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ON THE SOLUTIONS FOR AN EXTENSIBLE BEAM EQUATION WITH INTERNAL DAMPING AND SOURCE TERMS

DUCIVAL C. PEREIRA, HOANG NGUYEN*, CARLOS A. RAPOSO AND CELSA H. M. MARANHÃO

Dedicated to the 60th birthday of Jaime Rivera

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Abstract. In this manuscript, we consider the nonlinear beam equation with internal damping and source term

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t = |u|^{r-1}u$$

where $r > 1$ is a constant, $M(s)$ is a continuous function on $[0,+\infty)$. The global solutions are constructed by using the Faedo-Galerkin approximations, taking into account that the initial data is in appropriate set of stability created from the Nehari manifold. The asymptotic behavior is obtained by the Nakao method.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. In this paper, we study the existence and the energy decay estimate of global solutions for the initial boundary value problem of the following equation with internal damping and source terms

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t = |u|^{r-1}u \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = \frac{\partial u}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, t \geqslant 0, \tag{1.3}$$

where $r > 1$ is a constant, $M(s)$ is a continuous function on $[0,+\infty)$. In (1.3), $u = 0$ is the homogeneous Dirichlet boundary condition and the normal derivative $\partial u/\partial\eta = 0$ is the homogeneous Neumann boundary condition where η is the unit outward normal on

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* Corresponding author.

$\partial\Omega$. The physical meaning of the clamped boundary conditions (1.3) is that, with the natural boundary conditions, we imposed no a priori conditions on the function space and it turns out that a weak solution automatically satisfies the boundary conditions.

In 1955, Berger [10] established the equation

$$u_{tt} - \left(Q + \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = p(u, u_t, x), \tag{1.4}$$

which is called the Berger plate model [12], where the parameter Q describes in-plane forces applied to the plate and the function p represents transverse loads which may depend on the displacement u and the velocity u_t. If $n = 2$, the equation (1.4) represents the "Berger approximation" of the Von Kárman equations, modelling the nonlinear vibrations of a plate (see [15], pg. 501-507).

When $n = 1$ and $p = 0$, the corresponding equation had been introduced by Woinowsky-Krieger [13] as a model for the transverse motion of an extensible beam. It means that the equation (1.1) describes the transverse deflection of an extensible beam of the length L whose ends are attached at a fixed distance is the following equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} + \left(\beta + \int_0^L u_\varepsilon^2(\varepsilon, t)\, \mathrm{d}\,\varepsilon\right) \left(-\frac{\partial^2 u}{\partial u^2}\right) = f,$$

where α is a positive constant, β is a constant not necessarily positive and the nonlinear term represents the change in the tension of the beam due to its extensibility.

The physical origin of the problem here relates to the study of the dynamical buckling of the hinged extensible beam which is either stretched or compressed by an axial force. The readers could also see in Burgreen [24] and Eisley [25] for more physical justifications and the model background.

Cavalcanti et al [28] studied the equation

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u_t) + f(u) = 0 \tag{1.5}$$

with $g(s) = |s|^{\rho-1}s$ and $f(s) = |s|^{\gamma-1}s$ where ρ and γ are positive constants such that $1 < \rho, \gamma \leqslant n/(n-2)$ if $n \geqslant 3$; $\rho, \gamma > 1$ if $n = 1, 2$. The global existence and asymptotic stability were proved by means of the fixed point theorem and continuity arguments.

Zhijian [29] investigated the problem (1.5) more generally as follows

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u_t) + f(u) = h(x), \tag{1.6}$$

where the source terms $f, g \in C^1(\mathbb{R})$, $|f'(s)| \leqslant C(1 + |s|^{p-1})$ and $K_0|s|^{q-1} < g'(s) \leqslant C(1 + |s|^{q-1})$, $K_0, C > 0$ with $1 \leqslant p < \infty, 1 \leqslant q < \infty$ if $n \leqslant 4$; $1 \leqslant p \leqslant p^* = (n+4)/(n-4)$ and $p \leqslant q$ if $N \geqslant 5$. By Galerkin approximation combined with the monotone arguments, the author proved the existence of global solution.

The equation (1.1) without source terms was studied by several authors in different contexts. The problem without damping was considered in Dickey [5], Ball [1], Medeiros [6], Pereira [7] among others. When the damping term is considered, the problem was studied by, Brito [4], Biler [3], Ball [2] see also the references therein.

Beam equation with weak internal damping in domain with moving boundary was studied by Clark [8]. Extensible beam equation with nonlinearity of Kirchhoff type in domains with moving boundary and memory was studied in [11]. For coupled system of extensible beam models, we cite [9] and references therein.

In this work we use the potential well theory which approach is completely different from those in [28, 29]. In [28], the existence of global solutions was proved by means of the Fixed point theorem and the asymptotic behavior was obtained by using the perturbed energy method. In [29], the global existence and the longtime dynamics of solutions were considered by using semigroup theory. The outline of the paper is as follows. In the Section 2, we introduce some notations and the stability set created from the Nehari Manifold. In the Section 3, we prove the existence of solution through the Faedo-Galerkin method. By result of M. Nakao [20], the energy decay in the appropriate set of stability will be given in Section 4. In the last section, some final comments will be presented.

2. The potential well

It is well-known that the energy of a PDE system, in some sense, splits into the kinetic and the potential energy. By following the idea of Y. Ye [26], we are able to construct a set of stability. We will prove that there is a valley or a well of the depth d created in the potential energy. If d is strictly positive, then we find that, for solutions with the initial data in the good part of the potential well, the potential energy of the solution can never escape the potential well. In general, it is possible that the energy from the source term to cause the blow-up in a finite time. However, in the good part of the potential well, it remains bounded. As a result, the total energy of the solution remains finite on any time interval [0; T), providing the global existence of the solution.

We started by introducing the functional $J : H_0^2(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u|^2 - \frac{1}{r+1}|u|_{r+1}^{r+1}.$$

For $u \in H_0^2(\Omega)$, we have

$$J(\lambda u) = \frac{\lambda^2}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u|^2 - \frac{\lambda^{r+1}}{r+1}|u|_{r+1}^{r+1}, \lambda > 0.$$

Associated with J, we have the well-known Nehari Manifold given by

$$\mathscr{N} \stackrel{def}{=} \left\{u \in H_0^2(\Omega)\setminus\{0\};\ \left[\frac{d}{d\lambda}J(\lambda u)\right]_{\lambda=1} = 0\right\}.$$

Equivalently,

$$\mathscr{N} = \left\{u \in H_0^2(\Omega)\setminus\{0\};\ \left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u|^2 = |u|_{r+1}^{r+1}\right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [14],

$$d \stackrel{def}{=} \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda \geqslant 0} J(\lambda u).$$

It is well-known for $1 < r \leqslant 5$ that the depth of the well is a real constant strictly positive ([15], theorem 4.2) and $d = \inf_{u \in \mathcal{N}} J(u)$.

We now introduce the potential well

$$W = \left\{u \in H_0^2(\Omega);\ \ J(u) < d\right\} \cup \{0\}$$

and partition it into two sets as follows

$$W_1 = \left\{u \in H_0^2(\Omega);\ \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u|^2 > \frac{1}{r+1}|u|_{r+1}^{r+1}\right\} \cup \{0\}$$

and

$$W_2 = \left\{u \in H_0^2(\Omega);\ \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)|\Delta u|^2 < \frac{1}{r+1}|u|_{r+1}^{r+1}\right\}.$$

We will refer to W_1 as the "good" part of the potential well. Then we define by W_1 the set of stability for the problem (1.1)-(1.3).

3. Existence of global solutions

We consider the following hypothesis

(H) $M \in C([0,\infty])$ with $M(\lambda) \geqslant -\beta, \forall\, \lambda \geqslant 0, 0 < \beta < \lambda_1$,
λ_1 is the first eigenvalue of the problem $\Delta^2 u - \lambda(-\Delta u) = 0$.

REMARK 1. Let λ_1 be the first eigenvalue of $\Delta^2 u - \lambda(-\Delta u) = 0$ with the clamped boundary conditions

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \eta}|_{\partial\Omega} = 0,$$

then (see Miklin [16]),

$$\lambda_1 = \inf_{u \in H_0^2(\Omega)} \frac{|\Delta u|^2}{|\nabla u|^2} > 0 \ \text{ and } \ |\nabla u|^2 \leqslant \frac{1}{\lambda_1}|\Delta u|^2.$$

THEOREM 1. *Let $u_0 \in W_1$, $E(0) < d$, $u_1 \in L^2(\Omega)$, $1 < r \leqslant 5$. If the hypothesis (H) holds, then there exists a function $u : [0,T] \to L^2(\Omega)$ in the class*

$$u \in L^\infty(0,T; H_0^2(\Omega)) \cap L^\infty(0,T; L^{r+1}(\Omega)) \tag{3.1}$$

$$u_t \in [L^\infty(0,T; L^2(\Omega)) \tag{3.2}$$

such that, for all $w \in H_0^2(\Omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t(t),w)+\langle\Delta u(t),\Delta w\rangle+M(|\nabla u|^2)(-\Delta u,w)+(u_t(t),w)-(|u(t)|^{r-1}u(t),w)=0, \tag{3.3}$$

$$u(0)=u_0,\, u_t(0)=u_1, \tag{3.4}$$

in $\mathscr{D}'(0,T)$.

Proof. We use the Faedo-Galerkin's method and potential well to prove the global existence of solutions.

3.1. Approximated problem

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^2(\Omega)$ from the eigenvectors of the operator $-\Delta$, and $V_m=\operatorname{span}\{w_1,w_2,\dots,w_m\}$. Let

$$u^m(t)=\sum_{j=1}^{m}k_{jm}(t)w_j$$

be a solution of the approximated problem

$$(u_{tt}^m(t),w)+(\Delta u^m(t),\Delta w)+M(|\nabla u^m(t)|^2)(-\Delta u^m(t),w)$$
$$+(u_t^m(t),w)-\big(|u^m(t)|^{r-1}u^m(t),w\big)=0,\forall\, w\in V_m, \tag{3.5}$$
$$u^m(0)=u_{0m}\longrightarrow u_0 \text{ strongly in } H_0^2(\Omega, \tag{3.6}$$
$$u_t^m(0)=u_{1m}\longrightarrow u_1 \text{ strongly in } L^2(\Omega). \tag{3.7}$$

The system (3.5)-(3.7) has a local solution in $[0,t_m)$, $0<t_m\leqslant T$, by virtue of Carathéodory's theorem, see [17]. The extension of the solution to the whole interval $[0,T]$ is a consequence of the following estimate.

3.2. A priori estimates

Let $w=u_t^m(t)$ in (3.5). We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|u_t^m(t)|^2+\frac{1}{2}|\Delta u^m(t)|^2+\frac{1}{2}\hat{M}|\nabla u^m(t)|^2-\frac{1}{r+1}|u^m(t)|_{r+1}^{r+1}\right]=-|u_t^m(t)|\leqslant 0, \tag{3.8}$$

where $\hat{M}(s)=\displaystyle\int_0^s M(\xi)d\xi$.

Integrating (3.8) from 0 to t, $0\leqslant t\leqslant t_m$, we obtain

$$\frac{1}{2}|u_t^m(t)|^2+\frac{1}{2}|\Delta u^m(t)|^2+\frac{1}{2}\hat{M}|\nabla u^m(t)|^2-\frac{1}{r+1}|u^m(t)|_{r+1}^{r+1}+\int_0^t|u_t^m(t)|^2\,\mathrm{d}s=\frac{1}{2}|u_{1m}|^2. \tag{3.9}$$

Now, by (H), since $\beta < \lambda_1$, we have

$$\hat{M}(|\nabla u^m(t)|^2) \geqslant -\frac{\beta}{\lambda_1}|\Delta u^m(t)|^2. \tag{3.10}$$

By (3.9) and (3.10), it follows

$$\begin{aligned}&\frac{1}{2}|u_t^m(t)|^2 + \frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)|\Delta u^m(t)|^2 - \frac{1}{r+1}|u^m(t)|_{r+1}^{r+1} + \int_0^t |u_t^m(s)|^2\,\mathrm{d}s\\ \leqslant&\frac{1}{2}|u_{1m}|^2 + \frac{1}{2}|\Delta u_{0m}|^2 + \frac{1}{2}\hat{M}\left(|\nabla u_{0m}|^2\right) - \frac{1}{r+1}|u_{1m}|_{r+1}^{r+1}.\end{aligned}$$

Now,

$$\hat{M}(|\nabla u_{0m}|^2) \leqslant m_0|\nabla u_{0m}|^2 \leqslant \frac{m_0}{\lambda_1}|\Delta u_{0m}|^2, \tag{3.11}$$

where $m_0 = \max\limits_{0\leqslant s\leqslant |\nabla u_{0m}|\leqslant C_0} M(s)$ and C_0 is a positive constant independent of m and t.

Therefore, the approximate energy

$$E_m(t) = \frac{1}{2}|u_t^m(t)|^2 + \frac{1}{2}|\Delta u^m(t)|^2 + \frac{1}{2}\hat{M}(|\nabla u^m(t)|^2) - \frac{1}{r+1}|u^m(t)|_{r+1}^{r+1}$$

satisfies

$$E_m(t) \leqslant E_(0) = \frac{1}{2}|u_{1m}|^2 + C_1 J(u_{0m}),$$

where $C_1 = C_1(m_0, \lambda_1, \beta) > 0$ is a constant independent of m and t.

We have that $J(u_{0m}) < d$ and by convergence (3.7), there exists a constant $C_2 > 0$ independent of m and t such that $\frac{1}{2}|u_{1m}|^2 \leqslant C_2$. So, there exists a constant $C_3 > 0$ such that

$$E_m(t) + \int_0^t |u_t^m(s)|^2\,\mathrm{d}s = E_m(0) < C_3,$$

that is,

$$\frac{1}{2}|u_t^m(t)|^2 + \frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)|\Delta u^m(t)|^2 - \frac{1}{r+1}|u^m(t)|_{r+1}^{r+1} \leqslant E_m(t) \leqslant C_3. \tag{3.12}$$

We can extend the approximate solutions $u^m(t)$ to the interval $[0,T]$, $T>0$ (see [18]).

Then by (3.12) we have

$$(u^m)\text{is bounded in} \quad L^\infty(0,T;H_0^2(\Omega)) \cap L^\infty(0,T;L^{r+1}(\Omega)) \tag{3.13}$$

$$(u_t^m)\text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^2(\Omega)) \tag{3.14}$$

$$(|u^m|^{r-1}u^m)\text{is bounded in} \quad L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)) \tag{3.15}$$

3.3. Passage to the limit

From the estimates (3.13)-(3.15), there exists a subsequence of (u^m), also denoted by (u^m), such that

$$u^m \overset{*}{\rightharpoonup} u \text{ weakly star in } L^\infty(0,T;H_0^2(\Omega)), \tag{3.16}$$

$$u_t^m \overset{*}{\rightharpoonup} u_t \text{ weakly star in } L^\infty(0,T;L^2(\Omega), \tag{3.17}$$

$$|u^m|^{r-1}u^m \rightharpoonup \chi \text{ weakly in } L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)). \tag{3.18}$$

Applying the Lions-Aubin Lemma [19], we get from (3.16)-(3.17)

$$u^m \longrightarrow u \text{ strongly in } L^2(0,T;H_0^1(\Omega))$$

and since M is continous, it follows

$$M(|\nabla u^m|^2) \longrightarrow M(|\nabla u|^2) \text{ strongly in } L^2(0,T).$$

Therefore,

$$M(|\nabla u^m|^2)(-\Delta u^m) \rightharpoonup M(|\nabla u|^2)(-\Delta u) \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{3.19}$$

Now we prove that $\chi = |u|^{r-1}u$. Observe that

$$\int_0^T \left||u^m(t)|^{r-1}u^m(t)\right|^{\frac{r+1}{r}} \mathrm{d}t = \int_0^T |u^m(t)|^{r+1}\,\mathrm{d}t \leqslant C.$$

So,

$$|u^m|^{r-1}u^m \longrightarrow |u|^{r-1}u \text{ a.e. in } \Omega\times[0,t).$$

Therefore, from [19] Lemma 1.3, we have

$$|u^m|^{r-1}u^m \rightharpoonup |u|^{r-1}u \text{ weakly in } L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)). \tag{3.20}$$

So, by (3.18) and (3.20), we have $\chi = |u|^{r-1}u$.

By the convergence (3.16), (3.17) and (3.20), we can pass to the limit in the approximate equation (3.5) and obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t(t),w) + \langle \Delta u(t), \Delta w\rangle + M(|\nabla u|^2)(-\Delta u, w) + (u_t(t),w) - (|u(t)|^{r-1}u(t),w) = 0$$

in $\mathscr{D}'(0,T)$. The proof of existence is complete. □

4. Asymptotic behavior

We use the following result due to Nakao (see [20], Lemma 2].

LEMMA 1. *Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\sup_{t\leqslant s\leqslant t+1} \operatorname{ess} \phi(s) \leqslant C_0[\phi(t)-\phi(t+1)],$$

for any $t \geqslant 0$, where C_0 is a positive constant. Then,

$$\phi(t) \leqslant Ce^{-\alpha t}, \forall\, t \geqslant 0,$$

where C and α are positive constants.

THEOREM 2. *Under the hypotheses of theorem 1, the solution of problem* (1.1)-(1.3) *satisfies:*

$$\frac{1}{2}|u_t(t)|^2+\frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)|\Delta u(t)|^2-\frac{1}{r+1}|u(t)|_{r+1}^{r+1}+\int_t^{t+1}|u_t(s)|^2\,\mathrm{d}s \leqslant Ce^{-\alpha t},$$

for any $t \geqslant 0$, where C and α are positive constants.

Proof. Let $w = u_t(t)$ in the equation (3.3). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}|u_t(t)|^2+\frac{1}{2}|\Delta u(t)|^2+\frac{1}{2}\hat{M}(|\nabla u(t)|^2)-\frac{1}{r+1}|u(t)|_{r+1}^{r+1}\right]+|u_t(t)|^2=0.$$

That is, $\frac{\mathrm{d}}{\mathrm{d}t}E(t)+|u_t(t)|^2=0$, where

$$E(t)=\frac{1}{2}|u_t(t)|^2+\frac{1}{2}|\Delta u(t)|^2+\hat{M}(|\nabla u(t)|^2)-\frac{1}{r+1}|u(t)|_{r+1}^{r+1}. \tag{4.1}$$

Integrating from t to $t+1$, we obtain

$$\int_t^{t+1}|u_t(s)|^2\,\mathrm{d}s=E(t)-E(t+1)\overset{\text{def}}{=}F^2(t). \tag{4.2}$$

Then there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$, $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$|u_t(t_i)| \leqslant 2F(t_i), i=1;2. \tag{4.3}$$

Let $w = u(t)$ in the equation (3.3). Integrating from t_1 to t_2 and observing the hypothesis (H), we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2}\left[\left(1-\frac{\beta}{\lambda_1}\right)|\Delta u(s)|^2-|u(s)|_{r+1}^{r+1}\right]\mathrm{d}s\\
&\leqslant |u_t(t_1)||u(t_1)|+|u_t(t_2)||u(t_2)|+\textstyle\int_{t_1}^{t_2}|u_t(s)|^2\,\mathrm{d}s+\displaystyle\int_{t_1}^{t_2}|u_t(s)||u(s)|\,\mathrm{d}s\\
&\leqslant C_1 \sup\operatorname{ess}_{t\leqslant s\leqslant t+1}|\Delta u(s)|\left[|u_t(t_1)|+|u_t(t_2)|\right]+F^2(t)+\int_{t_1}^{t_2}C_1|u_t(s)||\Delta u(s)|\,\mathrm{d}s\\
&\leqslant 4C_1F(t)\sup\operatorname{ess}_{t\leqslant s\leqslant t+1}|\Delta u(s)|+F^2(t)+\frac{C_1^2}{\delta}\int_{t_1}^{t_2}|u_t(s)|^2\,\mathrm{d}s+\delta\int_{t_1}^{t_2}|\Delta u(s)|^2\,\mathrm{d}s,
\end{aligned}$$

where $0<\delta\leqslant \frac{1}{2}-\frac{\beta}{\lambda_1}$ and $C_1>0$ is a constant such that $|u(t)|\leqslant C_1|\Delta u(t)|$. Then

$$\int_{t_1}^{t_2}\Big[\Big(1-\frac{\beta}{\lambda_1}-\delta\Big)|\Delta u(s)|^2-|u(s)|_{r+1}^{r+1}\Big]\mathrm{d}s\leqslant 4C_1F(t)\sup_{t\leqslant s\leqslant t+1}\operatorname{ess}|\Delta u(s)|+\Big(1+\frac{C_1^2}{\delta}\Big)F^2(t).$$

Whence,

$$\int_{t_1}^{t_2}\Big[\Big(1-\frac{\beta}{\lambda_1}-\delta\Big)|\Delta u(s)|^2-|u(s)|_{r+1}^{r+1}\Big]\mathrm{d}s\leqslant C_2\Big[F(t)\sup_{t\leqslant s\leqslant t+1}\operatorname{ess}|\Delta u(s)|+F^2(t)\Big]\stackrel{\text{def}}{=}G^2(t). \tag{4.4}$$

Thanks to (4.2) and (4.3), we have

$$\int_{t_1}^{t_2}\Big[\Big(1-\frac{\beta}{\lambda_1}-\delta\Big)|\Delta u(s)|^2-|u(s)|_{r+1}^{r+1}+|u_t(s)^2|\Big]\mathrm{d}s\leqslant F^2(t)+G^2(t).$$

Hence there exists $t^*\in[t_1,t_2]$ such that

$$|u_t(t^*)|+\Big(1-\frac{\beta}{\lambda_1}-\delta\Big)|\Delta(t^*)|^2-|u(t^*)|_{r+1}^{r+1}\leqslant 2[F^2(t)+G^2(t)]$$

or,

$$|u_t(t^*)|^2+|\Delta u(t^*)|^2-|u(t^*)|_{r+1}^{r+1}\leqslant C_3[F^2(t)+G^2(t)], \tag{4.5}$$

where C_3 is a positive constant. Now,

$$\hat{M}(|\nabla u(t^*)|^2)\leqslant m_0|\nabla u(t^*)|^2\leqslant \frac{m_0}{\lambda_1}|\Delta u(t^*)|^2,$$

where $m_0=\max_{0\leqslant s\leqslant|\nabla u(t^*)|^2}M(s)$. Therefore,

$$\hat{M}(|\nabla u(t^*)|^2)\leqslant C_4[F^2(t)+G^2(t)]. \tag{4.6}$$

From (4.1), (4.5) and (4.6), it follows that

$$E(t^*)\leqslant C_5[F^2(t)+G^2(t)]. \tag{4.7}$$

Now, by (4.2) and (4.7), we have

$$\begin{aligned}\sup\operatorname{ess}_{t\leqslant s\leqslant t+1}E(s)&\leqslant E(t^*)+\int_t^{t+1}|u_t(s)|^2\,\mathrm{d}s\leqslant C_5[F^2(t)+G^2(t)]+F^2(t)\\&\leqslant C_6F^2(t)+\frac{1}{2}\sup\operatorname{ess}_{t\leqslant s\leqslant t+1}E(s).\end{aligned}$$

Therefore,

$$\sup_{t\leqslant s\leqslant t+1}\operatorname{ess}E(s)\leqslant C_7\,[E(t)-E(t+1)],$$

where C_i, $i=4,5,6,7$ are positive constants. By Lemma 1, we have

$$E(t)\leqslant Ce^{-\alpha t},\ \forall\, t\geqslant 0, \tag{4.8}$$

where C and α are positive constants. So, by hypothesis (H), (4.2) and (4.8), it follows that

$$\frac{1}{2}|u_t(t)|^2+\frac{1}{2}\Big(1-\frac{\beta}{\lambda_1}\Big)|\Delta u(t)|^2-\frac{1}{r+1}|u(t)|_{r+1}^{r+1}+\int_t^{t+1}|u_t'(s)|^2\,\mathrm{d}s\leqslant Ce^{-\alpha t},\ \forall\, t\geqslant 0,$$

and the proof of theorem 2 is complete. □

Final comments

The force term can be more general that $|u|^{r-1}u$, type $f \in C^1(\mathbb{R})$ with $|f(u)| \leqslant c|u|^r$ for all $|u| \geqslant 1$, where $1 \leqslant r < 6$ with the suitable Sobolev imbeddings. For instance, for $n = 3$ we have $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and the Nemytski operator $f(u)$ is locally Lipschitz continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$ for $1 \leqslant r \leqslant 3$. The source is called subcritical if $1 \leqslant r < 3$, critical if $r = 3$ and supercritical if $3 < r \leqslant 5$. When $5 < r < 6$, the source is super-supercritical (see [27]) and in this situation, the potential energy may not be defined in the finite energy space, so the problem itself is no longer in the framework of the potential well theory.

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Ducival C. Pereira
Department of Mathematics
State University of Pará (UEPA)
Brazil
e-mail: ducival@uepa.br

Hoang Nguyen
LMAP (UMR E2S-UPPA CNRS 5142)
Bat. IPRA, Avenue de l'Université
64013 Pau, France
Institute of Mathematics and Campus Duque de Caxias
Federal University of Rio de Janeiro (UFRJ)
Brazil
e-mail: huy-hoang.nguyen@univ-pau.fr, nguyen@im.ufrj.br

Carlos A. Raposo
Department of Mathematics
Federal University of São João del-Rei (UFSJ)
Brazil
e-mail: raposo@ufsj.edu.br

Celsa H. M. Maranhão
Department of Mathematics
Federal University of Pará (UFPA)
Brazil
e-mail: celsa@ufpa.br