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POSITIVE SOLUTIONS FOR FRACTIONAL INTEGRO–BOUNDARY VALUE PROBLEM OF ORDER $(1,2)$ ON AN UNBOUNDED DOMAIN

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Abstract. In this manuscript, we study a system of fractional integro boundary value problem on unbounded domain. The solution of the system is defined in terms of the Green's function. We have established the existence and uniqueness results by utilizing the fixed point theorems. The main outcomes and assumptions are verified via some examples.

1. Introduction

The study of fractional calculus began as an exploration into whether the meaning of a derivative $\frac{d^n y}{dx^n}$ of integer order could be extended when n is a fractional value. There is an extensive list of research articles and books which are mostly written by mathematicians, dealing with the past history of this subject and its applications [1, 2, 3]. This subject has received great attention and variety of results under the finite region due to their important applications in several fields of science and engineering [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Despite the fact that most of the work on fractional calculus deals with finite domain hitherto there has been a significant development on the subject connecting with unbounded domain; see [14, 15, 16, 17].

Bicadze et al. in [9] initiated the concept of multipoint boundary-value problems and introduce some applications in various physical problems of applied sciences. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary-value problem.

The main goal of this paper is to decompose the existence of positive solutions for the following singular fractional boundary value problem (FBVP) on an infinite domain

$$\begin{aligned} {}^{RL}D^{\mu-\nu}y(t) &= Af(t,y(t)) + \sum_{i=1}^{k} B_i g_i(t,y(t),\mathfrak{B}y(t)),\ t\in[0,\infty), \\ y(0)=0, \quad \lim_{t\to\infty} {}^{RL}D^{\mu-\nu-1}y(t) &= a\left({}^{RL}D^{\frac{\mu-\nu-1}{2}}y(t)\right)\Big|_{t=\xi}, \end{aligned} \tag{1}$$

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where $\mu > \nu,\ 0 \leqslant \xi < \infty,\ a > 0,\ 1 < \mu - \nu < 2,\ A, B_i$ with $1 \leqslant i \leqslant k$, are real constants. The functions $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $g_i : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. The term ${}^{RL}D^{\mu-\nu}$ represents the Riemann-Liouville fractional derivative of order $\mu - \nu$, i.e

$$ {}^{RL}D^{\mu-\nu} y(t) = \frac{1}{\Gamma(n-\mu+\nu)} \frac{d^n}{dt^n} \int_0^t \frac{y(t)}{(t-s)^{\mu-\nu+1-n}} ds,\ n = [\mu - \nu] + 1, $$

and the term $\mathfrak{B}$ stand for the integral operator defined by

$$ \mathfrak{B}y(t) = \int_0^t H(t,s,y(s))ds, \tag{2} $$

where $H : [0,\infty) \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$.

Throughout this manuscript, positive solution means that a continuous function $y(t)$ such that $y(t) : \mathbb{R} \to \mathbb{R}$ which is positive on the half domain $[0,\infty)$ and satisfies all the boundary conditions of the system (1). In the last few decades, the literature on the existence of positive solutions for the non integer order is continuously exceeding in the finite domain. Several methods are introduced, such as topological degree theory, fixed point theorems, upper and lower solution method, monotonic iterative method [5, 6, 7, 8, 18] to established the existence results.

The system of fractional differential equations on a half-line have become a current research topic for the mathematicians and scientists due to its various applications in the field of applied mathematics and physics. For example, the models of gas pressure in a semi-infinite porous medium, see [19].

In [14] author's introduced the concept of Green function for the system FBVP on infinite domain and by using the Leray-Schauder nonlinear alternative theorem obtained the main results. In [20] authors study a model with resonance and established the results via Green's function and fixed point theorems. Further, in [21] author's study a system of FBVP on unbounded domain and established the existence and uniqueness results.

In [22] author's study on fractional model on unbounded domain and verified the claim results by using the coincidence degree theory of Mawthin. Author's in [23] analyzed the following integro-differential system in unbounded domain

$$ \begin{aligned} &{}^{RL}D^q u(t) + f(t,u(t),Tu(t),Su(t)) = 0,\ t \in [0,\infty),\ q \in (1,2), \\ &u(0) = 0, \quad D^{q-1}u(\infty) = u*, \end{aligned} $$

and established the claim results by combing the concept of cone theory and the monotone iterative techniques.

Nyamoradi et al. [5] intensively studied the following fractional model

$$ \begin{aligned} &{}^{RL}D^{\alpha} u(t) = Af(t,u(t)) + \sum_{i=1}^{k} B_i I^{\beta_i} g_i(t,u(t)),\ t \in (0,1),\ \alpha \in (1,2], \\ &D^{\delta}u(0) = 0,\ D^{\delta}u(1) = aD^{\frac{\alpha-\delta-1}{2}}(D^{\delta}u(t))|_{t=\xi},\ \xi \in (0,\frac{1}{2}],\ a \in [0,\infty), \end{aligned} $$

where $1 < \alpha - \delta < 2$, $0 < \beta_i < 1$, A, B_i, $1 \leqslant i \leqslant k$, are real constants.

In this paper, we analyze the positive solution for three point nonlocal BVP involving integro-differential equations of fractional order $(1,2)$ with two nonlinear functions on an infinite domain. The main contribution in this paper is to construct a suitable Green's function for the considered problem on unbounded domain by utilizing a few speculations from the papers [5, 14]. Finally, we have stated and proved the existence and uniqueness results of the considered system by using the Banach, Leray-Schauder's alternative and Schauder's fixed point theorems.

2. Technical background

The common notations of fractional calculus, properties of Riemann-Liouville fractional integral and derivatives used in this paper are taken from [2, 3]. Rest of the details are given below.

LEMMA 1. ([1]) *Let $\mu > 0$. Then the differential equation*

$$^{RL}D^{\mu}_{0^+} u = 0,$$

has a unique solution $u(t) = c_1 t^{\mu-1} + c_2 t^{\mu-2} + \ldots + c_n t^{\mu-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, there $n - 1 < \mu \leqslant n$.

LEMMA 2. ([1]) *Let $\mu > 0$. Then the following equality holds for $u \in L^1(\mathbb{R}^+)$, $D^{\mu}_{0^+} u \in L^1(\mathbb{R}^+)$;*

$$I^{\mu}_{0^+}\, {}^{RL}D^{\mu}_{0^+} u(t) = u(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \ldots + c_n t^{\mu-n},$$

$c_i \in \mathbb{R}$, $i = 1, \ldots, n$, there $n - 1 < \mu \leqslant n$.

DEFINITION 1. [15] A function y is called positive solution of the problem (1) if $y(t) \geqslant 0$, $\forall\, t \in \mathbb{R}_+$, and it satisfies the boundary conditions of the problem (1).

THEOREM 1. [24] *Suppose U is an open subset of a Banach space $\mathbb{X}$, $0 \in U$ and $F : \overline{U} \to X$ a contraction with $F(\overline{U})$ bounded. Then either F has a fixed point in $\overline{U}$, or there exists $\lambda \in (0,1)$ and $u \in \partial U$ with $u = \lambda F(u)$ holds.*

LEMMA 3. ([4]) *For $\lambda > -1$ and $\mu > 0$, $^{RL}D^{\mu}_{0^+} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\mu+1)} t^{\gamma-\mu}$.*

LEMMA 4. ([1]) *If $\alpha > \beta > 0$, then $^{RL}D^{\beta} I^{\alpha} f(x) = I^{\alpha-\beta} f(x)$.*

LEMMA 5. *For any $h \in C[0,\infty) \cap L(0,\infty]$, the BVP*

$$\begin{aligned} &^{RL}D^{\mu-\nu} y(t) + h(t) = 0,\ t \in [0,\infty), \\ &y(0) = 0, \quad \lim_{t\to\infty} {}^{RL}D^{\mu-\nu-1} y(t) = aD^{\frac{\mu-\nu-1}{2}} y(t))\big|_{t=\xi}, \end{aligned} \tag{3}$$

has a unique solution

$$y(t) = -I^{\mu-\nu}h(t) + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[\int_0^\infty h(s)ds$$
$$-a\int_0^\xi \frac{(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\Gamma(\frac{\mu-\nu+1}{2})}h(s)ds\Big].$$

The proof of this lemma is similar as in ([14] Lemma 3.2). Next, we explain an auxiliary lemma which will play an important role in the sequel.

LEMMA 6. *A continuous function y is a solution of the following integral equation*

$$y(t) = \int_0^\infty \mathcal{G}(t,s)h(s)ds,$$

iff y is a solution of the BVP (3) where $\mathcal{G}(t,s)$ is the Green's function defined by

$$\mathcal{G}(t,s) = \frac{1}{\Gamma(\mu-\nu)}\begin{cases} -(t-s)^{\mu-\nu-1} + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}} \\ -a\frac{t^{\mu-\nu-1}(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}, & 0\leqslant s\leqslant t\leqslant\infty,\ s\leqslant\xi, \\ -(t-s)^{\mu-\nu-1} \\ +\frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}, & 0\leqslant s\leqslant t\leqslant\infty,\ \xi\leqslant s, \\ \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}} \\ -a\frac{t^{\mu-\nu-1}(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}, & 0\leqslant t\leqslant s\leqslant\infty,\ s\leqslant\xi, \\ \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}, & 0\leqslant t\leqslant s\leqslant\infty,\ \xi\leqslant s. \end{cases} \tag{4}$$

such that $\Gamma(\frac{\mu-\nu+1}{2}) > a\xi^{\frac{\mu-\nu-1}{2}}$, $a>0$, $0<\xi\leqslant\infty$.

Proof. The unique solution of the problem (3) may defined for $t\leqslant\xi$, as

$$\begin{aligned} y(t) &= -\int_0^t \frac{(t-s)^{\mu-\nu-1}}{\Gamma(\mu-\nu)}h(s)ds \\ &\quad + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[\int_0^\infty h(s)ds - a\int_0^\xi \frac{(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\Gamma(\frac{\mu-\nu+1}{2})}h(s)ds\Big] \\ &= -\int_0^t \frac{(t-s)^{\mu-\nu-1}}{\Gamma(\mu-\nu)}h(s)ds + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \\ &\quad \Big[\Big(\int_0^t + \int_t^\xi + \int_\xi^\infty\Big)h(s)ds - a\Big(\int_0^t + \int_t^\xi\Big)\frac{(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\Gamma(\frac{\mu-\nu+1}{2})}h(s)ds\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\mu-\nu)} \int_0^t \Big[-(t-s)^{\mu-\nu-1} + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \\
&\quad -a \frac{t^{\mu-\nu-1}(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \Big] h(s)ds \\
&\quad + \frac{1}{\Gamma(\mu-\nu)} \int_t^\xi \Big[\frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} - a \frac{t^{\mu-\nu-1}(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \Big] h(s)ds \\
&\quad + \frac{1}{\Gamma(\mu-\nu)} \int_\xi^\infty \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} h(s)ds = \frac{1}{\Gamma(\mu-\nu)} \int_0^\infty \mathscr{G}(t,s)h(s)ds.
\end{aligned}
$$

Subsequently, for $t \geqslant \xi$, we have

$$
\begin{aligned}
y(t) &= -\Big(\int_0^\xi + \int_\xi^t \Big) \frac{(t-s)^{\mu-\nu-1}}{\Gamma(\mu-\nu)} h(s)ds + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \\
&\quad \Big[\Big(\int_0^\xi + \int_\xi^t + \int_t^\infty \Big) h(s)ds - a \int_0^\xi \frac{(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\Gamma(\frac{\mu-\nu+1}{2})} h(s)ds \Big] \\
&= \int_0^\xi \Big[-\frac{(t-s)^{\mu-\nu-1}}{\Gamma(\mu-\nu)} + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \\
&\quad -a \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \frac{(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\Gamma(\frac{\mu-\nu+1}{2})} \Big] h(s)ds \\
&\quad + \int_\xi^t \Big[-\frac{(t-s)^{\mu-\nu-1}}{\Gamma(\mu-\nu)} + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \Big] h(s)ds \\
&\quad + \int_t^\infty \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} h(s)ds \\
&= \frac{1}{\Gamma(\mu-\nu)} \int_0^\xi \Big[-(t-s)^{\mu-\nu-1} + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \\
&\quad -a \frac{t^{\mu-\nu-1}(\xi-s)^{\frac{\mu-\nu-1}{2}}}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \Big] h(s)ds \\
&\quad + \frac{1}{\Gamma(\mu-\nu)} \int_\xi^t \Big[-(t-s)^{\mu-\nu-1} + \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \Big] h(s)ds \\
&\quad + \frac{1}{\Gamma(\mu-\nu)} \int_t^\infty \frac{t^{\mu-\nu-1}\Gamma(\frac{\mu-\nu+1}{2})}{\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} h(s)ds = \frac{1}{\Gamma(\mu-\nu)} \int_0^\infty \mathscr{G}(t,s)h(s)ds.
\end{aligned}
$$

This completes the proof of the lemma. □

LEMMA 7. [25] *Suppose that $\mathscr{G}(t,s)$ be the Green's function for the linear system (3) given in the statement of Lemma 6. Then we find that:*

1. *$\mathscr{G}(t,s)$ is a continuous function on the unit square $[0,\infty]\times[0,\infty]$.*
2. *$\mathscr{G}(t,s)=0$ for each $(t,s)\in[0,\infty]\times[0,\infty]$ and*
3. *$\max_{t\in[0,1]}\mathscr{G}(t,s)=\mathscr{G}(\infty,s)$, for each $s\in[0,\infty]$.*

Inspired with the work of Zhao et al. [14], let us consider the space $\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$, defined by

$$\mathscr{C}_\infty([0,\infty),\ \mathbb{R})=\{u\in\mathscr{C}([0,\infty),\ \mathbb{R});\lim_{t\to\infty}\frac{u(t)}{1+t^{\mu-\nu-1}}\ \text{exists}\},$$

equipped with the norm

$$\|u\|_\infty=\sup_{t\in[0,\infty)}\left|\frac{u(t)}{1+t^{\mu-\nu-1}}\right|.$$

We can easily show that $\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$ is a Banach space. For $y\in\mathscr{C}_\infty$, consider an operator $T:\mathscr{C}_\infty\to\mathscr{C}_\infty$ defined by

$$Ty(t)=\int_0^\infty\mathscr{G}(t,s)[Af(s,y(s))+\sum_{i=1}^k B_ig_i(s,y(s),\mathfrak{B}y(s))]ds. \tag{5}$$

Clearly, the continuous function $y\in\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$, is the solution for BVP (1) iff $y(t)=Ty(t)$ for all $t\in\mathbb{R}$. To show the compactness of the operator T defined by equation (5), we prove that T has a fixed point on $\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$, see [15]. Due to the non compactness of the domain $[0,\infty)$, the Arzela-Ascoli theorem cannot be applied to the space $\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$. Henceforth, we need the following type of modification in the standard criteria.

LEMMA 8. [16] *Suppose that E is bounded subset of the Banach space $\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$. Then E is relatively compact in $\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$, provided that the following assumptions hold:*

- *For $y\in\mathscr{C}_\infty([0,\infty),\ \mathbb{R})$, $\frac{y(t)}{1+t^{\mu-\nu-1}}$ and $\frac{y'(t)}{1+t^{\mu-\nu-1}}$ are eqicontinuous on any compact half line $[0,\infty)$.*
- *For any $\varepsilon>0$, $\exists\,\Upsilon=\Upsilon(\varepsilon)$ such that for all $t_1,\ t_1\geqslant\Upsilon$,*

$$\left|\frac{y(t_1)}{1+t_1^{\mu-\nu-1}}-\frac{y(t_2)}{1+t_2^{\mu-\nu-1}}\right|\leqslant\varepsilon,\quad\left|\frac{y'(t_1)}{1+t_1^{\mu-\nu-1}}-\frac{y'(t_2)}{1+t_2^{\mu-\nu-1}}\right|\leqslant\varepsilon,$$

  for all $y(t)\in E$.
- *Then it is called equi-convergence at infinity for E.*

LEMMA 9. *Let us assume the following axiom hold*
(H_1) *Suppose the functions*

$$|f(t,(1+t^{\mu-\nu-1})u)| \leqslant \varphi(t)\omega(|u|),$$
$$|g_i(t,(1+t^{\mu-\nu-1})u,(1+t^{\mu-\nu-1})\widetilde{u})| \leqslant \varphi_i(t)\omega_i(|u|)+\overline{\varphi_i}(t)\overline{\omega_i}(|\widetilde{u}|),$$

where $\omega,\omega_i,\overline{\omega_i} \in C(\mathbb{R},(0,\infty))$, $i=1,\dots,k$ *are nondecreasing and* $\varphi,\varphi_i,\overline{\varphi_i} \in L^1([0,\infty),[0,\infty))$, $i=1,\dots,k$, *are nonnegative functions. Then the operator* $T:\mathscr{C}_\infty \to \mathscr{C}_\infty$ *is completely continuous.*

Proof. The proof of this Lemma is divided into three steps.
Step 1: In the first step, we prove that the operator $T:\mathscr{C}_\infty \to \mathscr{C}_\infty$ is continuous. Let $y_n \to y$ as $n\to\infty$ in $\mathscr{C}_\infty$, there exists r such that

$$\max\{\|y\|_\infty, \sup_{n\in N\setminus\{0\}} \|y_n\|_\infty\} < r,$$

we deduce

$$\begin{aligned}
&\int_0^\infty \frac{\mathscr{G}(t,s)}{1+t^{\mu-\nu-1}}[A|f(s,y_n(s))-f(s,y(s))| \\
&+\sum_{i=1}^k B_i|g_i(s,y_n(s),\mathfrak{B}y_n(s))-g_i(s,y(s),\mathfrak{B}y(s))|]ds \\
\leqslant\ & \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\int_0^\infty [A(|f(s,y_n(s))|+|f(s,y(s))|) \\
&+\sum_{i=1}^k B_i(|g_i(s,y_n(s),\mathfrak{B}y_n(s))|+|g_i(s,y(s),\mathfrak{B}y(s))|)]ds \\
=\ & \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\int_0^\infty [A(|f(s,(\frac{y_n(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}}))| \\
&+|f(s,\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}})|)+\sum_{i=1}^k B_i(|g_i(s,(\frac{y_n(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}}), \\
&(\frac{\mathfrak{B}y_n(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}}))|+|g_i(s,(\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}}),(\frac{\mathfrak{B}y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}}))|)]ds \\
=\ & \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[2A\omega(r)\int_0^\infty \varphi(s)ds \\
&+\sum_{i=1}^k 2B_i(\omega_i(r)\int_0^\infty \varphi_i(s)ds+\overline{\omega_i}(r)\int_0^\infty \overline{\varphi_i}(s)ds)\Big] < \infty.
\end{aligned}$$

Hence, we obtain

$$\|Ty_n - Ty\|_\infty \leqslant \sup_{t\in[0,\infty)}\int_0^\infty \frac{\mathscr{G}(t,s)}{1+t^{\mu-\nu-1}}A|f(s,y_n(s))-f(s,y(s))|$$

$$+\sum_{i=1}^{k} B_i|g_i(s,y_n(s),\mathfrak{B}y_n(s))-g_i(s,y(s),\mathfrak{B}y(s))|ds \to 0.$$

as $n \to +\infty$. Thus, T is continuous.

Step 2: Now we shall prove that $T:\mathscr{C}_\infty \to \mathscr{C}_\infty$ is relatively compact.

Suppose, on the contrary that $\aleph$ be any bounded subset of $\mathscr{C}_\infty$, then $\exists\, r_0>0$ s.t. $\|y\|_\infty \leqslant r_0$, we obtain

$$\begin{aligned}
\|Ty\|_\infty &= \sup_{t\in[0,\infty)} \int_0^\infty \Big|\frac{\mathscr{G}(t,s)}{1+t^{\mu-\nu-1}} Af(s,y(s)) + \sum_{i=1}^{k} B_i g_i(s,y(s),\mathfrak{B}y(s))\Big| ds\\
&\leqslant \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}} \int_0^\infty \Big|Af(s,y(s)) + \sum_{i=1}^{k} B_i g_i(s,y(s),\mathfrak{B}y(s))\Big| ds\\
&= \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}} \int_0^\infty \Big|Af(s,\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}})\\
&\quad + \sum_{i=1}^{k} B_i g_i(s,\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}},\frac{\mathfrak{B}y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}})\Big| ds\\
&= \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}} \Big[A\omega(r_0)\int_0^\infty \varphi(s)ds\\
&\quad + \sum_{i=1}^{k} B_i\{\omega_i(r_0)\int_0^\infty \varphi_i(s)ds + \overline{\omega_i}(r_0)\int_0^\infty \overline{\varphi_i}(s)ds\}\Big],
\end{aligned}$$

for $y \in \aleph$. Thus, $T\aleph$ is uniformly bounded.

Next, we prove that $T\aleph$ is equi-continuous on any compact domain of $[0,\infty)$. For any $T>0$, t_1, $t_2 \in [0,T]$, and $u \in \aleph$, without lose of generality, let $t_2>t_1$, Indeed

$$\begin{aligned}
&\Big|\frac{Ty(t_2)}{1+t_2^{\mu-\nu-1}} - \frac{Ty(t_1)}{1+t_1^{\mu-\nu-1}}\Big|\\
&= \Big|\int_0^\infty \frac{\mathscr{G}(t_2,s)}{1+t_2^{\mu-\nu-1}}[Af(s,y(s)) + \sum_{i=1}^{k} B_i g_i(s,y(s),\mathfrak{B}y(s))]ds\\
&\quad - \int_0^\infty \frac{\mathscr{G}(t_1,s)}{1+t_1^{\mu-\nu-1}}[Af(s,y(s)) + \sum_{i=1}^{k} B_i g_i(s,y(s),\mathfrak{B}y(s))]ds\Big|\\
&= \int_0^\infty \Big[\frac{\mathscr{G}(t_2,s)}{1+t_2^{\mu-\nu-1}} - \frac{\mathscr{G}(t_1,s)}{1+t_2^{\mu-\nu-1}} + \frac{\mathscr{G}(t_1,s)}{1+t_2^{\mu-\nu-1}} - \frac{\mathscr{G}(t_1,s)}{1+t_1^{\mu-\nu-1}}\Big(A\omega(r)\varphi(s)\\
&\quad + \sum_{i=1}^{k} B_i\{\omega_i(r)\varphi_i(s) + \overline{\omega_i}(r)\overline{\varphi_i}(s)\}\Big)ds\Big] \to 0,
\end{aligned}$$

uniformly as $t_1 \to t_2$ for all $y \in \aleph$. Therefore, $T\aleph$ is locally equi-continuous on the domain $[0,\infty)$.

Step 3: At last, with the help of the Lemma 8, we must prove that $T:\mathscr{C}_\infty \to \mathscr{C}_\infty$ is equi-convergent at ∞. For any $y \in \aleph$, we have

$$\int_0^\infty |Af(s,y(s)) + \sum_{i=1}^k B_i g_i(s,y(s),\mathfrak{B}y(s))|ds$$
$$\leqslant A\omega(r)\int_0^\infty \varphi(s)ds + \sum_{i=1}^k B_i\{\omega_i(r)\int_0^\infty \varphi_i(s)ds + \overline{\omega_i}(r)\int_0^\infty \overline{\varphi_i}(s)ds\} < \infty.$$

Thus, we obtain

$$\lim_{t\to+\infty} |\frac{Ty(t)}{1+t^{\mu-\nu-1}}| \to 0.$$

Hence $T\aleph$ is equi-convergent at ∞. By using Lemma 8, we examine that $T:\mathscr{C}_\infty \to \mathscr{C}_\infty$ is completely continuous. $\square$

3. Existence results

Now we state first existence result for a class of BVP of nonlinear FDE which is based on Leray-Schauder's alternative fixed point theorem.

THEOREM 2. *Let us assume that (H_1) hold and suppose the following condition is satisfy with $\rho > 0$ such that*

$$\rho > \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[A\omega(\rho)\int_0^\infty \varphi(s)ds$$
$$+\sum_{i=1}^k B_i\{\omega_i(\rho)\int_0^\infty \varphi_i(s)ds + \overline{\omega_i}(\rho)\int_0^\infty \overline{\varphi_i}(s)ds\}\Big]. \quad (6)$$

Then BVP (1) has an unbounded solution $y = y(t)$ such that $0 \leqslant \frac{y(t)}{1+t^{\mu-\nu-1}} \leqslant \rho$, for $t \in [0,\infty)$.

Proof. We address the following BVP of order $\mu-\nu \in (1,2)$ given as

$${}^{RL}D^{\mu-\nu}y(t) - \lambda Af(t,y(t)) - \lambda\sum_{i=1}^k B_i g_i(t,y(t),\mathfrak{B}y(t)) = 0,\ t\in[0,\infty),$$
$$y(0) = 0, \quad \lim_{t\to\infty} y(t) = a^{RL}D^{\frac{\mu-\nu-1}{2}}y(t))\big|_{t=\xi}. \quad (7)$$

for $\lambda \in (0,1)$. Solving the system (7) is similar to solving the fixed point problem $y = \lambda Ty$. Consider $U = \{y \in \mathscr{C}_\infty; \|y\|_{C_\infty} < \rho\}$. We plea that $y \neq \lambda Ty$ for $y \in \partial U$, $\lambda \in (0,1)$. Since if $\exists\, y \in \partial U$ with $y = \lambda Ty$, then the plea is adjoin for $\lambda \in (0,1)$, we occur

$$\|y\|_\infty = \sup_{t\in[0,\infty)}\left|\frac{(\lambda Ty)(t)}{1+t^{\mu-\nu-1}}\right| \leqslant \sup_{t\in[0,\infty)}\left|\frac{(Ty)(t)}{1+t^{\mu-\nu-1}}\right|$$

$$= \sup_{t\in[0,\infty)} \Big| \int_0^\infty \mathscr{G}(t,s)[Af(s,\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}})$$
$$+\sum_{i=1}^k B_i g_i(s,\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}},\frac{\mathfrak{B}y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}})]ds\Big|$$
$$\leqslant \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[A\omega(\rho)\int_0^\infty \varphi(s)ds$$
$$+\sum_{i=1}^k B_i\{\omega_i(\rho)\int_0^\infty \varphi_i(s)ds+\overline{\omega_i}(\rho)\int_0^\infty \overline{\varphi_i}(s)ds\}\Big].$$

Gathering with (6) and Lemma 9, we obtain

$$\rho = \|y\|_\infty \leqslant \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[A\omega(\rho)\int_0^\infty \varphi(s)ds$$
$$+\sum_{i=1}^k B_i\{\omega_i(\rho)\int_0^\infty \varphi_i(s)ds+\overline{\omega_i}(\rho)\int_0^\infty \overline{\varphi_i}(s)ds\}\Big] < \rho,$$

which contradicts with the inequality (6). By the Theorem 1 and Lemma 7, BVP (1) has an unbounded solution $y=y(t)$ s.t. $0\leqslant y(t)\leqslant 1+t^{\mu-\nu-1}\rho$, for $t\in[0,\infty)$. This completes the proof of the theorem. □

The next following existence result is based on Schauder's fixed point theorem.

THEOREM 3. *Under the condition* (H_1) *the BVP (1) has at least one solution on* $t\in[0,\infty)$.

Proof. Define an operator T as in (5). Now let us check that all the axioms of Schauder's fixed point theorem on $\mathscr{C}_\infty$ are satisfied. The given functions f, g_i, $\mathscr{G}(t,s)$ are continuous implies that the operator T is continuous. Rest of the proof of the Theorem 3 is distinguish into the following manner.

Claim 1: Consider a closed ball $B_{r_s}=\{y\in C_\infty : \|y\|\leqslant r_s\}$. Then we show that $T: B_{r_s}\to B_{r_s}$. We choose

$$r_s \geqslant \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[A\omega(r_s)\int_0^\infty \varphi(s)ds$$
$$+\sum_{i=1}^k B_i\{\omega_i(r_s)\int_0^\infty \varphi_i(s)ds+\overline{\omega_i}(r_s)\int_0^\infty \overline{\varphi_i}(s)ds\}\Big].$$

For any $y\in\mathscr{C}_\infty$, we show that $TB_{r_s}\subset B_{r_s}$. Then for $t\in[0,\infty)$, we have

$$\frac{|(Ty)(t)|}{1+t^{\mu-\nu-1}} = \sup_{t\in[0,\infty)}\int_0^\infty \Big|\frac{\mathscr{G}(t,s)}{1+t^{\mu-\nu-1}}Af(s,y(s))+\sum_{i=1}^k B_i g_i(s,y(s),\mathfrak{B}y(s))\Big|ds$$

$$
\begin{aligned}
&\leqslant \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\int_0^\infty \Big|Af(s,y(s)) \\
&\quad + \sum_{i=1}^k B_i g_i(s,y(s),\mathfrak{B}y(s))\Big|ds \\
&= \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\int_0^\infty \Big|Af(s,\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}}) \\
&\quad + \sum_{i=1}^k B_i g_i(s,\frac{y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}},\frac{\mathfrak{B}y(s)(1+s^{\mu-\nu-1})}{1+s^{\mu-\nu-1}})\Big|ds \\
&= \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2})-a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[A\omega(r_s)\int_0^\infty \varphi(s)ds \\
&\quad + \sum_{i=1}^k B_i\{\omega_i(r_s)\int_0^\infty \varphi_i(s)ds+\overline{\omega_i}(r_s)\int_0^\infty \overline{\varphi_i}(s)ds\}\Big] < r_s.
\end{aligned}
$$

Hence, we get that $\|Ty\|_{\mathscr{C}_\infty} \leqslant r_s$, which implies that $TB_{r_s} \subset B_{r_s}$, i.e the operator T maps B_{r_s} into B_{r_s}.

Claim 2: Secondly, with the help of Lemma 9, we can easily show that T is continuous and completely continuous on $\mathscr{C}_\infty$.

Therefore, one to conclude that from Schauder's fixed point theorem the operator T has a fixed point y in $\mathscr{C}_\infty$ which is a solution of BVP (1). □

4. Uniqueness result

THEOREM 4. *Assume that* (H_1) *and the following hypothesis hold*

(H_2) *There exist some positive functions* $l_1(t)$, $l_2(t)$, $l_3(t)$, $l_3^*(t)$ *with*

$$
\begin{aligned}
l_1^* &= \int_0^\infty (1+t^{\mu-\nu-1})l_1(t)dt < \infty, \\
l_2^* &= \int_0^\infty (1+t^{\mu-\nu-1})l_2(t)dt < \infty, \\
l_3^* &= \int_0^\infty (1+t^{\mu-\nu-1})l_3(t)dt < \infty, \\
l_3^{**} &= \int_0^\infty (1+t^{\mu-\nu-1})l_3^*(t)dt < \infty,
\end{aligned}
$$

such that

$$
\begin{aligned}
&|f(t,u)-f(t,v)| \leqslant l_1(t)|u-v|, \\
&|g_i(t,u,\widetilde{u})-g_i(t,v,\widetilde{v})| \leqslant l_2(t)|u-v|+l_3(t)|\widetilde{u}-\widetilde{v}|, \\
&|H(t,s,u)-H(t,s,v)| \leqslant l_3^*(t)|u-v|,
\end{aligned}
$$

$\forall$, $t \in [0,\infty)$, $u,v,\widetilde{u},\widetilde{v} \in \mathbb{R}$ *and* $\eta = \int_0^\infty |f(t,0)+g(t,0,0)|dt < \infty$.

Then BVP (1) contains a unique solution $y(t)$ in $\mathscr{C}_\infty$. Moreover, $\{y_n(t)\}$ be a monotone iterative sequence s.t. $y_n(t) \to y(t)$ as $n \to \infty$ which is uniformly on any unbounded sub domain of $t \in [0,\infty)$. In addition, there exists an error estimate for the approximation sequence

$$\|y_n - y\|_\infty \leqslant \frac{\Delta^n}{1-\Delta}\|y_1 - y_0\|_\infty. \tag{8}$$

where $n = 1,2,...$ and

$$\Delta = \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big(Al_1^* + \sum_{i=1}^{k} B_i(l_2^* + l_3^* l_3^{**})\Big) < 1.$$

Proof. Consider an operator T defined by the equation (5). Let $y_1,\ y_2 \in \mathscr{C}_\infty$. For $t \in [0,\infty)$, we have

$$\begin{aligned}
\frac{|(Ty_1)(t) - (Ty_2)(t)|}{1+t^{\mu-\nu-1}} &\leqslant \int_0^\infty \frac{\mathscr{G}(t,s)}{1+t^{\mu-\nu-1}}[A|f(s,y_1(s)) - f(s,y_2(s))| \\
&\quad + \sum_{i=1}^{k} B_i|g_i(s,y_1(s),\mathfrak{B}y_1(s)) - g_i(s,y_2(s),\mathfrak{B}y_2(s))|]ds \\
&\leqslant \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \\
&\quad \Big(A\int_0^\infty [(1+t^{\mu-\nu-1})l_1(t)]dt + \sum_{i=1}^{k} B_i \int_0^\infty [(1+t^{\mu-\nu-1})l_2(t) \\
&\quad + (1+t^{\mu-\nu-1})l_3(t)l_3^*(t)]dt\Big)\|y_1 - y_2\|_\infty.
\end{aligned}$$

By using the appropriate assumptions, it gives that

$$\begin{aligned}
|(Ty_1)(t) - (Ty_2)(t)| &\leqslant \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}} \\
&\quad \Big(Al_1^* + \sum_{i=1}^{k} B_i(l_2^* + l_3^* l_3^{**})\Big)\|y_1 - y_2\|_\infty \\
&= \Delta\|y_1 - y_2\|_\infty.
\end{aligned}$$

Thus, we collect that

$$\|Ty_1 - Ty_2\|_\infty \leqslant \Delta\|y_1 - y_2\|_\infty,\ \forall y_1,\ y_2 \in \mathscr{C}_\infty. \tag{9}$$

As $\Delta < 1$, from the Banach fixed point theorem we deduce that T has a unique fixed point y in $\mathscr{C}_\infty$. Therefore, the BVP (1) has a unique solution $y \in \mathscr{C}_\infty$. In addition, for any $y_0 \in \mathscr{C}_\infty$, $\|y_n - y\|_\infty \to 0$ as $n \to \infty$, where $y_n = Ty_{n-1}(n = 1,2...)$. From inequality (9), we have

$$\|y_n - y_{n-1}\|_\infty \leqslant \Delta^{n-1}\|y_1 - y_0\|_\infty,$$

and

$$\begin{aligned}\|y_n - y_j\|_\infty &\leqslant \|y_n - y_{n-1}\|_\infty + \|y_{n-1} - y_{n-2}\|_\infty + \dots + \|y_{j+1} - y_j\|_\infty \\ &= \frac{\Delta^n(1-\Delta^{n-j})}{1-\Delta}\|y_1 - y_0\|_\infty. \end{aligned} \tag{10}$$

Let $n \to \infty$ in both sides of equation (10), we can estimate

$$\|y_n - y\|_\infty \leqslant \frac{\Delta^n}{1-\Delta}\|y_1 - y_0\|_\infty.$$

Hence equation (8) holds, and this completes the proof of the theorem. □

5. Application

In this section, we determine the following BVP to illustrate our one result:

$$\begin{aligned} {}^{RL}D^{1.5}y(t) &= Af(t,y(t)) + \sum_{i=1}^{2} B_i g_i(t, y(t), \mathfrak{B}y(t)), \quad t \in [0,\infty), \\ y(0) &= 0, \quad \lim_{t\to\infty} y(t) = a({}^{RL}D^{0.25}y(t))\big|_{t=1/2}. \end{aligned} \tag{11}$$

Choose $\mu = 3$, $\nu = 1.5$, $\omega(y) = \sqrt{y}$, $\omega_i(y) = \sqrt{y}/2$, $\varphi(t) = \frac{e^{-t}}{e^t+1}$, $\varphi_i(t) = e^{-t}$ for $i = 1,2$. Here, $A = B_i = 1$, $(i = 1,2)$, $\beta_1 = 1/2$, $\beta_2 = 2/3$, $a = 1$. Clearly, we have

$$f(t,y) = \frac{e^{-t}}{(1+t^{1/2})^2}\cos(3t^2 + y(t)),$$

and

$$g_i(t,y,\mathfrak{B}y(t)) = \frac{e^t|y(t)|}{(9+e^t)(1+|y(t)|)} + \int_0^t \frac{e^{-(s-t)}}{10}|y(s)|ds.$$

From the given data, we obtain:

1. Suppose the functions

$$|f(t,(1+t^{\mu-\nu-1})u)| \leqslant \varphi(t)\omega(|u|),$$

and

$$|g_i(t,(1+t^{\mu-\nu-1})u,(1+t^{\mu-\nu-1})\widetilde{u})| \leqslant \varphi_i(t)\omega_i(|u|) + \overline{\varphi_i}(t)\overline{\omega_i}(|\widetilde{u}|),$$

where $\omega, \omega_i, \overline{\omega_i} \in C(\mathbb{R},(0,\infty))$, $i = 1,\dots,k$ are nondecreasing and $\varphi, \varphi_i, \overline{\varphi_i} \in L^1([0,\infty),[0,\infty))$, $i = 1,\dots,k$, are nonnegative functions.

2. Also

$$\begin{aligned}\rho > \frac{\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\mu-\nu)\{\Gamma(\frac{\mu-\nu+1}{2}) - a\xi^{\frac{\mu-\nu-1}{2}}\}}\Big[A\omega(\rho)\int_0^\infty \varphi(s)ds \\ + \sum_{i=1}^{k} B_i\{\omega_i(\rho)\int_0^\infty \varphi_i(s)ds + \overline{\omega_i}(\rho)\int_0^\infty \overline{\varphi_i}(s)ds\}\Big].\end{aligned}$$

Hence all conditions of Theorem 1 hold and we conclude that the problem (11) has at least one positive solution $y(t)$ such that $0 \leqslant \frac{y(t)}{1+t^{0.3}} \leqslant \rho$, $t \in [0,\infty)$.

6. Conclusion

This paper is motivated from the work carried out [5, 14]. Basically, In this paper three new features are added. First, by using the main properties of Riemann-Liouville fractional derivative and various characteristics of Green's function, we obtained the solution of the composed problem (1). Second, The technique of the fixed point theorems are used to shown the existence and uniqueness results for the positive solution of the considered fractional integro differential equation with boundary conditions on half line domain. Third, we emphasis on an important result in section 4 to exiting literature of the topic, in which we not only discuss the uniqueness but also introduce an explicit iterative sequence with an error estimate for approximating the solution. Compile with these three contributions, we also explain an example to verify one of the results.

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