C \left(\left(\frac{R}{R_0} \right)^\varepsilon \omega(R_0) + \varphi(R^\delta R_0^{1-\delta}) \right)$$

where $C = C(\vartheta, \theta)$ and $\varepsilon = \varepsilon(\vartheta, \theta, \delta)$ are positive constants.

4. Proof of Theorem 1

Suppose that the assumptions (A1)–(A4) are satisfied. Let $u \in \mathring{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ be a generalized solution of equation (2.1). Denote by \bar{u} the extension of u to 0 on $\mathbb{R}^n \setminus \Omega$. From the definition of $\mathring{W}_{2,p}^{1,q}(\Omega)$ it follows that for any open bounded set $\Omega' \supset \bar{\Omega}$, $\bar{u} \in \mathring{W}_{2,p}^{1,q}(\Omega')$ and for every $\alpha \in \Lambda$, $D^\alpha \bar{u} = D^\alpha u$ a.e. in Ω and $D^\alpha \bar{u} = 0$ a.e. in $\Omega' \setminus \Omega$. We set $M = \|u\|_{L^\infty(\Omega)}$, thus

$$|\bar{u}| \leq M < +\infty \quad \text{a.e. in } \mathbb{R}^n. \tag{4.1}$$

By c_i , $i = 0, 1, \dots$, we shall denote positive constants depending only on

$$\text{data} \equiv (n, p, q, \tau, R_*, |\Omega|, M, a(M), a_1(M), a_2(M), b(M), \max_{0 \leq i \leq 3} \|g_i\|_\tau).$$

Fixing $x_0 \in \partial\Omega$ for every $R > 0$ we set $\Omega_R(x_0) = \Omega \cap B_R(x_0)$,

$$\mu(R) = \text{ess inf}_{B_R(x_0)} \bar{u}, \quad M(R) = \text{ess sup}_{B_R(x_0)} \bar{u}, \quad \omega(R) = M(R) - \mu(R).$$

By (2.7), it is obvious that $\mu(R) \leq 0$ and $M(R) \geq 0$ for every $R > 0$. We fix a positive number r such that

$$r < \min \left\{ 1 - \frac{n}{q\tau}, \frac{q-2p}{q-p} \right\}. \tag{4.2}$$

For every $R \in (0, \min\{1, \frac{1}{2}R_*\}]$ we shall establish the inequality

$$\omega(R) \leq c_0 \omega(2R) + R^r \tag{4.3}$$

with $c_0 \in (0, 1)$. This inequality, Lemma 3.5 and [42, Theorem 2.3] imply the validity of Theorem 1.

To prove (4.3), we fix $R \in (0, \min\{1, \frac{1}{2}R_*\}]$ and define a function $\bar{v}_0 : B_{2R}(x_0) \rightarrow \mathbb{R}$ as follows:

$$\bar{v}_0(x) = \begin{cases} \ln \frac{2e\omega(2R)}{M(2R) - \bar{u}(x) + R^r} & \text{if } M(2R) \geq \omega(2R)/2, \\ \ln \frac{2e\omega(2R)}{\bar{u}(x) - \mu(2R) + R^r} & \text{if } M(2R) < \omega(2R)/2. \end{cases} \tag{4.4}$$

We denote by v_0 the restriction of the function \bar{v}_0 on $\Omega_{2R}(x_0)$. It is easy to see that (4.3) follows from the estimate

$$\|\bar{v}_0\|_{L^\infty(B_R(x_0))} \leq c_1 \tag{4.5}$$

or, which is the same thing, from the inequality $\|v_0\|_{L^\infty(\Omega_R(x_0))} \leq c_1$. For definiteness we assume that the function \bar{v}_0 is defined by the first line in (4.4). We can also assume that

$$\omega(2R) \geq R^r, \tag{4.6}$$

and therefore, $\bar{v}_0 \geq 1$ a.e. in $B_{2R}(x_0)$, otherwise inequality (4.3) holds.

To derive inequality (4.5), we need some integral estimates of the function \bar{u} and its derivatives in the balls B_ρ that intersect $\partial\Omega$. We set

$$\Phi = \sum_{|\alpha|=1} |D^\alpha \bar{u}|^q + \sum_{|\alpha|=2} |D^\alpha \bar{u}|^p.$$

LEMMA 4.1. *Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be a function such that*

$$\zeta = 0 \text{ in } \mathbb{R}^n \setminus B_\rho \text{ and } 0 \leq \zeta \leq 1. \tag{4.7}$$

Then there exists a positive constant c_2 such that

$$\int_{B_\rho} \Phi \zeta^q dx \leq c_2 \rho^n \left(1 + \rho^{-q} + \max_{B_\rho} \left\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \right\} \right). \tag{4.8}$$

Proof. We assume that $B_\rho \cap \Omega \neq \emptyset$, otherwise $\Phi \equiv 0$ in B_ρ and (4.8) is trivial. We set $\lambda = b(M)/a(M)$ and define the function $h : \mathbb{R} \rightarrow \mathbb{R}$ by

$$h(s) = (e^{\lambda|s|} - 1) \operatorname{sign} s, \quad s \in \mathbb{R}.$$

Elementary calculations show that

$$a(M)h' - b(M)|h| = b(M) \quad \text{in } \mathbb{R}. \tag{4.9}$$

For every $x \in \Omega$ we set

$$v_1(x) = h(u(x)) \zeta^q(x).$$

By Lemma 3.4, $h(u) \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$. Therefore, $v_1 \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:

(a) for every n -dimensional multi-index α , $|\alpha| = 1$,

$$D^\alpha v_1 = h'(u)D^\alpha u \zeta^q + qh(u)\zeta^{q-1}D^\alpha \zeta \quad \text{a.e. in } \Omega,$$

(b) for every n -dimensional multi-index α , $|\alpha| = 2$,

$$\begin{aligned} & |D^\alpha v_1 - h'(u)D^\alpha u \zeta^q| \\ & \leq |h''(u)| \left\{ \sum_{|\beta|=1} |D^\beta u|^2 \right\} \mathbb{I}_{\{u \neq 0\}} \zeta^q + 2qh'(u) \left\{ \sum_{|\beta|=1} |D^\beta u| \right\} \left\{ \sum_{|\beta|=1} |D^\beta \zeta| \right\} \zeta^{q-1} \\ & \quad + q(q-1)|h(u)| \left\{ \sum_{|\beta|=1} |D^\beta \zeta|^2 \right\} \zeta^{q-2} + q|h(u)| |D^\alpha \zeta| \zeta^{q-1} \quad \text{a.e. in } \Omega. \end{aligned}$$

By (2.6), we have

$$\int_{\Omega} \left\{ \sum_{\alpha \in \Lambda} A_\alpha(x, u, \nabla_2 u) D^\alpha v_1 + B(x, u, \nabla_2 u) v_1 \right\} dx = 0.$$

From this equality, using (2.2), (2.5), (4.1), (4.7), and assertions (a) and (b), we deduce that

$$\int_{\Omega_p} \Phi(a(M)h'(u) - b(M)|h(u)|) \zeta^q dx \leq \sum_{i=1}^4 I_i + e^{\lambda M} \int_{\Omega_p} (g_3 + \lambda g_0) \zeta^q dx$$

where $\Omega_p = \Omega \cap B_p$,

$$\begin{aligned} I_1 &= q \sum_{\alpha \in \Lambda} \int_{\Omega_p} |A_\alpha(x, u, \nabla_2 u)| |D^\alpha \zeta| |h(u)| \zeta^{q-1} dx, \\ I_2 &= \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{\Omega_p} |A_\alpha(x, u, \nabla_2 u)| |D^\beta u|^2 |h''(u)| \mathbb{I}_{\{u \neq 0\}} \zeta^q dx, \\ I_3 &= q^2 \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{\Omega_p} |A_\alpha(x, u, \nabla_2 u)| |D^\beta \zeta|^2 |h(u)| \zeta^{q-2} dx, \\ I_4 &= 2q \sum_{|\alpha|=2} \sum_{|\beta|=1} \sum_{|\gamma|=1} \int_{\Omega_p} |A_\alpha(x, u, \nabla_2 u)| |D^\beta u| |D^\gamma \zeta| |h'(u)| \zeta^{q-1} dx. \end{aligned}$$

From this and (4.9) it follows that

$$b(M) \int_{\Omega_p} \Phi \zeta^q dx \leq I_1 + I_2 + I_3 + I_4 + e^{\lambda M} \int_{\Omega_p} (g_3 + \lambda g_0) \zeta^q dx. \tag{4.10}$$

Further, to obtain suitable estimates for the terms on the right-hand side of (4.10), we argue similarly to the proof of [42, Lemma 4.1].

Estimate for I_1 . Using the Young's inequality with the exponents $q/(q-1)$ and q , (2.3), (4.1) and (4.7), we obtain

$$\begin{aligned} & q \sum_{|\alpha|=1} \int_{\Omega_p} |A_\alpha(x, u, \nabla_2 u)| |D^\alpha \zeta| |h(u)| \zeta^{q-1} dx \\ & \leq \frac{b(M)}{16} \int_{\Omega_p} \Phi \zeta^q dx + c_3 \int_{\Omega_p} g_1 dx + c_3 \rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^q. \end{aligned}$$

Using the Young's inequality with the exponents $p/(p-1)$ and p , (2.4), (4.1) and (4.7), we obtain

$$\begin{aligned} & q \sum_{|\alpha|=2} \int_{\Omega_p} |A_\alpha(x, u, \nabla_2 u)| |D^\alpha \zeta| |h(u)| \zeta^{q-1} dx \\ & \leq \frac{b(M)}{16} \int_{\Omega_p} \Phi \zeta^q dx + c_4 \int_{\Omega_p} g_2 dx + c_4 \rho^n \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p. \end{aligned}$$

Summing the last two inequalities, we obtain

$$I_1 \leq \frac{b(M)}{8} \int_{\Omega_p} \Phi \zeta^q dx + c_5 \int_{\Omega_p} (g_1 + g_2) dx + c_5 \rho^n \max_{B_\rho} \Phi_\zeta \tag{4.11}$$

where

$$\Phi_\zeta = \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p.$$

Estimates for I_2, I_3 and I_4 . It is obvious that

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1, \quad q-1 = (p-1)\frac{q}{p} + \left(\frac{q}{p}-1\right). \tag{4.12}$$

Using this equalities, the Young's inequality, (2.4), (4.1) and (4.7), we obtain

$$I_2 \leq \frac{b(M)}{8} \int_{\Omega_p} \Phi \zeta^q dx + c_6 \int_{\Omega_p} g_2 dx + c_6 \rho^n, \tag{4.13}$$

$$I_3 \leq \frac{b(M)}{8} \int_{\Omega_p} \Phi \zeta^q dx + c_7 \int_{\Omega_p} g_2 dx + c_7 \rho^n \max_{B_\rho} \Phi_\zeta + c_7 \rho^n, \tag{4.14}$$

$$I_4 \leq \frac{b(M)}{8} \int_{\Omega_p} \Phi \zeta^q dx + c_8 \int_{\Omega_p} g_2 dx + c_8 \rho^n \max_{B_\rho} \Phi_\zeta + c_8 \rho^n. \tag{4.15}$$

From (4.10), (4.11), (4.13)–(4.15) it follows that

$$\frac{b(M)}{2} \int_{\Omega_p} \Phi \zeta^q dx \leq c_9 \left(\int_{\Omega_p} g dx + \rho^n \max_{B_\rho} \Phi_\zeta + \rho^n \right)$$

where $g = g_0 + g_1 + g_2 + g_3$. By Hölder's inequality and the inequality $\tau > n/q$ we have

$$\int_{\Omega_p} g dx \leq \|g\|_{L^\tau(\Omega)} |B_\rho|^{(\tau-1)/\tau} \leq c_{10} \rho^{n-q}.$$

The last two inequalities and (4.1) imply inequality (4.8). The proof is complete. \square

LEMMA 4.2. Let $B_\rho \subset B_{2R}(x_0)$ and let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be a function such that condition (4.7) be satisfied. Then there exists a positive constant c_{11} such that

$$\begin{aligned} & \int_{B_\rho} \frac{\Phi \zeta^q dx}{(M(2R) - \bar{u} + R^r)^q} \\ & \leq c_{11} \rho^n \left(\rho^{-q} + \max_{B_\rho} \left\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \right\} \right. \\ & \quad \left. + \rho^{2p-q} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p + \rho^{-q(q-2p)/(q-p)} \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^{qp/(q-p)} \right). \end{aligned} \tag{4.16}$$

In particular, if $|D^\alpha \zeta| \leq K\rho^{-|\alpha|}$ for any $\alpha \in \Lambda$, then

$$\int_{B_\rho} \frac{\Phi \zeta^q dx}{(M(2R) - \bar{u} + R^r)^q} \leq c_{12} (K + 1) \rho^{n-q}. \tag{4.17}$$

Proof. We assume that $B_\rho \cap \Omega \neq \emptyset$, otherwise $\Phi \equiv 0$ in B_ρ and (4.16) is trivial. For every $x \in B_{2R}(x_0)$, we set $U(x) = M(2R) - \bar{u}(x) + R^r$. Note that due to the inequality $M(2R) \geq \omega(2R)/2$ we have

$$|U^{1-q} - (M(2R) + R^r)^{1-q}| \leq 2^{q-1} U^{1-q} \quad \text{a.e. in } \Omega_{2R}(x_0). \tag{4.18}$$

We define the function $v_2 : \Omega \rightarrow \mathbb{R}$ by

$$v_2(x) = \begin{cases} [U^{1-q}(x) - (M(2R) + R^r)^{1-q}] \zeta^q(x) & \text{if } x \in \Omega_{2R}(x_0), \\ 0 & \text{if } x \in \Omega \setminus B_{2R}(x_0). \end{cases}$$

Simple calculations show that $v_2 \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$ and due to (4.7) and (4.18) the following assertions hold:

(c) for every n -dimensional multi-index α , $|\alpha| = 1$,

$$|D^\alpha v_2 - (q - 1)U^{-q} \zeta^q D^\alpha u| \leq q2^{q-1} U^{1-q} \zeta^{q-1} |D^\alpha \zeta| \quad \text{a.e. in } \Omega \cap B_\rho,$$

(d) for every n -dimensional multi-index α , $|\alpha| = 2$,

$$\begin{aligned} & \left| D^\alpha v_2 - (q - 1)U^{-q} \zeta^q D^\alpha u \right| \\ & \leq q2^{q-1} U^{1-q} \zeta^{q-1} |D^\alpha \zeta| + q(q - 1)2^{q-1} U^{1-q} \zeta^{q-2} \sum_{|\beta|=1} |D^\beta \zeta|^2 \\ & \quad + 2q(q - 1)U^{-q} \zeta^{q-1} \left\{ \sum_{|\beta|=1} |D^\beta u| \right\} \left\{ \sum_{|\beta|=1} |D^\beta \zeta| \right\} \\ & \quad + q(q - 1)U^{-1-q} \zeta^q \sum_{|\beta|=1} |D^\beta u|^2 \quad \text{a.e. in } \Omega \cap B_\rho. \end{aligned}$$

In virtue of (2.6) we have

$$\int_\Omega \left\{ \sum_{\alpha \in \Lambda} A_\alpha(x, u, \nabla_2 u) D^\alpha v_2 + B(x, u, \nabla_2 u) v_2 \right\} dx = 0.$$

Hence by using (2.2), (2.5), (4.1), (4.7), (4.18) and assertions (c) and (d), we obtain

$$a(\mathbf{M}) \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx \leq c_{13} \left(\sum_{i=1}^5 I'_i + \int_{\Omega_\rho} (g_0 + g_3) U^{-q} dx \right) \quad (4.19)$$

where $\Omega_\rho = \Omega \cap B_\rho$,

$$\begin{aligned} I'_1 &= \sum_{\alpha \in \Lambda} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)| |D^\alpha \zeta| U^{1-q} \zeta^{q-1} dx, \\ I'_2 &= \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)| |D^\beta u|^2 U^{1-q} \zeta^q dx, \\ I'_3 &= \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)| |D^\beta \zeta|^2 U^{1-q} \zeta^{q-2} dx, \\ I'_4 &= \sum_{|\alpha|=2} \sum_{|\beta|=1} \sum_{|\gamma|=1} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)| |D^\beta u| |D^\gamma \zeta| U^{-q} \zeta^{q-1} dx, \\ I'_5 &= \int_{\Omega_\rho} \Phi U^{1-q} \zeta^q dx. \end{aligned}$$

Further, to obtain suitable estimates for the terms on the right-hand side of (4.19), we argue similarly to the proof of [42, Lemma 4.2].

Estimate for I'_1 . Using the Young's inequality with the exponents $q/(q-1)$ and q , (2.3), (4.1) and (4.7), we obtain

$$\begin{aligned} & \sum_{|\alpha|=1} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)| |D^\alpha \zeta| U^{1-q} \zeta^{q-1} dx \\ & \leq \frac{a(\mathbf{M})}{20c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{14} \int_{\Omega_\rho} g_1 U^{-q} dx + c_{14} \rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^q. \end{aligned} \quad (4.20)$$

We use the Young's inequality with the exponents $p/(p-1)$ and p , (2.4), (4.1) and (4.7) to obtain

$$\begin{aligned} & \sum_{|\alpha|=2} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)| |D^\alpha \zeta| U^{1-q} \zeta^{q-1} dx \\ & \leq \frac{a(\mathbf{M})}{20c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{15} \int_{\Omega_\rho} g_2 U^{-q} dx + c_{15} \sum_{|\alpha|=2} \int_{\Omega_\rho} |D^\alpha \zeta|^p U^{p-q} \zeta^{q-p} dx, \end{aligned}$$

whence, taking into account the inequalities $U \geq R^r$, $\rho/2 < R < 1$ and (4.2), we derive

$$\begin{aligned} & \sum_{|\alpha|=2} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)| |D^\alpha \zeta| U^{1-q} \zeta^{q-1} dx \\ & \leq \frac{a(\mathbf{M})}{20c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{15} \int_{\Omega_\rho} g_2 U^{-q} dx + c_{16} \rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p. \end{aligned} \quad (4.21)$$

Summing inequalities (4.20) and (4.21), we obtain

$$\begin{aligned}
 I'_1 \leq & \frac{a(M)}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{17} \int_{\Omega_\rho} (g_1 + g_2) U^{-q} dx \\
 & + c_{14} \rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^q + c_{16} \rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p.
 \end{aligned} \tag{4.22}$$

Estimate for I'_2 . We use (4.1), the first equality in (4.12), Young’s inequality, (2.4) and (4.7) to obtain

$$I'_2 \leq \frac{a(M)}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{18} \int_{\Omega_\rho} g_2 U^{-q} dx + c_{18} \int_{\Omega_\rho} U^{-q(q-p)/(q-2p)} \zeta^q dx.$$

Estimating the last integral in this inequality by means of the inequalities $U \geq R'$, $\rho/2 < R < 1$ and (4.2), we obtain

$$I'_2 \leq \frac{a(M)}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{18} \int_{\Omega_\rho} g_2 U^{-q} dx + c_{19} \rho^{n-q}. \tag{4.23}$$

Estimates for I'_3 and I'_4 . Using the reasoning similar the proof of (4.23), we obtain

$$\begin{aligned}
 I'_3 \leq & \frac{a(M)}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{20} \int_{\Omega_\rho} g_2 U^{-q} dx \\
 & + c_{20} \rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^q + c_{20} \rho^{n-q},
 \end{aligned} \tag{4.24}$$

$$\begin{aligned}
 I'_4 \leq & \frac{a(M)}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{21} \int_{\Omega_\rho} g_2 U^{-q} dx \\
 & + c_{21} \rho^{n-q(q-2p)/(q-p)} \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^{qp/(q-p)}.
 \end{aligned} \tag{4.25}$$

Estimate for I'_5 . We use Young’s inequality and Lemma 4.1, to obtain

$$I'_5 \leq \frac{a(M)}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q dx + c_{22} \rho^n \left(\rho^{-q} + \max_{B_\rho} \left\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \right\} \right).$$

Collecting (4.19), (4.22)–(4.25) and the above inequality, we obtain

$$\begin{aligned}
 & \frac{a(M)}{2} \int_{B_\rho} \Phi U^{-q} \zeta^q dx \\
 & \leq c_{23} \left(\rho^{n-q} + \rho^n \max_{B_\rho} \left\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \right\} \right) \\
 & \quad + \rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p + \rho^{n-q(q-2p)/(q-p)} \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^{qp/(q-p)} \\
 & \quad + \int_{\Omega_\rho} g U^{-q} dx
 \end{aligned}$$

where $g = g_0 + g_1 + g_2 + g_3$. Finally, to obtain (4.16), we estimate the last integral in this inequality by means of Holder's inequality and the relations (4.2), $U \geq R^r$ and $\rho/2 < R < 1$. Inequality (4.17) is a trivial consequence of (4.16). The proof is complete. \square

LEMMA 4.3. *For every $\kappa \geq 1$ there is a positive constant $c = c(\text{data}, \kappa)$ such that $\lim_{\kappa \rightarrow +\infty} c(\text{data}, \kappa) = +\infty$ and*

$$\int_{\Omega_{3R/2}(x_0)} v_0^\kappa dx \leq cR^n. \tag{4.26}$$

Proof. First, we estimate from above the average integral

$$(\bar{v}_0)_{B_{3R/2}(x_0)} = \frac{1}{|B_{3R/2}(x_0)|} \int_{B_{3R/2}(x_0)} \bar{v}_0 dx$$

by a constant depending only on data. For this purpose we choose a function $\zeta_1 \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \zeta_1 \leq 1 \text{ in } \mathbb{R}^n, \quad \zeta_1 = 1 \text{ in } B_{3R/2}(x_0), \quad \zeta_1 = 0 \text{ in } \mathbb{R}^n \setminus B_{7R/4}(x_0),$$

$$|D^\alpha \zeta_1| \leq K_1 R^{-|\alpha|} \quad \text{for } |\alpha| = 1, 2,$$

where K_1 is an absolute constant, not depending on R . We have

$$\bar{v}_0 \in W^{1,q}(B_{3R/2}(x_0)).$$

Moreover by (2.7)

$$|B_{3R/2}(x_0) \setminus \Omega| \geq c_{24} R^n$$

and by the inequalities (4.6) and $M(2R) \geq \omega(2R)/2$,

$$1 \leq \bar{v}_0 \leq 1 + \ln 4 \text{ on } B_{3R/2}(x_0) \setminus \Omega.$$

Taking into account these facts and using Holder's inequality, Lemmas 3.2 and 4.2 and the properties of the function ζ_1 , we obtain

$$(\bar{v}_0)_{B_{3R/2}(x_0)} \leq c_{25} R^{-n/q} \left(\int_{B_{3R/2}(x_0)} \bar{v}_0^q dx \right)^{1/q}$$

$$\leq c_{26} R^{1-n/q} \left(\int_{B_{7R/4}(x_0)} \frac{\Phi \zeta_1^q dx}{(M(2R) - \bar{u} + R^r)^q} + R^{n-q} \right)^{1/q} \leq c_{27}. \tag{4.27}$$

Next, let $B_{2\rho} \subset B_{2R}(x_0)$, and let $\zeta_2 \in C_0^\infty(\mathbb{R}^n)$ be a function such that

$$0 \leq \zeta_2 \leq 1 \text{ in } \mathbb{R}^n, \quad \zeta_2 = 1 \text{ in } B_\rho, \quad \zeta_2 = 0 \text{ in } \mathbb{R}^n \setminus B_{2\rho},$$

$$|D^\alpha \zeta_2| \leq K_2 \rho^{-|\alpha|} \quad \text{for } |\alpha| = 1, 2,$$

where K_2 is an absolute constant, not depending on ρ . Using Holder’s inequality, Lemma 4.2 and the properties of the function ζ_2 , we derive that

$$\begin{aligned} \sum_{|\alpha|=1} \int_{B_\rho} |D^\alpha \bar{v}_0| dx &\leq c_{28} \rho^{n-n/q} \left(\sum_{|\alpha|=1} \int_{B_\rho} |D^\alpha \bar{v}_0|^q dx \right)^{1/q} \\ &\leq c_{28} \rho^{n-n/q} \left(\int_{B_{2\rho}} \frac{\Phi \zeta_2^q dx}{(M(2R) - \bar{u} + R^r)^q} \right)^{1/q} \leq c_{29} \rho^{n-1}. \end{aligned}$$

Hence, by Lemma 3.3, we have

$$\int_{B_{3R/2}(x_0)} \exp\left(c_{30} |\bar{v}_0 - (\bar{v}_0)_{B_{3R/2}(x_0)}|\right) dx \leq c_{31} R^n. \tag{4.28}$$

Now let $\kappa \geq 1$. Then inequalities (4.27) and (4.28) imply (4.26). The proof is complete. \square

Now we are ready to prove inequality (4.5).

LEMMA 4.4. *There exists a positive constant c_1 such that inequality (4.5) holds.*

Proof. The proof is based on obtaining the estimate $R^{-n/k} \|\bar{v}_0\|_{L^k(B_R(x_0))} \leq c_1$ at $k \rightarrow +\infty$ by adapting the Moser’s iterative technique. We divide the proof in four steps.

Step 1. Let us introduce some auxiliary functions.

We fix a function $\psi_0 \in C_0^\infty(\mathbb{R})$ such that

$$0 \leq \psi_0 \leq 1 \text{ on } \mathbb{R}, \quad \psi_0 = 1 \text{ in } [-1, 1], \quad \psi_0 = 0 \text{ in } \mathbb{R} \setminus (-3/2, 3/2).$$

We define the function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\psi(x) = \psi_0\left(\frac{|x-x_0|}{R}\right)$. Let $E \subset \Omega_{2R}(x_0)$ be a set such that $|E| = 0$ and for every $x \in \Omega_{2R}(x_0) \setminus E$,

$$\mu(2R) \leq u(x) \leq M(2R). \tag{4.29}$$

Let $\mathcal{U} : \Omega \rightarrow \mathbb{R}$ be the function such that

$$\mathcal{U}(x) = \begin{cases} M(2R) - u(x) + R^r & \text{if } x \in \Omega_{2R}(x_0) \setminus E, \\ 2\omega(2R) & \text{if } x \in (\Omega \setminus B_{2R}(x_0)) \cup E. \end{cases}$$

Let now

$$\varkappa = 2qp/(q - 2p), \tag{4.30}$$

$$k \geq \bar{k} := \max\{q, 12Mb(M)/a(M)\}, \tag{4.31}$$

$$l := \max\{q, \varkappa\} < t \leq C_0k, \tag{4.32}$$

and $C_0 = C_0(n, p, q, \tau) > 1$ is a constant that will be specified below. Define

$$G = \max\{\mathcal{U}^{(1-q)/\varkappa} - (M(2R) + R^r)^{(1-q)/\varkappa}, 0\},$$

$$v_0 = \ln(2e\omega(2R)\mathcal{U}^{-1}), \quad w = G^\varkappa v_0^k \psi^t.$$

Due to (4.6), (4.29), $2p < q$ we have $w \in \dot{W}_{2,p}^{1,q}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned} |D^\alpha w - (q-1)v_0^k \psi^t G^{\varkappa-1} \mathcal{U}^{(1-q)/\varkappa-1} D^\alpha u - k v_0^{k-1} \psi^t G^\varkappa \mathcal{U}^{-1} D^\alpha u| \\ \leq c_{32} k R^{-1} G^\varkappa v_0^k \psi^{t-1} \quad \text{if } |\alpha| = 1, \end{aligned} \quad (4.33)$$

$$\begin{aligned} |D^\alpha w - (q-1)v_0^k \psi^t G^{\varkappa-1} \mathcal{U}^{(1-q)/\varkappa-1} D^\alpha u - k v_0^{k-1} \psi^t G^\varkappa \mathcal{U}^{-1} D^\alpha u| \\ \leq c_{32} k^2 v_0^k \psi^{t-2} G^{\varkappa-2} \mathcal{U}^{2(1-q)/\varkappa} \left\{ R^{-2} + \mathcal{U}^{-2} \sum_{|\beta|=1} |D^\beta u|^2 \right\} \quad \text{if } |\alpha| = 2. \end{aligned} \quad (4.34)$$

Putting the function w in (2.6) instead of v and using (2.2), (2.5), (4.33), (4.34) and the inequalities $G \leq \mathcal{U}^{(1-q)/\varkappa}$, $\mathcal{U} \leq 2M+1$ we deduce that

$$\begin{aligned} (q-1)a(M) \int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa-1} \mathcal{U}^{(1-q)/\varkappa-1} v_0^k \psi^t dx \\ + k a(M) \int_{\Omega_{2R}(x_0)} \Phi G^\varkappa \mathcal{U}^{-1} v_0^{k-1} \psi^t dx \\ \leq b(M) \int_{\Omega_{2R}(x_0)} \Phi G^\varkappa v_0^k \psi^t dx + \int_{\Omega_{2R}(x_0)} k g_4 v_0^k \psi^t \mathcal{U}^{-q} dx + \mathcal{I}_1 + \mathcal{I}_2, \end{aligned} \quad (4.35)$$

where $g_4 = 2g_0 + (2M+1)g_3$,

$$\mathcal{I}_1 = \frac{c_{32}k}{R} \sum_{|\alpha|=1} \int_{\Omega_{2R}(x_0)} |A_\alpha(x, u, \nabla_2 u)| G^\varkappa v_0^k \psi^{t-1} dx, \quad (4.36)$$

$$\begin{aligned} \mathcal{I}_2 = c_{32}k^2 \sum_{|\alpha|=2} \int_{\Omega_{2R}(x_0)} |A_\alpha(x, u, \nabla_2 u)| \\ \times G^{\varkappa-2} \mathcal{U}^{2(1-q)/\varkappa} \left\{ R^{-2} + \mathcal{U}^{-2} \sum_{|\beta|=1} |D^\beta u|^2 \right\} v_0^k \psi^{t-2} dx. \end{aligned} \quad (4.37)$$

Step 2. We show that the first term on the right-hand side of inequality (4.35) is absorbed by the second term in its left-hand side. For this we need the inequality

$$\mathcal{U} v_0 \leq 12M \quad \text{a.e. in } \Omega_{2R}(x_0), \quad (4.38)$$

which follows from (4.1) and from the fact that $\ln s < s$ for every $s > 0$.

Using (4.38), the first term on the right-hand side of inequality (4.35) is estimated in the following way

$$b(M) \int_{\Omega_{2R}(x_0)} \Phi G^\varkappa v_0^k \psi^t dx \leq 12M b(M) \int_{\Omega_{2R}(x_0)} \Phi G^\varkappa \mathcal{U}^{-1} v_0^{k-1} \psi^t dx. \quad (4.39)$$

Now (4.31), (4.35) and (4.39) imply the inequality

$$(q-1)a(M) \int_{\Omega_{2R}(x_0)} \Phi G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} v_0^k \psi^t dx \leq k \int_{\Omega_{2R}(x_0)} g_4 v_0^k \psi^t \mathcal{U}^{-q} dx + \mathcal{I}_1 + \mathcal{I}_2. \quad (4.40)$$

Step 3. Let us estimate from above the quantities \mathcal{I}_1 and \mathcal{I}_2 , which are defined by (4.36) and (4.37) respectively.

Estimate for \mathcal{I}_1 . We use (2.3), the inequality $G \leq \mathcal{U}^{(1-q)/\alpha}$ and Young’s inequality $|yz| \leq \varepsilon|y|^{q/(q-1)} + \varepsilon^{1-q}|z|^q$, where

$$y = |A_\alpha(x, u, \nabla_2 u)| (G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} \psi^t)^{(q-1)/q}, \quad |\alpha| = 1, \\ z = kR^{-1} G^{(\alpha+q-1)/q} \mathcal{U}^{((q-1)/\alpha+1)(q-1)/q} \psi^{t/q-1}$$

and ε is an appropriate positive number, to obtain

$$\mathcal{I}_1 \leq \frac{(q-1)a(M)}{4} \int_{\Omega_{2R}(x_0)} \Phi G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} v_0^k \psi^t dx + c_{33} \int_{\Omega_{2R}(x_0)} g_1 v_0^k \psi^t \mathcal{U}^{-q} dx + \frac{c_{33} k^q}{R^q} \int_{\Omega_{2R}(x_0)} v_0^k \psi^{t-q} dx. \quad (4.41)$$

Estimate for \mathcal{I}_2 . Using the first equality in (4.12), Young’s inequality, the inequality $G \leq \mathcal{U}^{(1-q)/\alpha}$, (4.30) and (4.2), we establish that if $\varepsilon > 0$, $\alpha, \beta \in \Lambda$, $|\alpha| = 2$ and $|\beta| = 1$, then

$$k^2 R^{-2} |A_\alpha(x, u, \nabla_2 u)| G^{\alpha-2} \mathcal{U}^{2(1-q)/\alpha} \psi^{t-2} = |A_\alpha(x, u, \nabla_2 u)| (G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} \psi^t)^{(p-1)/p} k^2 R^{-2} \psi^{2t/q} \times G^{(\alpha-p-1)/p} \mathcal{U}^{(p-1)/p-(q-1)(p+1)/\alpha p} \psi^{2t/\alpha-2} \leq \varepsilon |A_\alpha(x, u, \nabla_2 u)|^{p/(p-1)} G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} \psi^t + \varepsilon (1 + \varepsilon^{-\alpha/2}) k^q R^{-q} \psi^{t-\alpha} \quad \text{a.e. in } \Omega_{2R}(x_0), \quad (4.42)$$

$$k^2 |A_\alpha(x, u, \nabla_2 u)| G^{\alpha-2} \mathcal{U}^{2(1-q)/\alpha-2} |D^\beta u|^2 \psi^{t-2} = |A_\alpha(x, u, \nabla_2 u)| (G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} \psi^t)^{(p-1)/p} \times |D^\beta u|^2 (G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} \psi^t)^{2/q} \times k^2 G^{(\alpha-2)/\alpha} \mathcal{U}^{(\alpha+2)(1-q-\alpha)/\alpha^2} \psi^{2t/\alpha-2} \leq \varepsilon (|A_\alpha(x, u, \nabla_2 u)|^{p/(p-1)} + |D^\beta u|^q) G^{\alpha-1} \mathcal{U}^{(1-q)/\alpha-1} \psi^t + \varepsilon^{1-\alpha/2} k^\alpha R^{-q} \psi^{t-\alpha} \quad \text{a.e. in } \Omega_{2R}(x_0). \quad (4.43)$$

From (2.4), (4.37), (4.42), (4.43), taking into account (4.32) and the suitable choice of

ε , we deduce the estimate

$$\begin{aligned} \mathcal{J}_2 &\leq \frac{(q-1)a(M)}{4} \int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa-1} \mathcal{U}^{(1-q)/\varkappa-1} v_0^k \psi^t dx \\ &\quad + c_{34} \int_{\Omega_{2R}(x_0)} g_2 v_0^k \psi^t \mathcal{U}^{-q} dx + \frac{c_{34} k^l}{R^q} \int_{\Omega_{2R}(x_0)} v_0^k \psi^{t-l} dx. \end{aligned} \tag{4.44}$$

From (4.40), (4.41), (4.44), (4.32) and (4.2) it follows that

$$\begin{aligned} &\int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa-1} \mathcal{U}^{(1-q)/\varkappa-1} v_0^k \psi^t dx \\ &\leq \frac{c_{35} k^l}{R^q} \int_{\Omega_{2R}(x_0)} v_0^k \psi^{t-l} dx + \frac{c_{35} k^l}{R^{q-n/\tau}} \int_{\Omega_{2R}(x_0)} (g_1 + g_2 + g_4) v_0^k \psi^{t-l} dx. \end{aligned}$$

Estimating the last two integrals by Hölder’s inequality with the exponents τ and $\tau/(\tau-1)$ and taking into account (4.31) and (4.32), we obtain that for every $k \geq \bar{k}$ and $t \in (l, C_0 k]$ the following inequality holds:

$$\begin{aligned} &\int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa-1} \mathcal{U}^{(1-q)/\varkappa-1} v_0^k \psi^t dx \\ &\leq \frac{c_{36} k^l}{R^{q-n/\tau}} \left(\int_{\Omega_{2R}(x_0)} (v_0^k \psi^{t-l})^{\tau/(\tau-1)} dx \right)^{(\tau-1)/\tau}. \end{aligned} \tag{4.45}$$

Define $E_R(x_0) = \{x \in \Omega_{3R/2}(x_0) : \mathcal{U}(x) \leq 2^{\varkappa/(1-q)}(M(2R) + R^r)\}$.

We shall suppose that

$$|E_R(x_0)| \neq 0. \tag{4.46}$$

If (4.46) is not true, then (4.5) is a simple consequence of the inequalities

$$M(2R) \geq \omega(2R)/2 \quad \text{and} \quad \mathcal{U}(x) > 2^{\varkappa/(1-q)}(M(2R) + R^r) \quad \text{a.e. in } \Omega_{3R/2}(x_0).$$

For every $x \in E_R(x_0)$ we have $G(x) \geq \frac{1}{2}[\mathcal{U}(x)]^{(1-q)/\varkappa}$. Using this fact from (4.45), we deduce that

$$\int_{E_R(x_0)} \Phi \mathcal{U}^{-q} v_0^k \psi^t dx \leq \frac{c_{37} k^l}{R^{q-n/\tau}} \left(\int_{\Omega_{2R}(x_0)} (v_0^k \psi^{t-l})^{\tau/(\tau-1)} dx \right)^{(\tau-1)/\tau}. \tag{4.47}$$

Step 4. At this step, using inequality (4.47), we organize an iterative Moser-type process for the functions $v_0^k \psi^t$ in $\Omega_{2R}(x_0)$, which completes the proof of the lemma. We set

$$L = \ln \frac{2^{\varkappa/(q-1)+1} e \omega(2R)}{M(2R) + R^r},$$

$$J(k, t) = \frac{1}{R^n} \int_{\Omega_{2R}(x_0)} v_0^k \psi^t dx + L^k, \quad k > 0, t > 0, \tag{4.48}$$

$$\theta = \frac{\tau}{\tau - 1} \cdot \frac{q}{q^*}, \quad \tilde{l} = \frac{(q + l)q^*}{q}. \tag{4.49}$$

We now prove the following assertion: if $k \geq \bar{k}q^*/q$ and $\tilde{l} < t \leq C_0k$, then

$$J(k, t) \leq c_{38} k^{\tilde{l}} [J(k\theta, t\theta - \tilde{l})]^{1/\theta}. \tag{4.50}$$

Let $k \geq \bar{k}q^*/q$ and $\tilde{l} < t \leq C_0k$. We set $\bar{v} = (\max\{\bar{v}_0^k, L^k\})^{1/q^*} \psi^{t/q^*}$. It is easy to see that

$$\int_{\Omega_{2R}(x_0)} v_0^k \psi^t dx \leq \int_{B_{2R}(x_0)} \bar{v}^{q^*} dx. \tag{4.51}$$

Applying inequality (3.1) to the function $\bar{v} \in \dot{W}^{1,q}(B_{2R}(x_0))$ and taking into account (4.32) and the definitions of the functions \bar{v} and ψ , we obtain

$$\begin{aligned} \int_{B_{2R}(x_0)} \bar{v}^{q^*} dx &\leq c_{39} k^{q^*} L^k R^n + c_{39} k^{q^*} \left(\int_{E_R(x_0)} \Phi \mathcal{U}^{-q} v_0^{kq/q^*} \psi^{tq/q^* - q} dx \right. \\ &\quad \left. + R^{-q} \int_{\Omega_{2R}(x_0)} v_0^{kq/q^*} \psi^{tq/q^* - q} dx \right)^{q^*/q}. \end{aligned}$$

From this inequality, estimating the first addend in the brackets by means of (4.47) and the second addend by means of Hölder’s inequality and using (4.48) and (4.49), we deduce that

$$\int_{B_{2R}(x_0)} \bar{v}^{q^*} dx \leq c_{40} R^n k^{\tilde{l}} [J(k\theta, t\theta - \tilde{l})]^{1/\theta}.$$

The last inequality, (4.48) and (4.51) imply (4.50).

Now, we choose a number $i_0 \in \mathbb{N}$ such that $\theta^{-i_0} > \bar{k}q^*/q$ and set $C_0 = \tilde{l}/(1 - \theta)$,

$$k_i = \theta^{-i_0 - i}, \quad t_i = \frac{\tilde{l}(\theta^{-i_0 - i} - 1)}{1 - \theta}, \quad J_i = J(k_i, t_i), \quad i = 0, 1, 2, \dots$$

Then (4.50) and the inequality $\theta < 1$ imply that for every $i = 0, 1, 2, \dots$,

$$J_i^{1/k_i} \leq c_{41} J_0^{\theta^{i_0}}. \tag{4.52}$$

Due to Lemma 4.3 and to the inequality $M(2R) \geq \omega(2R)/2$ we have

$$J_0^{\theta^{i_0}} \leq c_{42}. \tag{4.53}$$

From (4.52) and (4.53) it follows that

$$\|\bar{v}_0\|_{L^\infty(B_R(x_0))} = \lim_{i \rightarrow \infty} \left(\frac{1}{R^n} \int_{B_R(x_0)} \bar{v}_0^{k_i} dx \right)^{1/k_i} \leq \limsup_{i \rightarrow \infty} J_i^{1/k_i} \leq c_1.$$

The proof is complete. \square

Thus, the validity of (4.3) is established. Then using Lemma 3.5 and the interior regularity result of [42, Theorem 2.3], we come to the conclusion of Theorem 1. The proof of Theorem 1 is complete.

REFERENCES

- [1] L. BOCCARDO, F. MURAT, J. P. PUEL, *Résultats d'existence pour certains problèmes elliptiques quasilineaires*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **11**, 2 (1984), 213–235.
- [2] L. BOCCARDO, F. MURAT, J. P. PUEL, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl. (4), **152** (1988), 183–196.
- [3] L. BOCCARDO, F. MURAT, J. P. PUEL, *L^∞ -estimate for some nonlinear elliptic partial differential equations and application to an existence result*, SIAM J. Math. Anal. **23**, 2 (1992), 326–333.
- [4] S. BONAFEDE, *Hölder continuity of bounded generalized solutions for some degenerated quasilinear elliptic equations with natural growth terms*, Comment. Math. Univ. Carolin. **59**, 1 (2018), 45–64.
- [5] S. BONAFEDE, *On the behaviour near the boundary of the bounded generalized solutions of quasilinear degenerate elliptic equations with natural growth terms*, preprint, (2018).
- [6] S. BONAFEDE, V. CATALDO, S. D'ASERO, *Hölder continuity up to the boundary of minimizers for some integral functionals with degenerate integrands*, J. Appl. Math. **2007**, Article ID 31819 doi:10.1155/2007/31819 (2007), 14 pages.
- [7] S. BONAFEDE, F. NICOLOSI, *On the sets of regularity of solutions for a class of degenerate nonlinear elliptic fourth-order equations with L^1 data*, Bound. Value Probl. **2007**, (2007), 1–15.
- [8] S. BONAFEDE, F. NICOLOSI, *On regularity up to the boundary of solutions to a system of degenerate nonlinear elliptic fourth-order equations*, Complex Var. Elliptic Equ. **53**, 2 (2008), 101–116.
- [9] V. CATALDO, S. D'ASERO, F. NICOLOSI, *Regularity of minimizers of some integral functionals with degenerate integrands*, Nonlinear Anal. **68**, 11 (2008), 3283–3293.
- [10] P. CIANCI, G. R. CIRMI, S. D'ASERO, S. LEONARDI, *Morrey estimates for solutions of singular quadratic nonlinear equations*, Ann. Mat. Pura Appl. (4), **196**, 5 (2017), 1739–1758.
- [11] G. R. CIRMI, *Nonlinear elliptic equations with lower order terms and $L^{1,\lambda}$ -data*, Nonlinear Anal. **68**, 9 (2008), 2741–2749.
- [12] G. R. CIRMI, S. D'ASERO, S. LEONARDI, *Fourth-order nonlinear elliptic equations with lower order term and natural growth conditions*, Nonlinear Anal. **108**, (2014), 66–86.
- [13] G. R. CIRMI, S. D'ASERO, S. LEONARDI, *Gradient estimate for solutions of nonlinear singular elliptic equations below the duality exponent*, Math. Methods Appl. Sci. **41**, 1 (2018), 261–269.
- [14] G. R. CIRMI, S. LEONARDI, *Regularity results for the gradient of solutions linear elliptic equations with $L^{1,\lambda}$ data*, Ann. Mat. Pura Appl. (4), **185**, 4 (2006), 537–553.
- [15] G. R. CIRMI, S. LEONARDI, *Regularity results for solutions of nonlinear elliptic equations with $L^{1,\lambda}$ data*, Nonlinear Anal. **69**, 1 (2008), 230–244.
- [16] G. R. CIRMI, S. LEONARDI, *Higher differentiability for solutions of linear elliptic systems with measure data*, Discrete Contin. Dyn. Syst. **26**, 1 (2010), 89–104.
- [17] G. R. CIRMI, S. LEONARDI, *Higher differentiability for the solutions of nonlinear elliptic systems with lower-order terms and $L^{1,\theta}$ -data*, Ann. Mat. Pura Appl. (4), **193**, 1 (2014), 115–131.
- [18] G. R. CIRMI, S. LEONARDI, J. STARÁ, *Regularity results for the gradient of solutions of a class of linear elliptic systems with $L^{1,\lambda}$ data*, Nonlinear Anal. **68**, 12 (2008), 3609–3624.
- [19] S. D'ASERO, *On Harnack inequality for degenerate nonlinear higher-order elliptic equations*, Appl. Anal. **85**, 8 (2006), 971–985.
- [20] S. D'ASERO, *On removability of the isolated singularity for solutions of high-order elliptic equations*, Complex Var. Elliptic Equ. **55**, 5–6 (2010), 525–536.
- [21] S. D'ASERO, D. V. LARIN, *Degenerate nonlinear higher-order elliptic problems in domains with fine-grained boundary*, Nonlinear Anal. **64**, 4 (2006), 788–825.
- [22] R. GARIEPY, W. P. ZIEMER, *Behavior at the boundary of solutions of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **56**, 4 (1974), 372–384.
- [23] R. GARIEPY, W. P. ZIEMER, *A regularity condition at the boundary for solutions of quasilinear elliptic equations*, Arch. Rational Mech. Anal. **67**, 1 (1977), 25–39.
- [24] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.
- [25] F. JOHN, L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14**, 3 (1961), 415–426.
- [26] T. KILPELÄINEN, J. MALY, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta. Math. **172**, 1 (1994), 137–161.

- [27] A. A. KOVALEVSKII, *Entropy solutions of the Dirichlet problem for a class of fourth-order nonlinear elliptic equations with L^1 -right-hand sides*, *Izv. Math.* **65**, 2 (2001), 231–283.
- [28] A. KOVALEVSKY, *Entropy solutions of Dirichlet problem for a class of nonlinear elliptic high-order equations with L^1 -data*, *Nonlinear Boundary Value Problems* **12**, (2002), 119–127.
- [29] A. KOVALEVSKY, F. NICOLOSI, *On regularity up to the boundary of solutions to degenerate nonlinear elliptic high-order equations*, *Nonlinear Anal.* **40**, 1–8 (2000), 365–379.
- [30] O. LADYZHENSKAYA AND N. URAL'TSEVA, *Linear and quasilinear elliptic equations*, Academic Press, New York and London, 1968.
- [31] S. LEONARDI, J. KOTTAS, J. STARÁ, *Hölder regularity of the solutions of some classes of elliptic systems in convex nonsmooth domains*, *Nonlinear Anal.* **60**, 5 (2005), 925–944.
- [32] V. G. MAZ'JA, *On the continuity at a boundary point of the solution of quasi-linear elliptic equations*, (Russian), *Vestnik Leningrad Univ.* **25**, 13 (1970), 42–55.
- [33] J. M. RAKOTOSON, *Résultats de régularité et d'existence pour certaines équations elliptiques quasi linéaires*, *C. R. Acad. Sci. Paris, Série I*, **302**, 16 (1986), 567–570.
- [34] J. M. RAKOTOSON, *Réarrangement relatif dans les équations elliptiques quasi-linéaires avec un second membre distribution: Application à un théorème d'existence et de régularité*, *J. Differential Equations* **66**, 3 (1987), 391–419.
- [35] I. V. SKRYPNIK, *Nonlinear higher order elliptic equations*, (Russian), *Naukova dumka*, Kiev, 1973.
- [36] I. V. SKRYPNIK, *Higher order quasilinear elliptic equations with continuous generalized solutions*, *Differential Equations* **14**, 6 (1978), 786–795.
- [37] I. V. SKRYPNIK, *A criterion for regularity of a boundary point for quasilinear elliptic equations*, (Russian), *Dokl. Akad. Nauk SSSR*, **274**, 5 (1984), 1040–1044.
- [38] I. V. SKRYPNIK, *Regularity of a boundary point for a higher-order quasilinear elliptic equation*, *Proc. Steklov Inst. Math.* **200**, 2 (1993), 339–351.
- [39] M. V. VOITOVICH, *Existence of bounded solutions for a class of nonlinear fourth-order equations*, *Differ. Equ. Appl.* **3**, 2 (2011), 247–266.
- [40] M. V. VOITOVICH, *Existence of bounded solutions for nonlinear fourth-order elliptic equations with strengthened coercivity and lower-order terms with natural growth*, *Electron. J. Differential Equations* **2013**, 102 (2013), 1–25.
- [41] M. V. VOITOVICH, *On the existence of bounded generalized solutions of the Dirichlet problem for a class of nonlinear high-order elliptic equations*, *J. Math. Sci. (N. Y.)*, **210**, 1 (2015), 86–113.
- [42] M. V. VOITOVYCH, *Hölder continuity of bounded generalized solutions for nonlinear fourth-order elliptic equations with strengthened coercivity and natural growth terms*, *Electron. J. Differential Equations* **2017**, 63 (2017), 1–18.

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