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DYNAMICS OF THERMOELASTIC PLATE SYSTEM WITH TERMS CONCENTRATED IN THE BOUNDARY

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Abstract. In this paper we show the existence, uniform boundedness and upper semicontinuity of the global attractors of autonomous thermoelastic plate systems with Neumann boundary conditions when some reaction terms are concentrated in a neighborhood of the boundary and this neighborhood shrinks to boundary as a parameter ε goes to zero.

1. Introduction

In this work we analyze the behavior of the global compact attractors of autonomous thermoelastic plate systems with Neumann boundary conditions when some reaction terms are concentrated in a neighborhood of the boundary and this neighborhood shrinks to boundary as a parameter ε goes to zero.

There has been numerous studies to investigate the dynamics, in the sense of global compact attractors, of systems when reaction terms are concentrated in a neighborhood of the boundary and this neighborhood shrinks to boundary as a parameter ε goes to zero. For instance, concentrated terms equations on the strip were initially studied in [9], where linear elliptic equations with terms concentrated were considered and convergence results of the solutions were proved. Later, the asymptotic behavior of the attractors of a parabolic problem was analyzed in [14], [15], where the upper semicontinuity of attractors was proved. In [4], [5], some results of [9], [14], [15] were extended to reaction-diffusion problems with dealy. In [6] some results of [9] were adapted to a semilinear elliptic equation posed on an open square contained in $\mathbb{R}^2$ and considering a strip with oscillatory behavior. Remarks on semilinear parabolic systems with terms concentrating in the boundary can be found in [21]. The work [7] was the first to consider the lower semicontinuity of attractors for parabolic problems with terms concentrating on the boundary. Recently, in [2], [3], the upper semicontinuity of the pullback attractors and the continuity of the set equilibria were proved for a non-autonomous damped wave equation with Neumann boundary conditions and terms concentrating on the boundary. But, in this works, thermoelastic plate systems with Neumann boundary conditions were not considered with concentration technique.

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Here, the Neumann boundary condition for the hyperbolic equation, as well as for the parabolic equation it will bring us extra difficulties that we need to work around.

To better describe the problem we introduce some notations, let Ω be an open bounded smooth set in $\mathbb{R}^N$, $N \geqslant 2$, with a smooth boundary $\Gamma = \partial\Omega$. We define the strip of width ε and base $\partial\Omega$ as

$$\omega_\varepsilon = \{x - \sigma \vec{n}\,(x) : \; x \in \Gamma \; \text{ and } \; \sigma \in [0, \varepsilon)\},$$

for sufficiently small ε, say $0 < \varepsilon \leqslant \varepsilon_0$, where $\vec{n}\,(x)$ denotes the outward normal vector at $x \in \Gamma$. We note that the set ω_ε has Lebesgue measure $|\omega_\varepsilon| = O(\varepsilon)$ with $|\omega_\varepsilon| \leqslant k|\Gamma|\varepsilon$, for some $k > 0$ independent of ε, and that for small ε, the set ω_ε is a neighborhood of Γ in $\overline{\Omega}$, that collapses to the boundary when the parameter ε goes to zero, see Figure 1.

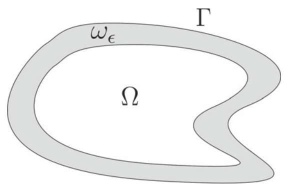

Figure 1: The set $\omega_\varepsilon \subset \overline{\Omega}$.

We are interested in the behavior, for small ε, of the solutions of the autonomous thermoelastic plate systems with concentrated terms given by

$$\begin{cases} \partial_t^2 u^\varepsilon + \Delta^2 u^\varepsilon + u^\varepsilon + \Delta\theta^\varepsilon - \theta^\varepsilon = f(u^\varepsilon) + \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u^\varepsilon) & \text{in } \Omega \times (0, +\infty), \\ \partial_t \theta^\varepsilon - \Delta\theta^\varepsilon + \theta^\varepsilon - \Delta\partial_t u^\varepsilon + \partial_t u^\varepsilon = 0 & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial u^\varepsilon}{\partial \vec{n}} = 0, \;\; \dfrac{\partial(\Delta u^\varepsilon)}{\partial \vec{n}} = 0, \;\; \dfrac{\partial \theta^\varepsilon}{\partial \vec{n}} = 0 & \text{on } \Gamma \times (0, +\infty), \\ u^\varepsilon(0) = u_0 \in H^2(\Omega), \;\; u_t^\varepsilon(0) = v_0 \in L^2(\Omega), \;\; \theta^\varepsilon(0) = \theta_0 \in L^2(\Omega), \end{cases} \tag{1.1}$$

where $\chi_{\omega_\varepsilon}$ denotes the characteristic function of the set ω_ε. We refer to $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u^\varepsilon)$ as the concentrating reaction in ω_ε.

We take $j : \mathbb{R} \to \mathbb{R}$ to be $\mathscr{C}^2$, and assume that it satisfies the growth estimates

$$|j(s)| + |j'(s)| + |j''(s)| \leqslant K, \quad \forall s \in \mathbb{R}, \tag{1.2}$$

for some constant $K > 0$, we also assume the standard dissipative assumption given by

$$\limsup_{|s| \to +\infty} \frac{j(s)}{s} \leqslant 0, \tag{1.3}$$

with $j = f$ or $j = g$. We note that (1.3) is equivalent to saying that for any $\gamma > 0$ there exists $c_\gamma > 0$ such that

$$s j(s) \leqslant \gamma s^2 + c_\gamma, \quad \forall s \in \mathbb{R}. \tag{1.4}$$

REMARK 1. The condition (1.2) on the nonlinearities does not represent any restriction. Since the nonlinearities are assumed dissipative in (1.3), we have $L^\infty(\Omega)$ estimates of the attractors of the system (1.1) and these estimates are uniform in the parameter ε. In particular, all solutions of (1.1) are bounded with a bound independent of ε, see [8, proposition 3.2] and [8, theorem 3.4]. In case (1.2) is not satisfied we can cut off the nonlinearity without modifying the solutions of the equation so that (1.2) is satisfied.

As in (1.1) the nonlinear term $g(u^\varepsilon)$ is only effective on the region ω_ε which collapses to Γ as $\varepsilon \to 0$, then it is reasonable to expect that the family of solutions u^ε of (1.1) will converge to a solution of an equation of the same type with nonlinear boundary condition on Γ. Indeed, we will show that the "limit problem" for the problem (1.1) is given by

$$\begin{cases} \partial_t^2 u + \Delta^2 u + u + \Delta\theta - \theta = f(u) & \text{in } \Omega \times (\tau, +\infty), \\ \partial_t \theta - \Delta\theta + \theta - \Delta\partial_t u + \partial_t u = 0 & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial \vec{n}} = 0, \ \ \dfrac{\partial(\Delta u)}{\partial \vec{n}} = -g(u), \ \ \dfrac{\partial \theta}{\partial \vec{n}} = 0 & \text{on } \Gamma \times (0, +\infty), \\ u(0) = u_0 \in H^2(\Omega), \ \ u_t(0) = v_0 \in L^2(\Omega), \ \ \theta(0) = \theta_0 \in L^2(\Omega). \end{cases} \tag{1.5}$$

Moreover, we will show the existence, uniform boundedness and upper semicontinuity of the global attractors at $\varepsilon = 0$ of the problems (1.1) and (1.5).

Initial boundary-value problems associated with thermoelastic plate systems on bounded smooth domains in Euclidian spaces has been extensively discussed for several authors in different contexts. For instance, in [10] the authors studied the existence of almost periodic solutions for an evolution systems like (1.1) and (1.5), in [11] the authors studied the pullback dynamics of evolution systems like (1.1) and (1.5), in [19] the authors proved that the linear semigroup defined by systems like (1.1) and (1.5) with $f, g \equiv 0$ with clamped boundary condition for u and Dirichlet boundary condition for θ is analytic. The typical difficulties in thermoelasticity comes from the boundary condition, which make more complicated the task of getting estimates to show the exponential stability of the solutions or analyticity of the corresponding semigroup. In that direction we have the works of [20], [18] to free - clamped boundary condition. In this last work the authors show the exponential stability and analyticity of the semigroup associated with the systems like (1.1) and (1.5). We refer to the book [20] for a general survey on those topics.

This paper is organized as follow: in section 2, we will define the abstract problems associated with the initial-boundary value problems (1.1) and (1.5). After we will show the local existence and the uniqueness of the solutions of these abstract problems and that the solutions are continuously differentiable with respect to initial conditions. In section 3, we will show that the solutions are globally defined and the dissipativity of the nonlinear semigroups associated with the solutions. Finally, in section 4 , we will prove the existence and upper semicontinuous of the global attractors of the problems (1.1) and (1.5) at $\varepsilon = 0$.

2. Local well-posedness and differentiability

To better explain the results in the paper, initially, we will define the abstract problems associated to (1.1) and (1.5). After we will prove local existence and uniqueness of the solutions of these abstract problems and that the solutions are continuously differentiable with respect to initial conditions.

2.1. Abstract setting

Let us consider the Hilbert space $Y := L^2(\Omega)$ and the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ defined by

$$\Lambda u = (-\Delta)^2 u, \quad u \in D(\Lambda),$$

with domain

$$D(\Lambda) := \left\{ u \in H^4(\Omega) \; : \; \frac{\partial u}{\partial \vec{n}} = \frac{\partial(\Delta u)}{\partial \vec{n}} = 0 \text{ on } \Gamma \right\}.$$

The operator Λ has a discrete spectrum formed of eigenvalues satisfying

$$0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n \leqslant \cdots, \quad \lim_{n\to\infty} \mu_n = \infty.$$

Since this operator turns out to be sectorial in Y in the sense of D. Henry [17, definition 1.3.1] and J. Cholewa and T. Dłotko [13, example 1.3.9], associated to it there is a scale of Banach spaces (the fractional power spaces) Y^α, $\alpha \in \mathbb{R}$, denoting the domain of the fractional power operators associated with Λ, that is, $Y^\alpha := D(\Lambda^\alpha), \alpha \geqslant 0$. Let us consider Y^α endowed with the norm $\|\cdot\|_{Y^\alpha} = \|\Lambda^\alpha \cdot\|_Y + \|\cdot\|_Y, \alpha \geqslant 0$. The fractional power spaces are related to the Bessel Potentials spaces $H^s(\Omega)$, $s \in \mathbb{R}$, and it is well know that

$$Y^\alpha \hookrightarrow H^{2\alpha}(\Omega), \quad Y^{-\alpha} = (Y^\alpha)', \;\; \alpha \geqslant 0, \tag{2.1}$$

with

$$Y^{\frac{1}{2}} = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma \right\}.$$

We also have

$$Y^{-\frac{1}{2}} = (Y^{\frac{1}{2}})', \quad Y = Y^0 = L^2(\Omega) \quad \text{and} \quad Y^1 = D(\Lambda).$$

Since the problem (1.5) has a nonlinear term on boundary, choosing $\frac{1}{2} < s \leqslant 1$ and using the standard trace theory results that for any function $v \in H^s(\Omega)$, the trace of v is well defined and lies in $L^2(\Gamma)$. Moreover, the scale of negative exponents $Y^{-\alpha}$, for $\alpha > 0$, is necessary to introduce the nonlinear term of (1.5) in the abstract equation, since we are using the operator Λ with homogeneous boundary conditions. Considering the realizations of Λ in this scale, the operator $\Lambda_{-\frac{1}{2}} \in \mathscr{L}(Y^{\frac{1}{2}}, Y^{-\frac{1}{2}})$ is given by

$$\langle \Lambda_{-\frac{1}{2}} u, v\rangle_Y = \int_\Omega \Delta u \Delta v dx + \int_\Omega uv dx, \quad \text{for } u, v \in Y^{\frac{1}{2}}.$$

With some abuse of notation we will identify all different realizations of this operator and we will write them all as Λ.

We also consider the operator $\Lambda+I : D(\Lambda+I)\subset Y\to Y$, it is a positive defined and sectorial operator in Y in the sense of D. Henry [17, definition 1.3.1] and J. Cholewa and T. Dłotko [13, example 1.3.9], associated to it there is a scale of Banach spaces (the fractional power spaces) $D((\Lambda+I)^{\alpha})$, $\alpha\geqslant 0$, domain of the operator $(\Lambda+I)^{\alpha}$. Let us consider $D((\Lambda+I)^{\alpha})$ endowed with the graph norm $\|(\cdot)\|_{D((\Lambda+I)^{\alpha})}=\|(\Lambda+I)^{\alpha}(\cdot)\|_Y, \alpha\geqslant 0$ $(0\in\rho((\Lambda+I)^{\alpha}))$. Since $D(\Lambda+I)=D(\Lambda)$, we also have that $D((\Lambda+I)^{\alpha})=Y^{\alpha}$ for $0\leqslant\alpha\leqslant 1$ endowed with equivalent norms.

The operator $\Lambda+I$ has a discrete spectrum formed of eigenvalues satisfying

$$1=\mu_1^I\leqslant\mu_2^I\leqslant\cdots\leqslant\mu_n^I\leqslant\cdots,\quad \lim_{n\to\infty}\mu_n^I=\infty.$$

Also, let us consider the following Hilbert space

$$X=X^0=Y^{\frac{1}{2}}\times Y\times Y$$

equipped with the inner product

$$\left\langle\begin{pmatrix}u_1\\v_1\\\theta_1\end{pmatrix},\begin{pmatrix}u_2\\v_2\\\theta_2\end{pmatrix}\right\rangle_X=\langle u_1,u_2\rangle_{Y^{\frac{1}{2}}}+\langle v_1,v_2\rangle_Y+\langle\theta_1,\theta_2\rangle_Y,$$

where $\langle\cdot,\cdot\rangle_Y$ is the usual inner product in $L^2(\Omega)$.

We define the unbounded linear operator $\mathbb{A}: D(\mathbb{A})\subset X\to X$ by

$$\mathbb{A}\begin{pmatrix}u\\v\\\theta\end{pmatrix}=\begin{pmatrix}0 & I & 0\\ -\Lambda-I & 0 & \Lambda^{\frac{1}{2}}+I\\ 0 & -\Lambda^{\frac{1}{2}}-I & -\Lambda^{\frac{1}{2}}-I\end{pmatrix}\begin{pmatrix}u\\v\\\theta\end{pmatrix}=\begin{pmatrix}v\\ -\Lambda u-u+\Lambda^{\frac{1}{2}}\theta+\theta\\ -\Lambda^{\frac{1}{2}}v-v-\Lambda^{\frac{1}{2}}\theta-\theta\end{pmatrix},\tag{2.2}$$

for any $\begin{pmatrix}u\\v\\\theta\end{pmatrix}\in D(\mathbb{A})$, with domain

$$D(\mathbb{A})=Y^1\times Y^{\frac{1}{2}}\times Y^{\frac{1}{2}}.\tag{2.3}$$

For each $\varepsilon\in(0,\varepsilon_0]$, we write (1.1) in the abstract form as

$$\begin{cases}\dfrac{dw^{\varepsilon}}{dt}=\mathbb{A}w^{\varepsilon}+F_{\varepsilon}(w^{\varepsilon}), & t>0,\\ w^{\varepsilon}(0)=w_0,\end{cases}\tag{2.4}$$

with

$$w^{\varepsilon}=\begin{pmatrix}u^{\varepsilon}\\\partial_t u^{\varepsilon}\\\theta^{\varepsilon}\end{pmatrix},\quad w_0=\begin{pmatrix}u_0\\v_0\\\theta_0\end{pmatrix}\in X$$

and nonlinear map $F_{\varepsilon}: X\to H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)$, with $\frac{1}{2}<s\leqslant 1$, defined by

$$F_{\varepsilon}(w)=\begin{pmatrix}0\\ f_{\Omega}(u)+\dfrac{1}{\varepsilon}\chi_{\omega_{\varepsilon}}g_{\Omega}(u)\\ 0\end{pmatrix},\quad \text{for } w=\begin{pmatrix}u\\v\\\theta\end{pmatrix}\in X,$$

where $f_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega : H^2(\Omega) \to H^{-s}(\Omega)$ are the operators, respectively, given by

$$\langle f_\Omega(u), \varphi\rangle = \int_\Omega f(u)\varphi dx, \quad \forall u \in H^2(\Omega) \text{ and } \forall \varphi \in H^s(\Omega) \tag{2.5}$$

and

$$\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u), \varphi \right\rangle = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} g(u)\varphi dx, \quad \forall u \in H^2(\Omega) \text{ and } \forall \varphi \in H^s(\Omega). \tag{2.6}$$

While the problem (1.5) can be written in the abstract form as

$$\begin{cases} \dfrac{dw}{dt} = \mathbb{A}w + F_0(w), & t > 0, \\ w(0) = w_0, \end{cases} \tag{2.7}$$

with

$$w = \begin{pmatrix} u \\ \partial_t u \\ \theta \end{pmatrix}$$

and nonlinear map $F_0 : X \to H^2(\Omega) \times H^{-s}(\Omega) \times L^2(\Omega)$, with $\frac{1}{2} < s \leqslant 1$, defined by

$$F_0(w) = \begin{pmatrix} 0 \\ f_\Omega(u) + g_\Gamma(u) \\ 0 \end{pmatrix}, \quad \text{for } w = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in X,$$

where f_Ω is defined in (2.5) and $g_\Gamma : H^2(\Omega) \to H^{-s}(\Omega)$ is the operator given by

$$\langle g_\Gamma(u), \varphi\rangle = \int_\Gamma \gamma(g(u))\gamma(\varphi) dS, \quad \forall u \in H^2(\Omega) \text{ and } \forall \varphi \in H^s(\Omega), \tag{2.8}$$

where $\gamma : H^s(\Omega) \to L^2(\Gamma)$ is the trace operator, to according with H. Triebel [22].

Concerning analyticity of a C_0- semigroup of contractions on a Hilbert space, we have following result.

THEOREM 1. *Let* $\{S(t) : t \geqslant 0\}$ *be a* C_0*-semigroup of contractions of linear operators in a Hilbert space with infinitesimal generator* $\mathscr{B}$*. Suppose that* $i\mathbb{R} \subset \rho(\mathscr{B})$*. Then,* $\{S(t) : t \geqslant 0\}$ *is analytic if and only if* $\limsup_{|\beta|\to+\infty} \|\beta(i\beta I - \mathscr{B})^{-1}\| < \infty$.

Proof. For the proof, see Liu and Zheng [20, theorem 1.3.3]. □

LEMMA 1. *The unbounded linear operator* $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ *defined in* (2.2)-(2.3) *satisfy the following equality*

$$Re\left\langle \mathbb{A}\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right\rangle_X = -\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 - \|\theta\|_Y^2 \leqslant 0, \quad \forall \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in D(\mathbb{A}). \tag{2.9}$$

Proof. Note that

$$\left\langle \mathbb{A}\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right\rangle_X = \left\langle \begin{pmatrix} v \\ -\Lambda u - u + \Lambda^{\frac{1}{2}}\theta + \theta \\ -\Lambda^{\frac{1}{2}} v - v - \Lambda^{\frac{1}{2}}\theta - \theta \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right\rangle_{Y^{\frac{1}{2}} \times Y \times Y}$$

$$= \langle v, u \rangle_{Y^{\frac{1}{2}}} - \langle \Lambda u - \Lambda^{\frac{1}{2}}\theta, v \rangle_Y - \langle u - \theta, v \rangle_Y - \langle \Lambda^{\frac{1}{2}} v + \Lambda^{\frac{1}{2}}\theta, \theta \rangle_Y - \langle v + \theta, \theta \rangle_Y$$

$$= \overline{\langle \Lambda^{\frac{1}{2}} u, \Lambda^{\frac{1}{2}} v \rangle_Y} - \langle \Lambda^{\frac{1}{2}} u, \Lambda^{\frac{1}{2}} v \rangle_Y + \overline{\langle u, v \rangle_Y} - \langle u, v \rangle_Y + \langle \Lambda^{\frac{1}{2}}\theta, v \rangle_Y - \overline{\langle \Lambda^{\frac{1}{2}}\theta, v \rangle_Y} + \overline{\langle v, \theta \rangle_Y}$$
$$- \langle v, \theta \rangle_Y - \|\Lambda^{\frac{1}{4}}\theta\|_Y^2 - \|\theta\|_Y^2.$$

Finally, from this we get (2.9). □

THEOREM 2. *The unbounded linear operator* $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ *defined in* (2.2)-(2.3) *is closed and densely defined.*

Proof. Let $w_n = [u_n \ \ v_n \ \ \theta_n]^T \in D(\mathbb{A})$ with $w_n \to [u \ \ v \ \ \theta]^T$ in X as $n \to \infty$, and $\mathbb{A} w_n \to \varphi = [\varphi_1 \ \ \varphi_2 \ \ \varphi_3]^T$ in X as $n \to \infty$, or equivalently

$$\begin{cases} v_n \to \varphi_1 & \text{in } Y^{\frac{1}{2}} \text{ as } n \to \infty; \\ -\Lambda u_n - u_n + \Lambda^{\frac{1}{2}}\theta_n + \theta_n \to \varphi_2 & \text{in } Y \text{ as } n \to \infty; \\ -\Lambda^{\frac{1}{2}} v_n - v_n - \Lambda^{\frac{1}{2}}\theta_n - \theta_n \to \varphi_3 & \text{in } Y \text{ as } n \to \infty, \end{cases}$$

then $v = \varphi_1 \in Y^{\frac{1}{2}}$. Since $-(\Lambda^{\frac{1}{2}} + I)\theta_n = [-(\Lambda^{\frac{1}{2}} + I)v_n - (\Lambda^{\frac{1}{2}} + I)\theta_n] + (\Lambda^{\frac{1}{2}} + I)v_n \to \varphi_3 + \Lambda^{\frac{1}{2}}\varphi_1 + \varphi_1$ in Y as $n \to \infty$, we have

$$\theta \in D(\Lambda^{\frac{1}{2}} + I) = Y^{\frac{1}{2}} \quad \text{and} \quad -(\Lambda^{\frac{1}{2}} + I)\theta = \varphi_3 + \Lambda^{\frac{1}{2}}\varphi_1 + \varphi_1.$$

Finally, since $-(\Lambda + I)u_n = [-(\Lambda + I)u_n + (\Lambda^{\frac{1}{2}} + I)\theta_n] - (\Lambda^{\frac{1}{2}} + I)\theta_n \to \varphi_2 + \varphi_3 + \Lambda^{\frac{1}{2}}\varphi_1 + \varphi_1$ in Y as $n \to \infty$, we conclude

$$u \in D(\Lambda + I) = Y^1 \quad \text{and} \quad -(\Lambda + I)u = \varphi_2 + \varphi_3 + \Lambda^{\frac{1}{2}}\varphi_1 + \varphi_1,$$

that is, $[u \ \ v \ \ \theta]^T \in D(\mathbb{A})$ and $[\varphi_1 \ \ \varphi_2 \ \ \varphi_3]^T = [v \ \ -(\Lambda + I)u + (\Lambda^{\frac{1}{2}} + I)\theta \ \ -(\Lambda^{\frac{1}{2}} + I)v - (\Lambda^{\frac{1}{2}} + I)\theta]^T = \mathbb{A}[u \ \ v \ \ \theta]^T$. □

THEOREM 3. *The unbounded linear operator* $-\mathbb{A}$ *such that* $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ *is defined in* (2.2)-(2.3) *is sectorial.*

Proof. First, we will show that $i\mathbb{R} \subset \rho(\mathbb{A})$. We show this result by a contradiction argument. That is, let us suppose that there exists $0 \neq \beta \in \mathbb{R}$, such that $i\beta$ is in the spectrum of $\mathbb{A}$. Then $i\beta$ must be an eigenvalue of $\mathbb{A}$, because $\mathbb{A}^{-1}$ is compact. Thus there is a vector function $w = [u \ \ v \ \ \theta]^T \in D(\mathbb{A})$, $\|w\|_X = 1$, such that

$$(i\beta I - \mathbb{A})w = 0 \text{ in } X$$

or equivalenty

$$\begin{cases} i\beta u - v = 0, \\ i\beta v + (\Lambda + I)u - (\Lambda^{\frac{1}{2}} + I)\theta = 0, \\ i\beta\theta + (\Lambda^{\frac{1}{2}} + I)v + (\Lambda^{\frac{1}{2}} + I)\theta = 0, \end{cases} \tag{2.10}$$

and so

$$Re\langle \mathbb{A}w, w\rangle_X = -\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 - \|\theta\|_Y^2 = 0.$$

Thus $\theta = 0$ and by (2.10), $u = v = 0$, which give the contradiction. Therefore, $i\mathbb{R} \subset \rho(\mathbb{A})$.

Finally, we show that there exists a positive constant C such that

$$|\beta| \left\| \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right\|_X \leqslant C\|\mathscr{F}\|_X, \text{ for all } \mathscr{F} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in X, \ \beta \in \mathbb{R},$$

where $w = [u \ \ v \ \ \theta]^T = (i\beta I - \mathbb{A})^{-1}\mathscr{F} \in D(\mathbb{A})$. In fact, multiplying equation

$$(i\beta I - \mathbb{A})w = \mathscr{F} \text{ in } X \tag{2.11}$$

with $w = [u \ \ v \ \ \theta]^T$; that is, in terms of its components yields

$$\begin{cases} i\beta u - v = f_1, \\ i\beta v + (\Lambda + I)u - (\Lambda^{\frac{1}{2}} + I)\theta = f_2, \\ i\beta\theta + (\Lambda^{\frac{1}{2}} + I)v + (\Lambda^{\frac{1}{2}} + I)\theta = f_3, \end{cases}$$

we get

$$i\beta\|w\|_X^2 - \langle \mathbb{A}w, w\rangle_X = \langle \mathscr{F}, w\rangle_X. \tag{2.12}$$

Taking the real part in (2.12) it follows that

$$|Re\langle \mathbb{A}w, w\rangle_X| = \|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2 \leqslant \|\mathscr{F}\|_X\|w\|_X, \tag{2.13}$$

and taking the imaginary parts in (2.12), and using (2.13) and Young's inequality we have

$$\begin{aligned} |\beta|\|w\|_X^2 &\leqslant 2|\langle \Lambda^{\frac{1}{2}}u, \Lambda^{\frac{1}{2}}v\rangle_Y| + 2|\langle u, v\rangle_Y| + 2|\langle \Lambda^{\frac{1}{2}}v, \theta\rangle_Y| + 2|\langle v, \theta\rangle_Y| + 2\|\mathscr{F}\|_X\|w\|_X \\ &= 2|\langle \Lambda^{\frac{3}{4}}u, \Lambda^{\frac{1}{4}}v\rangle_Y| + 2|\langle u, v\rangle_Y| + 2|\langle \Lambda^{\frac{1}{4}}v, \Lambda^{\frac{1}{4}}\theta\rangle_Y| + 2|\langle v, \theta\rangle_Y| + 2\|\mathscr{F}\|_X\|w\|_X \\ &\leqslant \|\Lambda^{\frac{3}{4}}u\|_Y^2 + \|u\|_Y^2 + 2\|\Lambda^{\frac{1}{4}}v\|_Y^2 + 2\|v\|_Y^2 + \|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2 + 2\|\mathscr{F}\|_X\|w\|_X \\ &\leqslant \|\Lambda^{\frac{3}{4}}u\|_Y^2 + \|u\|_Y^2 + 2\|\Lambda^{\frac{1}{4}}v\|_Y^2 + 2\|v\|_Y^2 + \|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2 + 2\|\mathscr{F}\|_X\|w\|_X. \end{aligned} \tag{2.14}$$

Thanks to (2.13) and (2.14) we obtain that

$$|\beta|\|w\|_X^2 \leqslant \|\Lambda^{\frac{3}{4}}u\|_Y^2 + \|u\|_Y^2 + 2(\|\Lambda^{\frac{1}{4}}v\|_Y^2 + \|v\|_Y^2) + 3\|\mathscr{F}\|_X\|w\|_X. \tag{2.15}$$

Multiplying (2.11) by $[0 \ \ \theta \ \ 0]^T$, in the sense of X, and multiplying (2.11) by $[0 \ \ 0 \ \ v]^T$, in the sense of X, and using the Young's inequality we have

$$\begin{aligned}
&2(\|\Lambda^{\frac{1}{4}}v\|_Y^2+\|v\|_Y^2)\\
\leqslant &2(\|f_2\|_Y\|\theta\|_Y+\|f_3\|_Y\|v\|_Y)+\|\Lambda^{\frac{3}{4}}u\|_Y^2+\|u\|_Y^2+\left(1+\frac{1}{\gamma_0}\right)(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2+\|\theta\|_Y^2)\\
&+\gamma_0(\|\Lambda^{\frac{1}{4}}v\|_Y^2+\|v\|_Y^2),
\end{aligned}$$

for some constant $\gamma_0>0$ to be chosen later.

Thus

$$\begin{aligned}
(2-\gamma_0)(\|\Lambda^{\frac{1}{4}}v\|_Y^2+\|v\|_Y^2)\leqslant &2(\|f_2\|_Y\|\theta\|_Y+\|f_3\|_Y\|v\|_Y)+\|\Lambda^{\frac{3}{4}}u\|_Y^2+\|u\|_Y^2\\
&+\left(1+\frac{1}{\gamma_0}\right)(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2+\|\theta\|_Y^2),
\end{aligned}$$

for some constant $\gamma_0>0$ to be chosen later.

With this, by (2.13) and choosing $0<\gamma_0<2$ we get

$$(2-\gamma_0)(\|\Lambda^{\frac{1}{4}}v\|_Y^2+\|v\|_Y^2)\leqslant C_1\|\mathscr{F}\|_X\|w\|_X+\|\Lambda^{\frac{3}{4}}u\|_Y^2+\|u\|_Y^2, \tag{2.16}$$

for some constant $C_1>0$.

Now, multiplying (2.11) by $[0 \ \ \Lambda^{\frac{1}{2}}u+u \ \ 0]^T$, in the sense of X, we have

$$\langle i\beta v+(\Lambda+I)u-(\Lambda^{\frac{1}{2}}+I)\theta,\Lambda^{\frac{1}{2}}u+u\rangle_Y=\langle f_2,\Lambda^{\frac{1}{2}}u+u\rangle_Y,$$

that is,

$$\begin{aligned}
&\|\Lambda^{\frac{3}{4}}u\|_Y^2+\|\Lambda^{\frac{1}{2}}u\|_Y^2+\|\Lambda^{\frac{1}{4}}u\|_Y^2+\|u\|_Y^2\\
\leqslant &(\|\Lambda^{\frac{1}{2}}f_1\|_Y+\|f_1\|_Y)\|v\|_Y+\|f_2\|_Y(\|\Lambda^{\frac{1}{2}}u\|_Y+\|u\|_Y)+\frac{2+\gamma_1}{2\gamma_1}\|\Lambda^{\frac{1}{4}}\theta\|_Y^2+\frac{1}{2}\|\Lambda^{\frac{3}{4}}u\|_Y^2\\
&+\gamma_1\|\Lambda^{\frac{1}{4}}u\|_Y^2+\frac{1}{2}\|\theta\|_Y^2+\frac{1}{2}\|u\|_Y^2-\|\Lambda^{\frac{1}{4}}v\|_Y^2-\|v\|_Y^2\\
\leqslant &(\|\Lambda^{\frac{1}{2}}f_1\|_Y+\|f_1\|_Y)\|v\|_Y+\|f_2\|_Y(\|\Lambda^{\frac{1}{2}}u\|_Y+\|u\|_Y)+\frac{2+\gamma_1}{2\gamma_1}(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2+\|\theta\|_Y^2)\\
&+\frac{1}{2}\|\Lambda^{\frac{3}{4}}u\|_Y^2+\gamma_1\|\Lambda^{\frac{1}{4}}u\|_Y^2+\frac{1}{2}\|u\|_Y^2,
\end{aligned}$$

for some constant $\gamma_1>0$ to be chosen later.

Thus

$$\begin{aligned}
&\frac{1}{2}(\|\Lambda^{\frac{3}{4}}u\|_Y^2+\|u\|_Y^2)+(1-\gamma_1)\|\Lambda^{\frac{1}{4}}u\|_Y^2\\
\leqslant &(\|\Lambda^{\frac{1}{2}}f_1\|_Y+\|f_1\|_Y)\|v\|_Y+\|f_2\|_Y(\|\Lambda^{\frac{1}{2}}u\|_Y+\|u\|_Y)+\frac{2+\gamma_1}{2\gamma_1}(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2+\|\theta\|_Y^2),
\end{aligned}$$

for some constant $\gamma_1>0$ to be chosen later.

Now, take $0<\gamma_1<1$ and see that by (2.13) we get

$$\|\Lambda^{\frac{3}{4}}u\|_Y^2+\|u\|_Y^2\leqslant C_2\|\mathscr{F}\|_X\|w\|_X,$$

for some constant $C_2>0$.

Thanks to (2.16) we have

$$\|\Lambda^{\frac{1}{4}}v\|_Y^2+\|v\|_Y^2\leqslant C_3\|\mathscr{F}\|_X\|w\|_X,$$

for some constant $C_3>0$.

Finally, from (2.15) we obtain

$$|\beta|\|w\|_X^2\leqslant c_0\|\mathscr{F}\|_X\|w\|_X,$$

for some constant $c_0>0$, and we conclude that $-\mathbb{A}$ is a sectorial operator. □

Below we have some remarks thanks to sectoriality of $-\mathbb{A}$ which are important in the sense of existence and regularity of solutions of initial value problems as (2.4) and (2.7), where $\mathbb{A}$ defines the main part of the differential equation.

REMARK 2. The following startments are hold.

(i) Zero is in the resolvent set of $\mathbb{A}$ and

$$\mathbb{A}^{-1}=\begin{pmatrix}-(\Lambda+I)^{-1}(\Lambda^{\frac{1}{2}}+I) & -(\Lambda+I)^{-1} & -(\Lambda+I)^{-1}\\ I & 0 & 0\\ -I & 0 & -(\Lambda^{\frac{1}{2}}+I)^{-1}\end{pmatrix}.$$

(ii) Denote by X_{-1} the extrapolation space of $X=Y^{\frac{1}{2}}\times Y\times Y$ generated by the operator $\mathbb{A}^{-1}$. The following equality holds

$$X_{-1}=Y\times Y^{-\frac{1}{2}}\times Y^{-\frac{1}{2}}.$$

In fact, recall first that X_{-1} is the completion of the normed space $(X,\|\mathbb{A}^{-1}\cdot\|)$. Now, note that

$$\left\|\mathbb{A}^{-1}\begin{pmatrix}u\\ v\\ \theta\end{pmatrix}\right\|_X=\left\|\begin{pmatrix}-(\Lambda+I)^{-1}(\Lambda^{\frac{1}{2}}+I)u-(\Lambda+I)^{-1}v-(\Lambda+I)^{-1}\theta\\ u\\ -u-(\Lambda^{\frac{1}{2}}+I)^{-1}\theta\end{pmatrix}\right\|_X$$
$$\leqslant C_1\left\|\begin{pmatrix}u\\ v\\ \theta\end{pmatrix}\right\|_{X_{-1}},$$

for any $\begin{pmatrix}u\\ v\\ \theta\end{pmatrix}\in X_{-1}$ and for some constant $C_1>0$. We also have that

$$\left\|\begin{pmatrix}u\\ v\\ \theta\end{pmatrix}\right\|_{X_{-1}}\leqslant C_2\left\|\mathbb{A}^{-1}\begin{pmatrix}u\\ v\\ \theta\end{pmatrix}\right\|_X,$$

for some constant $C_2>0$.

So we conclude that the completion of $(X,\|\mathbb{A}^{-1}\cdot\|_X)$ and $(X,\|\cdot\|_{X_{-1}})$ coincide.

REMARK 3. The operator $\mathbb{A}$ can be extended to its closed $X_{-1}-$realization (see Amann [1]), which we will still denote by the same symbol so that $\mathbb{A}$ considered in X_{-1} is then sectorial positive operator. Our next concern will be to obtain embedding of the spaces from the fractional powers scale $X_{\alpha-1}$, $\alpha \geqslant 0$, generated by $(\mathbb{A}, X_{-1})$.

Below we have a partial description of the fractional power spaces scale for $\mathbb{A}$: for convenience we denote X by X_0, then

$$X_0 \hookrightarrow X_{\alpha-1} \hookrightarrow X_{-1}, \text{ for all } 0 < \alpha < 1,$$

where

$$X_{\alpha-1} = [X_{-1}, X_0]_\alpha = Y^{\frac{\alpha}{2}} \times Y^{\frac{\alpha-1}{2}} \times Y^{\frac{\alpha-1}{2}},$$

where $[\cdot,\cdot]_\alpha$ denotes the complex interpolation functor (see [22]). The first equality follows from theorem 3 (since $0 \in \rho(\mathbb{A})$) see [1, example 4.7.3 (b)] and the second equality follows from [12, proposition 2].

REMARK 4. The operator $\mathbb{A}$ or, more precisely, a suitable realization of it, generates an analytic semigroup, $\{e^{\mathbb{A}t} : t \geqslant 0\}$, in X_{-1}, this semigroup is order preserving and satisfies the smoothing estimates. Thanks to [17, theorem 1.4.3] we have

$$\left\| e^{\mathbb{A}t} w \right\|_X \leqslant M e^{-\omega t} t^{-1} \|w\|_{X_{-1}}$$

for any $t > 0$, $w \in X_{-1}$, for some constants $M > 0$ and $\omega > 0$.

Finally, thanks to (2.1) we have $Y^{\frac{1}{2}} \hookrightarrow H^s(\Omega)$, and consequently, $H^2(\Omega) \times H^{-s}(\Omega) \times L^2(\Omega) \hookrightarrow X_{-1}$ and

$$\left\| e^{\mathbb{A}t} w \right\|_X \leqslant M e^{-\omega t} t^{-1} \|w\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} \tag{2.17}$$

for any $t > 0$, $w \in H^2(\Omega) \times H^{-s}(\Omega) \times L^2(\Omega)$, for some constants $M > 0$ and $\omega > 0$.

2.2. Local well-posedness

We are interested in obtaining the local well-posedness of the parabolic problems (2.4) and (2.7) (or (1.1) and (1.5)), for this it is necessary to study the behavior of nonlinearity F_ε, $\varepsilon \in [0, \varepsilon_0]$.

The next lemma is one of the crucial result in our analysis.

LEMMA 2. *Assme that $v \in H^{s,p}(\Omega)$ with $\frac{1}{p} < s \leqslant 2$ and $s - \frac{N}{p} \geqslant -\frac{N-1}{q}$, or $v \in H^{1,1}(\Omega)$, i.e., $s = 1 = p$ and $q = 1$ below. Then for sufficiently small ε_0, we have*

(i) The map

$$[0, \varepsilon_0] \ni \sigma \mapsto \int_{\Gamma_\sigma} |v|^q dS$$

is continuous, where for sufficiently small $\sigma \geqslant 0$, $\Gamma_\sigma = \{x - \sigma \vec{n}(x) : x \in \Gamma\}$ is the "parallel" interior boundary.

(ii) *There exists $C>0$ independent of ε and v such that for any $0<\varepsilon\leqslant\varepsilon_0$, we have*

$$\sup_{\sigma\in[0,\varepsilon)}\|v\|_{L^q(\Gamma_\sigma)}\leqslant C\|v\|_{H^{s,p}(\Omega)},$$

$$\int_{\omega_\varepsilon}|v|^q dx=\int_0^\varepsilon\Big(\int_{\Gamma_\sigma}|v|^q dS\Big)d\sigma,$$

with the same equality, without the absolute value, if $q=1$.

In particular

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|v|^q dx\leqslant C\|v\|^q_{H^{s,p}(\Omega)},$$

and

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|v|^q dx=\int_\Gamma|v|^q dS.$$

Proof. See [9, lemma 2.1]. □

LEMMA 3. *Suppose that f and g satisfy the growth estimate (1.2) and $\frac{1}{2}<s\leqslant 1$. Then:*

(i) *There exists $C>0$, independent of ε, such that*

$$\|F_\varepsilon(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}\leqslant C,\quad \text{for all}\quad w=\left(\begin{smallmatrix}u\\ v\\ \theta\end{smallmatrix}\right)\in X\quad \text{and}\quad 0\leqslant\varepsilon\leqslant\varepsilon_0. \tag{2.18}$$

(ii) *For each $0\leqslant\varepsilon\leqslant\varepsilon_0$, the map $F_\varepsilon:X\to H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)$ is globally Lipschitz, uniformly in ε.*

(iii) *For each $w=\left(\begin{smallmatrix}u\\ v\\ \theta\end{smallmatrix}\right)\in X$, we have*

$$\|F_\varepsilon(w)-F_0(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}\to 0,\quad as\ \varepsilon\to 0.$$

Furthermore, this limit is uniform for $w\in X$ such that $\|w\|_X\leqslant R$, for some $R>0$.

(iv) *If $w_\varepsilon\to w$ in X, as $\varepsilon\to 0$, then*

$$\|F_\varepsilon(w_\varepsilon)-F_0(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}\to 0,\quad as\ \varepsilon\to 0.$$

Proof.

(i) Initially note that

$$\|F_\varepsilon(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}=\left\|f_\Omega(u)+\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u)\right\|_{H^{-s}(\Omega)},\quad \varepsilon\in(0,\varepsilon_0],$$

$$\|F_0(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}=\|f_\Omega(u)+g_\Gamma(u)\|_{H^{-s}(\Omega)},$$

with f_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega$ and g_Γ defined, respectively, by (2.5), (2.6) and (2.8).

Using (1.2), Cauchy-Schwarz inequality and Sobolev embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$ with $\frac{1}{2} < s \leqslant 1$, we have

$$|\langle f_\Omega(u), \varphi\rangle| \leqslant \int_\Omega |f(u(x))||\varphi(x)|dx \leqslant \int_\Omega K|\varphi(x)|dx$$
$$\leqslant cK\,\|\varphi\|_{L^2(\Omega)} \leqslant k_1\,\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega).$$

Thus,

$$\|f_\Omega(u)\|_{H^{-s}(\Omega)} \leqslant k_1. \tag{2.19}$$

Using (1.2), Cauchy-Schwarz inequality, $|\omega_\varepsilon| \leqslant k|\Gamma|\varepsilon$ for some $k > 0$ independent of ε, and lemma 2, we have

$$\left|\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u), \varphi\right\rangle\right| \leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g(u(x))||\varphi(x)|dx \leqslant \frac{K}{\varepsilon}\int_{\omega_\varepsilon} |\varphi(x)|dx$$
$$\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} 1dx\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi(x)|^2dx\right]^{\frac{1}{2}}$$
$$\leqslant k_2\,\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega),$$

with $k_2 > 0$ independent of ε. Thus,

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u)\right\|_{H^{-s}(\Omega)} \leqslant k_2. \tag{2.20}$$

Now, using (1.2), Cauchy-Schwarz inequality and the continuity of the trace operator
$\gamma : H^s(\Omega) \to L^2(\Gamma)$ with $\frac{1}{2} < s \leqslant 1$, we have

$$|\langle g_\Gamma(u), \varphi\rangle| \leqslant \int_\Gamma |\gamma(g(u(x)))||\gamma(\varphi(x))|dS \leqslant K\int_\Gamma |\gamma(\varphi(x))|dS$$
$$\leqslant cK\left[\int_\Gamma |\gamma(\varphi(x))|^2dS\right]^{\frac{1}{2}} = cK\,\|\gamma(\varphi)\|_{L^2(\Gamma)}$$
$$\leqslant k_3\,\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega).$$

Thus,

$$\|g_\Gamma(u)\|_{H^{-s}(\Omega)} \leqslant k_3. \tag{2.21}$$

Now, (2.18) follows in a straightforward from (2.19), (2.20) and (2.21).

(ii) Initially, note that

$$\|F_\varepsilon(w_1) - F_\varepsilon(w_2)\|_{H^2_N(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}$$
$$= \left\|[f_\Omega(u_1) - f_\Omega(u_2)] + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}[g_\Omega(u_1) - g_\Omega(u_2)]\right\|_{H^{-s}(\Omega)}, \quad \varepsilon \in (0, \varepsilon_0],$$

and

$$\begin{aligned}&\|F_0(w_1)-F_0(w_2)\|_{H^2_N(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}\\&=\|[f_\Omega(u_1)-f_\Omega(u_2)]+[g_\Gamma(u_1)-g_\Gamma(u_2)]\|_{H^{-s}(\Omega)},\end{aligned}$$

with f_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega$ and g_Γ defined, respectively, by (2.5), (2.6) and (2.8).

Using (1.2), Cauchy-Schwarz inequality and Sobolev embeddings $H^2(\Omega)\hookrightarrow L^2(\Omega)$ and $H^s(\Omega)\hookrightarrow L^2(\Omega)$ with $\frac{1}{2}<s\leqslant 1$, we have

$$\begin{aligned}|\langle f_\Omega(u_1)-f_\Omega(u_2),\varphi\rangle| &\leqslant \int_\Omega |f(u_1(x))-f(u_2(x))||\varphi(x)|dx\\&\leqslant \int_\Omega |f'(\sigma(x)u_1(x)+(1-\sigma(x))u_2(x))|\\&\qquad\times|u_1(x)-u_2(x)||\varphi(x)|dx\\&\leqslant K\left[\int_\Omega |u_1(x)-u_2(x)|^2dx\right]^{\frac{1}{2}}\left[\int_\Omega|\varphi(x)|^2dx\right]^{\frac{1}{2}}\\&=K\|u_1-u_2\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}\\&\leqslant c_1\|u_1-u_2\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},\quad \forall\varphi\in H^s(\Omega),\end{aligned}$$

for some $0\leqslant\sigma(x)\leqslant 1, x\in\bar{\Omega}$. Thus,

$$\|f_\Omega(u_1)-f_\Omega(u_2)\|_{H^{-s}(\Omega)}\leqslant c_1\|u_1-u_2\|_{H^2(\Omega)}. \tag{2.22}$$

Using (1.2), Cauchy-Schwarz inequality and lemma 2, we have

$$\begin{aligned}&\left|\left\langle\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}[g_\Omega(u_1)-g_\Omega(u_2)],\varphi\right\rangle\right|\leqslant\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g(u_1(x))-g(u_2(x))||\varphi(x)|dx\\&\leqslant\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g'(\sigma(x)u_1(x)+(1-\sigma(x))u_2(x))||u_1(x)-u_2(x)||\varphi(x)|dx\\&\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u_1(x)-u_2(x)|^2dx\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi(x)|^2dx\right]^{\frac{1}{2}}\\&\leqslant c_2\|u_1-u_2\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},\quad\forall\varphi\in H^s(\Omega),\end{aligned}$$

with $c_2>0$ independent of ε and for some $0\leqslant\sigma(x)\leqslant 1, x\in\bar{\Omega}$. Thus,

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}[g_\Omega(u_1)-g_\Omega(u_2)]\right\|_{H^{-s}(\Omega)}\leqslant c_2\|u_1-u_2\|_{H^2(\Omega)}. \tag{2.23}$$

Now, using (1.2), Cauchy-Schwarz inequality and the continuity of the trace operators $\gamma : H^2(\Omega) \to L^2(\Gamma)$ and $\gamma : H^s(\Omega) \to L^2(\Gamma)$ with $\frac{1}{2} < s \leqslant 1$, we have

$$\begin{aligned}
&|\langle g_\Gamma(u_1) - g_\Gamma(u_2), \varphi\rangle| \\
\leqslant &\int_\Gamma |\gamma(g(u_1(x)) - g(u_2(x)))||\gamma(\varphi(x))|dS \\
\leqslant &\int_\Gamma |\gamma(g'(\sigma(x)u_1(x) + (1-\sigma(x))u_2(x)))||\gamma(u_1(x) - u_2(x))||\gamma(\varphi(x))|dS \\
\leqslant &K\left[\int_\Gamma |\gamma(u_1(x) - u_2(x))|^2 dS\right]^{\frac{1}{2}} \left[\int_\Gamma |\gamma(\varphi(x))|^2 dS\right]^{\frac{1}{2}} \\
\leqslant &c_3 \|u_1 - u_2\|_{H^2(\Omega)} \|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega),
\end{aligned}$$

for some $0 \leqslant \sigma(x) \leqslant 1, x \in \Gamma$. Thus,

$$\|g_\Gamma(u_1) - g_\Gamma(u_2)\|_{H^{-s}(\Omega)} \leqslant c_3 \|u_1 - u_2\|_{H^2(\Omega)}. \tag{2.24}$$

Now, (ii) follows in a straightforward from (2.22), (2.23) and (2.24).

(iii) Notice that

$$\|F_\varepsilon(w) - F_0(w)\|_{H^2_N(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} = \left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u)\right\|_{H^{-s}(\Omega)}.$$

As in [15, lemma 5.2] we can prove that there exists $M(\varepsilon, R)$ with $M(\varepsilon, R) \to 0$ as $\varepsilon \to 0$ such that

$$\begin{aligned}
\left|\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u), \varphi\right\rangle\right| &= \left|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g(u(x))\varphi(x)dx - \int_\Gamma \gamma(g(u(x)))\gamma(\varphi(x))dS\right| \\
&\leqslant M(\varepsilon, R)\|\varphi\|_{H^1(\Omega)}, \quad \forall \varphi \in H^1(\Omega).
\end{aligned}$$

Thus,

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u)\right\|_{H^{-1}(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0, \tag{2.25}$$

uniformly for $u \in H^2(\Omega)$ such that $\|u\|_{H^2(\Omega)} \leqslant R$.

Now, fix $\frac{1}{2} < s_0 < 1$. Then for any s such that $-1 < -s < -s_0 < -\frac{1}{2}$, using interpolation we have

$$\begin{aligned}
&\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u)\right\|_{H^{-s}(\Omega)} \\
\leqslant &\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u)\right\|^{\theta}_{H^{-s_0}(\Omega)} \left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u)\right\|^{1-\theta}_{H^{-1}(\Omega)},
\end{aligned}$$

for some $0 < \theta < 1$. By (2.20) and (2.21), the first term in the right hand side above is uniformly bounded while, by (2.25), the second goes to zero, both uniformly for $u \in H^2(\Omega)$ such that $\|u\|_{H^2(\Omega)} \leqslant R$.

(iv) This item follows from (ii) and (iii), adding and subtracting $F_\varepsilon(w)$. In fact

$$\begin{aligned}&\|F_\varepsilon(w_\varepsilon)-F_0(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}\\ \leqslant&\|F_\varepsilon(w_\varepsilon)-F_\varepsilon(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}+\|F_\varepsilon(w)-F_0(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}\\ \leqslant&L\|w_\varepsilon-w\|_X+\|F_\varepsilon(w)-F_0(w)\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)}\to 0,\quad \text{as } \varepsilon\to 0,\end{aligned}$$

where $L>0$ is the constant of Lipschitz, with this we conclude the proof of lemma 3. □

From lemma 3 follows that the map $F_\varepsilon : X \to H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)$ is bounded, uniformly in ε, in bounded set of X, and it is globally Lipschitz, uniformly in ε. Thus, it follows from [16, theorem 4.2.1] that given $w_0\in X$, there is an unique local solution $w^\varepsilon(t,w_0)$ of (2.4), with $\varepsilon\in(0,\varepsilon_0]$, defined on a maximal interval of existence $[0,t^\varepsilon_{max}(w_0))$, and there is an unique local solution $w(t,w_0)$ of (2.7) defined on a maximal interval of existence $[0,t_{max}(w_0))$. Moreover, these solutions depend continuously on the initial data.

Note that the results obtained in the lemma 3 are more general than is necessary here, but they will be used throughout the paper.

2.3. The differentiability

We will prove that the solutions of (2.4) and (2.7) are continuously differentiable with respect to initial conditions, for this it is necessary to prove the Fréchet differentiability of $F_\varepsilon : X\to H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)$, $\varepsilon\in[0,\varepsilon_0]$. It is enough to prove the Fréchet differentiability of $f_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega, g_\Gamma : H^2(\Omega)\to H^{-s}(\Omega)$.

We define the maps $Df_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega, Dg_\Gamma : H^2(\Omega)\to\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))$, with $\frac{1}{2}<s\leqslant 1$, respectively by

$$\langle Df_\Omega(u)\cdot h,\varphi\rangle=\int_\Omega f'(u)h\varphi dx,\quad \forall u,h\in H^2(\Omega) \text{ and } \forall\varphi\in H^s(\Omega), \tag{2.26}$$

$$\langle\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega(u)\cdot h,\varphi\rangle=\frac{1}{\varepsilon}\int_{\omega_\varepsilon}g'(u)h\varphi dx,\quad \forall u,h\in H^2(\Omega) \text{ and } \forall\varphi\in H^s(\Omega) \tag{2.27}$$

and

$$\langle Dg_\Gamma(u)\cdot h,\varphi\rangle=\int_\Gamma\gamma(g'(u)h)\gamma(\varphi)dS,\quad \forall u,h\in H^2(\Omega) \text{ and } \forall\varphi\in H^s(\Omega), \tag{2.28}$$

where $\gamma : H^s(\Omega)\to L^2(\Gamma)$ is the trace operator.

LEMMA 4. *Suppose that f and g satisfy the growth estimate (1.2). Then, f_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega$, $g_\Gamma : H^2(\Omega)\to H^{-s}(\Omega)$ are Fréchet differentiable, uniformly in ε, and your Fréchet differentials are respectively given by (2.26), (2.27) and (2.28). Consequently, for each $\varepsilon\in[0,\varepsilon_0]$, $F_\varepsilon : X\to H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)$ is also Fréchet differentiable, uniformly in ε.*

Proof. First we check that (2.26), (2.27) and (2.28) are well defined. In fact, for $h \in H^2(\Omega)$, using (1.2), Cauchy-Schwarz inequality and Sobolev embeddings, we get

$$\begin{aligned} |\langle Df_\Omega(u)\cdot h,\varphi\rangle| &\leqslant \int_\Omega |f'(u)h||\varphi|dx \leqslant K\int_\Omega |h||\varphi|dx \leqslant K\left[\int_\Omega |h|^2dx\right]^{\frac{1}{2}}\left[\int_\Omega |\varphi|^2dx\right]^{\frac{1}{2}} \\ &= K\|h\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \leqslant k_1\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega). \end{aligned}$$

Thus,

$$\|Df_\Omega(u)\cdot h\|_{H^{-s}(\Omega)} \leqslant k_1\|h\|_{H^2(\Omega)}, \quad \forall h \in H^2(\Omega),$$

and $Df_\Omega(u) \in \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$.

Using (1.2), Cauchy-Schwarz inequality and lemma 2, we have

$$\begin{aligned} \left|\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)\cdot h,\varphi\rangle\right| &\leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g'(u)h||\varphi|dx \leqslant \frac{K}{\varepsilon}\int_{\omega_\varepsilon} |h||\varphi|dx \\ &\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |h|^2dx\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^2dx\right]^{\frac{1}{2}} \\ &\leqslant k_2\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega), \end{aligned}$$

where the positive constant k_2 is independent of ε. Thus,

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)\cdot h\right\|_{H^{-s}(\Omega)} \leqslant k_2\|h\|_{H^2(\Omega)}, \quad \forall h \in H^2(\Omega),$$

and $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) \in \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$.

Now, using (1.2), Cauchy-Schwarz inequality and trace theorem, we get

$$\begin{aligned} |\langle Dg_\Gamma(u)\cdot h,\varphi\rangle| &\leqslant \int_\Gamma |\gamma(g'(u)h)||\gamma(\varphi)|dS \leqslant K\int_\Gamma |\gamma(h)||\gamma(\varphi)|dS \\ &\leqslant K\left[\int_\Gamma |\gamma(h)|^2dS\right]^{\frac{1}{2}}\left[\int_\Gamma |\gamma(\varphi)|^2dS\right]^{\frac{1}{2}} \\ &\leqslant k_3\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega). \end{aligned}$$

Thus,

$$\|Dg_\Gamma(u)\cdot h\|_{H^{-s}(\Omega)} \leqslant k_3\|h\|_{H^2(\Omega)}, \quad \forall h \in H^2(\Omega),$$

and $Dg_\Gamma(u) \in \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$.

Now, let $u,h \in H^2(\Omega)$ and using (1.2), Cauchy-Schwarz inequality and Sobolev embeddings, we have

$$\begin{aligned}|\langle f_\Omega(u+h)-f_\Omega(u)-Df_\Omega(u)\cdot h,\varphi\rangle| &\leqslant \int_\Omega |f(u+h)-f(u)-f'(u)h||\varphi|dx\\ &= \int_\Omega |f'(u+\sigma h)-f'(u)||h||\varphi|dx\\ &= \int_\Omega |f''(\theta(u+\sigma h)+(1-\theta)u)||\sigma h||h||\varphi|dx\\ &\leqslant K\int_\Omega |h|^2|\varphi|dx \leqslant K\|h\|^2_{L^4(\Omega)}\|\varphi\|_{L^2(\Omega)}\\ &\leqslant c_1\|h\|^2_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},\quad \forall\varphi\in H^s(\Omega),\end{aligned}$$

where $\sigma=\sigma(x)\in[0,1]$ and $\theta=\theta(x)\in[0,1]$, $x\in\bar{\Omega}$. Thus,

$$\|f_\Omega(u+h)-f_\Omega(u)-Df_\Omega(u)\cdot h\|_{H^{-s}(\Omega)} \leqslant c_1\|h\|^2_{H^2(\Omega)}.$$

This proves that f_Ω is Fréchet diferentiable and your Fréchet diferential is given by (2.26).

Let $u,h\in H^2(\Omega)$ and using (1.2), Cauchy-Schwarz and lemma 2, we have

$$\begin{aligned}&\Big|\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u+h)-\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u)-\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega(u)\cdot h,\varphi\rangle\Big|\\ &\leqslant\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g(u+h)-g(u)-g'(u)h||\varphi|dx=\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g'(u+\sigma h)-g'(u)||h||\varphi|dx\\ &=\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g''(\theta(u+\sigma h)+(1-\theta)u)||\sigma h||h||\varphi|dx\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|h|^4dx\right]^{\frac12}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^2dx\right]^{\frac12}\\ &\leqslant c_2\|h\|^2_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},\quad\forall\varphi\in H^s(\Omega),\end{aligned}$$

where $\sigma=\sigma(x)\in[0,1]$ and $\theta=\theta(x)\in[0,1]$, $x\in\bar{\Omega}$, and with $c_2>0$ independent of ε. Thus,

$$\Big\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u+h)-\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u)-\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega(u)\cdot h\Big\|_{H^{-s}(\Omega)}\leqslant c_2\|h\|^2_{H^2(\Omega)}.$$

This proves that $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega$ is Fréchet differentiable, uniformly in ε, and your Fréchet differential is given by (2.27).

Now, let $u,h\in H^2(\Omega)$ and using (1.2), Cauchy-Schwarz and trace theorems, we have

$$\begin{aligned}|\langle g_\Gamma(u+h)-g_\Gamma(u)-Dg_\Gamma(u)\cdot h,\varphi\rangle| &\leqslant\int_\Gamma|\gamma(g(u+h))-\gamma(g(u))-\gamma(g'(u)h)||\gamma(\varphi)|dS\\ &=\int_\Gamma|\gamma(g''(\theta(u+\sigma h)+(1-\theta)u))||\gamma(h)|^2|\gamma(\varphi)|dS\\ &\leqslant K\|\gamma(h)\|^2_{L^4(\Gamma)}\|\gamma(\varphi)\|_{L^2(\Gamma)}\\ &\leqslant c_3\|h\|^2_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},\quad\forall\varphi\in H^s(\Omega),\end{aligned}$$

where $\sigma = \sigma(x) \in [0,1]$ and $\theta = \theta(x) \in [0,1]$, $x \in \Gamma$. Thus,

$$\|g_\Gamma(u+h) - g_\Gamma(u) - Dg_\Gamma(u)\cdot h\|_{H^{-s}(\Omega)} \leqslant c_3\|h\|^2_{H^2(\Omega)}.$$

This proves that g_Γ is Fréchet differentiable and your Fréchet differential is given by (2.28).

The Fréchet differentiability of F_ε, uniformly in ε, follows immediately. □

LEMMA 5. *Suppose that f and g satisfy the growth estimate (1.2). Then, Df_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega$, $Dg_\Gamma : H^2(\Omega) \to \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$ are globally Lipschitz, uniformly in ε. Consequently, for each $\varepsilon \in [0,\varepsilon_0]$, $DF_\varepsilon : X \to \mathscr{L}(X, H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega))$ is also globally Lipschitz, uniformly in ε.*

Proof. Let $u, v \in H^2(\Omega)$ and using (1.2), Hölder's inequality and Sobolev embeddings, we have

$$\begin{aligned}
|\langle Df_\Omega(u)\cdot h - Df_\Omega(v)\cdot h, \varphi\rangle| &\leqslant \int_\Omega |f'(u)h - f'(v)h||\varphi|dx \\
&= \int_\Omega |f''(u+\sigma v)||u-v||h||\varphi|dx \leqslant K\int_\Omega |u-v||h||\varphi|dx \\
&\leqslant K\|u-v\|_{L^6(\Omega)}\|h\|_{L^3(\Omega)}\|\varphi\|_{L^2(\Omega)} \\
&\leqslant k_1\|u-v\|_{H^2(\Omega)}\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},
\end{aligned}$$

for any $h \in H^2(\Omega)$ and $\varphi \in H^s(\Omega)$, where $\sigma = \sigma(x) \in [0,1]$, $x \in \bar{\Omega}$. Thus,

$$\|Df_\Omega(u) - Df_\Omega(v)\|_{\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))} \leqslant k_1\|u-v\|_{H^2(\Omega)}.$$

Let $u, v \in H^2(\Omega)$ and using (1.2), Hölder's inequality and lemma 2, we have

$$\begin{aligned}
&\left|\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega(u)\cdot h - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega(v)\cdot h, \varphi\right\rangle\right| \leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g'(u)h - g'(v)h||\varphi|dx \\
=&\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g''(u+\sigma v)||u-v||h||\varphi|dx \\
\leqslant& K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u-v|^4dx\right]^{\frac14}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|h|^4dx\right]^{\frac14}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^2dx\right]^{\frac12} \\
\leqslant& k_2\|u-v\|_{H^2(\Omega)}\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},
\end{aligned}$$

for any $h \in H^2(\Omega)$ and $\varphi \in H^s(\Omega)$, where $k_2 > 0$ is independent of ε and $\sigma = \sigma(x) \in [0,1]$, $x \in \bar{\Omega}$. Thus,

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega(u) - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}Dg_\Omega(v)\right\|_{\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))} \leqslant k_2\|u-v\|_{H^2(\Omega)}.$$

Now, let $u,v \in H^2(\Omega)$ and using (1.2), Hölder's inequality and trace theorems, we have

$$\begin{aligned}\Big|\langle Dg_\Gamma(u)\cdot h - Dg_\Gamma(v)\cdot h,\varphi\rangle\Big| &\leqslant \int_\Gamma |\gamma(g'(u)h)-\gamma(g'(v)h)||\gamma(\varphi)|dS\\ &= \int_\Gamma |\gamma(g''(u+\sigma v))||\gamma(u-v)||\gamma(h)||\gamma(\varphi)|dS\\ &\leqslant K\|\gamma(u-v)\|_{L^4(\Gamma)}\|\gamma(h)\|_{L^4(\Gamma)}\|\gamma(\varphi)\|_{L^2(\Gamma)}\\ &\leqslant k_3\|u-v\|_{H^2(\Omega)}\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},\end{aligned}$$

for any $h\in H^2(\Omega)$ and $\varphi \in H^s(\Omega)$, where $\sigma=\sigma(x)\in[0,1]$, $x\in\Gamma$. Thus,

$$\|Dg_\Gamma(u)-Dg_\Gamma(v)\|_{\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))} \leqslant k_3\|u-v\|_{H^2(\Omega)}.$$

Consequently, it is immediate that for each $\varepsilon\in[0,\varepsilon_0]$, DF_ε is globally Lipschitz, uniformly in ε. □

Under the assumptions of lemma 4 and lemma 5, we have that the map F_ε is continuously Fréchet differentiable. Now, it follows from [16, theorem 4.2.1] that the solutions of (2.4) and (2.7) are continuously differentiable with respect to initial conditions.

3. Global well-posedness and dissipativity

In this section we will wish to prove that the solutions $w^\varepsilon(t,w_0)$, $\varepsilon\in(0,\varepsilon_0]$, and $w(t,w_0)$ of the problems (2.4) and (2.7), respectively, are globally defined, that is, that for each $w_0\in X$, $t^\varepsilon_{max}(w_0)=\infty$ and $t_{max}(w_0)=\infty$. Moreover, we will show that the semigroups associated to solutions are strongly bounded dissipativite. To prove this, we will assume the previous hypotheses and additional dissipativity assumption (1.3)(which is equivalent to (1.4)) and we will consider continuous functionals on X which are bounded in bounded subsets of X and non-increasing along solutions of these problems.

3.1. Perturbed problems

Let $V_\varepsilon : X\to\mathbb{R}$ be the continuous functional defined by

$$\begin{aligned}V_\varepsilon\begin{pmatrix}u\\v\\\theta\end{pmatrix} =&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)}+\frac{1}{2}\|u\|^2_{L^2(\Omega)}+\frac{1}{2}\|v\|^2_{L^2(\Omega)}+\frac{1}{2}\|\theta\|^2_{L^2(\Omega)}-\int_\Omega\int_0^u f(s)dsdx\\ &-\frac{1}{\varepsilon}\int_{\omega_\varepsilon}\int_0^u g(s)dsdx,\end{aligned} \tag{3.1}$$

where $\varepsilon\in(0,\varepsilon_0]$.

It follows from (1.3) that for any $\gamma_1>0$ and $\gamma_2>0$, there exist $k_1=k_1(\gamma_1)>0$ and $k_2=k_2(\gamma_2)>0$ such that

$$\int_0^u f(s)ds \leqslant \int_0^u\left[\frac{\gamma_1 s}{2}+k_1\right]ds \leqslant \frac{\gamma_1 u^2}{4}+k_1 u\leqslant \gamma_1 u^2+c_1 \tag{3.2}$$

and

$$\int_0^u g(s)ds \leqslant \int_0^u \left[\frac{\gamma_2 s}{2} + k_2\right] ds \leqslant \frac{\gamma_2 u^2}{4} + k_2 u \leqslant \gamma_2 u^2 + c_2, \tag{3.3}$$

where $c_1 = c_1(\gamma_1) > 0$ and $c_2 = c_2(\gamma_2) > 0$ are independent of ε.

Using (3.2) and (3.3), it follows that

$$\begin{aligned}
&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\
=& V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \int_\Omega \int_0^u f(s)dsdx + \frac{1}{\varepsilon}\int_{\omega_\varepsilon}\int_0^u g(s)dsdx \\
\leqslant& V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \int_\Omega (\gamma_1|u|^2 + c_1)dx + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} (\gamma_2|u|^2 + c_2)dx.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \gamma_1\right)\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\
\leqslant& V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \frac{\gamma_2}{\varepsilon}\int_{\omega_\varepsilon} |u|^2 dx + c_2 k|\Gamma| + c_1|\Omega|,
\end{aligned}$$

and from [9, lemma 2.1] there exists $C > 0$ independent of ε such that

$$\frac{\gamma_2}{\varepsilon}\int_{\omega_\varepsilon} |u|^2 dx \leqslant \gamma_2 C\|u\|^2_{H^2(\Omega)}, \tag{3.4}$$

and this implies that

$$\begin{aligned}
&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \gamma_1\right)\|u\|^2_{L^2(\Omega)} - \gamma_2 C\|u\|^2_{H^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\
\leqslant& V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + c_2 k|\Gamma| + c_1|\Omega|.
\end{aligned}$$

Consequently, for $w^\varepsilon(t) = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \\ \theta^\varepsilon \end{pmatrix}(t)$ being the solution of the problem (1.1) we have

$$\begin{aligned}
&\frac{1}{2}\|\Delta u^\varepsilon\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \gamma_1\right)\|u^\varepsilon\|^2_{L^2(\Omega)} - \gamma_2 C\|u^\varepsilon\|^2_{H^2(\Omega)} + \frac{1}{2}\|v^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta^\varepsilon\|^2_{L^2(\Omega)} \\
\leqslant& V_\varepsilon\begin{pmatrix} u^\varepsilon \\ v^\varepsilon \\ \theta^\varepsilon \end{pmatrix} + c_2 k|\Gamma| + c_1|\Omega|.
\end{aligned}$$

For $0 < \gamma_1 < \frac{1}{2}$ and choosing γ_2 sufficiently small in the inequality above, we obtain

$$\|w^\varepsilon(t)\|^2_X \leqslant C_1 V_\varepsilon(w^\varepsilon(t)) + C_2, \tag{3.5}$$

for some $C_1, C_2 > 0$ independent of ε.

We note that by section 2.3 we obtain that a map $t \mapsto w^\varepsilon(t, w_0)$ is differentiable.

It is clear that for $w^{\varepsilon}(t) = \begin{pmatrix} u^{\varepsilon} \\ v^{\varepsilon} \\ \theta^{\varepsilon} \end{pmatrix}(t)$ being the solution of the problem (1.1) we have $[0, t_{max}(w_0)) \ni t \mapsto V_{\varepsilon}(w^{\varepsilon}(t, w_0)) \in \mathbb{R}$ is non-increasing because

$$\frac{dV_{\varepsilon}}{dt}(t) = -\|\nabla\theta^{\varepsilon}(t)\|^2_{L^2(\Omega)} - \|\theta^{\varepsilon}(t)\|^2_{L^2(\Omega)} \leqslant 0,$$

for $V_{\varepsilon}(t) = V_{\varepsilon}(w^{\varepsilon}(t, w_0))$, $t \in [0, t_{max}(w_0))$.

Using lemma 2 we can prove that V_{ε} is continuous and uniformly bounded in uniformly bounded subsets of X. From (3.5) we have that given $r > 0$, there is a constant $C(r) > 0$ independent of ε such that

$$\sup\{\|w^{\varepsilon}(t, w_0)\|_X : \|w_0\|_X \leqslant r,\ t \in [0, t^{\varepsilon}_{max}(w_0))\} \leqslant C. \tag{3.6}$$

From (3.6) and [16, theorem 4.2.1] we have that for each $w_0 \in X$, the solution of (2.4) is defined for all $t \geqslant 0$, that is, $t^{\varepsilon}_{max}(w_0) = \infty$. Consequently, for each $\varepsilon \in [0, \varepsilon_0)$, we can to define a nonlinear semigroup $\{S_{\varepsilon}(t) : t \geqslant 0\}$ in X by

$$S_{\varepsilon}(t)w_0 = w^{\varepsilon}(t, w_0), \quad t \geqslant 0.$$

This also implies that each uniformly bounded subset of X has orbit and global orbit uniformly bounded in ε.

Note that the nonlinear semigroups are given by the variation of constants formula

$$S_{\varepsilon}(t)w_0 = e^{\mathbb{A}t}w_0 + \int_0^t e^{\mathbb{A}(t-s)}F_{\varepsilon}(S_{\varepsilon}(s)w_0)ds, \quad t \geqslant 0,$$

see [17] for details.

3.2. Limit problem

Let $V_0 : X \to \mathbb{R}$ be the continuous functional defined by

$$V_0\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} - \int_{\Omega}\int_0^u f(s)dsdx - \int_{\Gamma}\int_0^u g(s)dsdx. \tag{3.7}$$

Using (3.2) and (3.3), it follows that

$$\begin{aligned}
&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\
=&V_0\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \int_{\Omega}\int_0^u f(s)dsdx + \int_{\Gamma}\int_0^u g(s)dsdS \\
\leqslant&V_0\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \int_{\Omega}(\gamma_1|u|^2 + c_1)dx + \int_{\Gamma}(\gamma_2|\gamma(u)|^2 + c_2)dS.
\end{aligned}$$

Thus

$$\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \gamma_1\right)\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)}$$
$$\leqslant V_0\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \gamma_2 \int_\Gamma |\gamma(u)|^2 dS + c_2|\Gamma| + c_1|\Omega|,$$

and from trace theorem there exist $C > 0$ such that

$$\gamma_2 \int_\Gamma |\gamma(u)|^2 dS \leqslant \gamma_2 C\|u\|^2_{H^2(\Omega)},$$

and this implies that

$$\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \gamma_1\right)\|u\|^2_{L^2(\Omega)} - \gamma_2 C\|u\|^2_{H^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)}$$
$$\leqslant V_0\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + c_2|\Gamma| + c_1|\Omega|.$$

Consequently, for $w(t) = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}(t)$ being the solution of the problem (1.5) we have

$$\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \gamma_1\right)\|u\|^2_{L^2(\Omega)} - \gamma_2 C\|u\|^2_{H^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)}$$
$$\leqslant V_0\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + c_2|\Gamma| + c_1|\Omega|.$$

For $0 < \gamma_1 < \frac{1}{2}$ and choosing γ_2 sufficiently small in the inequality above, we have

$$\|w(t)\|^2_X \leqslant C_1 V_0(w(t)) + C_2, \tag{3.8}$$

for some C_1, $C_2 > 0$.

Again in the section 2.3 we obtain that a map $t \mapsto w(t, w_0)$ is differentiable.

It is clear that for $w(t) = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}(t)$ being the solution of the problem (1.5) we have that
$[0, t_{max}(w_0)) \ni t \mapsto V_0(w(t, w_0)) \in \mathbb{R}$ is non-increasing because

$$\frac{dV_0}{dt}(w(t)) = -\|\nabla\theta(t)\|^2_{L^2(\Omega)} - \|\theta(t)\|^2_{L^2(\Omega)} \leqslant 0,$$

for $V_0(t) = V_0(w(t, w_0))$ and $t \in [0, t_{max}(w_0))$.

Using trace theorem we can prove that V is continuous and uniformly bounded in uniformly bounded subsets of X. From (3.8) we have that given $r > 0$, there is a constant $C(r) > 0$ such that

$$\sup\{\|w(t, w_0)\|_X : \|w_0\|_X \leqslant r,\ t \in [0, t_{max}(w_0))\} \leqslant C. \tag{3.9}$$

From (3.9) and [16, theorem 4.2.1] we have that for each $w_0 \in X$, the solution of (2.7) is defined for all $t \geqslant 0$, that is $t_{max}(w_0) = \infty$. Consequently, we can to define a nonlinear semigroup $\{S_0(t) :\ t \geqslant 0\}$ in X by

$$S_0(t)w_0 = w(t, w_0), \quad t \geqslant 0.$$

This also implies that each uniformly bounded subset of X has orbit and global orbit uniformly bounded.

Note that the nonlinear semigroup is given by the variation of constants formula

$$S_0(t)w_0 = e^{\mathbb{A}t}w_0 + \int_0^t e^{\mathbb{A}(t-s)}F_0(S_0(s)w_0)ds, \quad t \geqslant 0,$$

see [17] for details.

4. Existence and upper semicontinuity of global attractors

From this section onwards we will be assuming all the previous hypotheses. The results obtained in the previous sections and smoothing effect of the equations assure us that the nonlinear semigroups generated by our problems (2.4) and (2.7) have global compact attractors $\mathscr{A}_\varepsilon$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Moreover, we get a result of boundedness uniform in ε of the attractores, the convergence of the nonlinear semigroups and upper semicontinuity of the global attractors.

4.1. Existence of the global attractors

In this subsection, we will establish the existence and characterization of the global compact attractors for the nonlinear semigroups generated by our problems (2.4) and (2.7) using the results of Hale [16, Chapter 3]. Moreover, we will obtain uniform boundedness of the attractors.

THEOREM 4. *For sufficiently small $\varepsilon \geqslant 0$, the parabolic problems* (2.4) *and* (2.7) *have a global compact attractor $\mathscr{A}_\varepsilon$ and $\mathscr{A}_\varepsilon = W^u(\mathscr{E}_\varepsilon)$, where*

$$W^u(\mathscr{E}_\varepsilon) = \Big\{w \in X : S_\varepsilon(-t)w \text{ is defined for } t \geqslant 0 \text{ and } \lim_{t\to+\infty} dist(S_\varepsilon(-t)w, \mathscr{E}_\varepsilon) = 0\},$$

and $\mathscr{E}_\varepsilon$ denotes the set of equilibria of the nonlinear semigroup $\{S_\varepsilon(t) : t \geqslant 0\}$ generated by our problems (2.4) *and* (2.7)*. Moreover, $\mathscr{A}_\varepsilon$ is connected.*

Proof. Using the functionals V_ε and V_0 defined in (3.1) and (3.7), respectively, for $\varepsilon \geqslant 0$ enough small, from the smoothing effect of the systems and the [16, theorem 3.8.5] we get that the problems (2.4) and (2.7) have a global compact attractor $\mathscr{A}_\varepsilon$ in X with the characterization $\mathscr{A}_\varepsilon = W^u(\mathscr{E}_\varepsilon)$, for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Moreover, $\mathscr{A}_\varepsilon$ is connected because X is a Hilbert space. □

Here we will present a result on the uniform bounds of the attractors that we will use to show the upper semicontinuity at $\varepsilon = 0$ of the attractors.

THEOREM 5. *For sufficiently small $\varepsilon \geqslant 0$, the union of the global attractors $\bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathscr{A}_\varepsilon$ is a bounded set in X.*

Proof. For sufficiently small $\varepsilon \geqslant 0$, it is important to note that for global bounded solutions of (2.4) in (3.5), we can estimate $V_\varepsilon(w^\varepsilon(t))$ by a constant independent of ε thanks to (3.4), as well as, the constant $C_2 > 0$ in (3.5) is independent of ε. Hence, this boundedness uniform in ε jointly with (3.6), (3.9), and the invariance of the attractors by the semigroups, allows to conclude that the union of the global attractors $\bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathscr{A}_\varepsilon$ is a bounded set in X. $\square$

4.2. Convergence of the nonlinear semigroups

From now on we will show the convergence of the nonlinear semigroups as $\varepsilon \to 0$. With this convergence result we concluded that the limit problem for the autonomous thermoelastic plate system (1.1) is given by (1.5). Initially, we will estimate the linear semigroup.

We use the remark 4 to show that the nonlinear semigroups behave continuously at $\varepsilon \to 0$.

PROPOSITION 1. *Under the above hypothesis, let $\frac{1}{2} < s \leqslant 1$ and some fixed $\tau > 0$. Then, there exists a function $C(\varepsilon) \geqslant 0$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for $w_0 \in B$, where $B \subset X$ is a bounded set, we have*

$$\left\| S_\varepsilon(t)w_0 - S_0(t)w_0 \right\|_X \leqslant M(\tau,B)C(\varepsilon), \quad \forall\, t \in [0,\tau], \tag{4.1}$$

for some constant $M(\tau,B) > 0$.

Proof. Let $B \subset X$ be a bounded set, and let $w_0 \in B$. Fixed $\tau > 0$, we consider the nonlinear semigroups given by the variation of constant formula

$$S_\varepsilon(t)w_0 = e^{\mathbb{A}t}w_0 + \int_0^t e^{\mathbb{A}(t-\xi)} F_\varepsilon(S_\varepsilon(\xi)w_0)d\xi,\ \varepsilon \in [0,\varepsilon_0] \tag{4.2}$$

associated with (2.4) and (2.7).

Note that from (4.2), for $t \in (0,\tau]$, we have

$$\begin{aligned} &\left\| S_\varepsilon(t)w_0 - S_0(t)w_0 \right\|_X \\ &\leqslant \int_0^t \left\| e^{\mathbb{A}(t-\xi)} \right\|_{\mathscr{L}(H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega),X)} \\ &\qquad \times \left\| F_\varepsilon(S_\varepsilon(\xi)w_0) - F_0(S_0(\xi)w_0) \right\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} d\xi. \end{aligned} \tag{4.3}$$

Adding and subtracting the term $F_\varepsilon(S_0(\xi)w_0)$ in the second norm on right side of

(4.3), from (2.17) we can to write the inequality above of the following form

$$\begin{aligned}
&\left\|S_\varepsilon(t)w_0 - S_0(t)w_0\right\|_X \\
\leqslant &\int_0^t \left\|e^{\mathbb{A}(t-\xi)}\right\|_{\mathscr{L}(H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega),X)} \\
&\quad \times \left\|F_\varepsilon(S_\varepsilon(\xi)w_0) - F_\varepsilon(S_0(\xi)w_0)\right\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} d\xi \\
&+ \int_0^t \left\|e^{\mathbb{A}(t-\xi)}\right\|_{\mathscr{L}(H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega),X)} \\
&\quad \times \left\|F_\varepsilon(S_0(\xi)w_0) - F_0(S_0(\xi)w_0)\right\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} d\xi \\
\leqslant &M_\omega \int_0^t (t-\xi)^{-1} e^{-\omega(t-\xi)} \left\|F_\varepsilon(S_\varepsilon(\xi)w_0) - F_\varepsilon(S_0(\xi)w_0)\right\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} d\xi \\
&+ M_\omega \int_0^t (t-\xi)^{-1} e^{-\omega(t-s)} \left\|F_\varepsilon(S_0(\xi)w_0) - F_0(S_0(\xi)w_0)\right\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} d\xi.
\end{aligned} \tag{4.4}$$

We will analyze each term on right side of (4.4) separately.

From (3.6) and (3.9) we have that there exists $C = C(w_0) > 0$ independent of ε, such that

$$\|S_\varepsilon(\xi)w_0\|_X \leqslant C, \quad \forall\, \varepsilon \in [0,\varepsilon_0] \quad \text{and} \quad \forall\, \xi \in [0,\tau].$$

Now, from lemma 3 item (ii), F_ε is globally Lipschitz, uniformly in ε, thus there exists $L > 0$ idependent of ε, such that

$$\begin{aligned}
&\int_0^t (t-\xi)^{-1} e^{-\omega(t-\xi)} \left\|F_\varepsilon(S_\varepsilon(\xi)w_0) - F_\varepsilon(S_0(\xi)w_0)\right\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} d\xi \\
\leqslant &L \int_0^t (t-\xi)^{-1} e^{-\omega(t-\xi)} \left\|S_\varepsilon(\xi)w_0 - S_0(\xi)w_0\right\|_X d\xi.
\end{aligned} \tag{4.5}$$

Since $\{S_0(s)w_0 : s \in [0,\tau]\}$ is bounded set contained in X. Thanks to lemma 3 item (iii), there exists a function $C(\varepsilon) \geqslant 0$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\begin{aligned}
&\int_0^t (t-\xi)^{-1} e^{-\omega(t-\xi)} \left\|F_\varepsilon(S_0(\xi)w_0) - F_0(S_0(\xi)w_0)\right\|_{H^2(\Omega)\times H^{-s}(\Omega)\times L^2(\Omega)} d\xi \\
\leqslant &M(\tau,w_0)C(\varepsilon) \int_0^t (t-\xi)^{-1} e^{-\omega(t-\xi)} d\xi \leqslant M(\tau,w_0)C(\varepsilon) \int_0^{+\infty} z^{-1} e^{-z} dz \\
= &M(\tau,w_0)C(\varepsilon)\Gamma(0),\ (\Gamma(0) = 1),
\end{aligned} \tag{4.6}$$

where $M(\tau,w_0) > 0$ and $\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz$ is the gamma function.

Combining (4.4) with (4.5) and (4.6), we get for all $t \in (0,\tau]$,

$$\begin{aligned}
&\left\|S_\varepsilon(t)w_0 - S_0(t)w_0\right\|_X \\
\leqslant &C(\varepsilon)M(\tau,w_0)M + LM_\omega \int_0^t (t-\xi)^{-1} e^{-\omega(t-\xi)} \left\|S_\varepsilon(\xi)w_0 - S_0(\xi)w_0\right\|_X d\xi,
\end{aligned}$$

where $C(\varepsilon) \geqslant 0$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

From Gronwall's inequality [17, lemma 7.1.1] it follows that

$$\left\| S_\varepsilon(t)w_0 - S_0(t)w_0 \right\|_X \leqslant M(\tau,\omega,L,B)C(\varepsilon)e^{-\omega t},$$

and consequently we conclude that (4.1) holds. □

Similarly, we can prove the following result.

PROPOSITION 2. *Under the above hypothesis, let* $\frac{1}{2} < s \leqslant 1$ *and some fixed* $\tau > 0$*.Then, there exists a function* $C(\varepsilon) \geqslant 0$ *with* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that for* $w_\varepsilon \in \mathscr{A}_\varepsilon$*,* $\varepsilon \in (0,\varepsilon_0]$*, we have*

$$\left\| S_\varepsilon(t)w_\varepsilon - S_0(t)w_\varepsilon \right\|_X \leqslant M(\tau)C(\varepsilon), \quad \forall\, t \in [0,\tau], \tag{4.7}$$

for some constant $M(\tau) > 0$.

4.3. Upper semicontinuity of the global attractors

Finally, in this subsection we will show the upper semicontinuity of global compact attractors at $\varepsilon = 0$, in the sense of Hausdorff semi-distance in X.

THEOREM 6. *The family of global attractors* $\mathscr{A}_\varepsilon$ *is upper semicontinuous at* $\varepsilon = 0$*; that is,*

$$\operatorname{dist}_X(\mathscr{A}_\varepsilon,\mathscr{A}_0) \to 0, \quad \textit{as} \quad \varepsilon \to 0,$$

where

$$\operatorname{dist}_X(\mathscr{A}_\varepsilon,\mathscr{A}_0) := \sup_{w_\varepsilon \in \mathscr{A}_\varepsilon} \operatorname{dist}(w_\varepsilon,\mathscr{A}_0) = \sup_{w_\varepsilon \in \mathscr{A}_\varepsilon} \inf_{w_0 \in \mathscr{A}_0} \{\|w_\varepsilon - w_0\|_X\}.$$

Proof. Thanks to theorem 5, there exists $B_0 \subset X$ a bounded set such that $B_0 \supset \bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathscr{A}_\varepsilon$ for some $\varepsilon_0 > 0$. Hence, $\mathscr{A}_0$ attracts $\bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathscr{A}_\varepsilon \supset \mathscr{A}_\varepsilon$ under the nonlinear semigroup $S_0(\cdot)$. Thus, given $\delta > 0$, there exists $\tau = \tau(\delta) > 0$ such that

$$\operatorname{dist}(S_0(\tau)w_\varepsilon,\mathscr{A}_0) < \frac{\delta}{2}, \quad \forall w_\varepsilon \in \mathscr{A}_\varepsilon. \tag{4.8}$$

Since $\mathscr{A}_\varepsilon$ is invariant then given $\varphi_\varepsilon \in \mathscr{A}_\varepsilon$ there exists $\vartheta_\varepsilon \in \mathscr{A}_\varepsilon$ such that $\varphi_\varepsilon = S_\varepsilon(\tau)\vartheta_\varepsilon$. Thus,

$$\begin{aligned}\operatorname{dist}(\varphi_\varepsilon,\mathscr{A}_0) &= \inf_{w_0\in\mathscr{A}_0} \|\varphi_\varepsilon - w_0\|_X \leqslant \inf_{w_0\in\mathscr{A}_0} \{\|\varphi_\varepsilon - S_0(\tau)\vartheta_\varepsilon\|_X + \|S_0(\tau)\vartheta_\varepsilon - w_0\|_X\} \\ &= \|S_\varepsilon(\tau)\vartheta_\varepsilon - S_0(\tau)\vartheta_\varepsilon\|_X + \operatorname{dist}(S_0(\tau)\vartheta_\varepsilon,\mathscr{A}_0).\end{aligned}$$

From proposition 2, for ε enough small, we get

$$\|S_\varepsilon(\tau)\vartheta_\varepsilon - S_0(\tau)\vartheta_\varepsilon\|_X \leqslant \frac{\delta}{2}. \tag{4.9}$$

Using (4.8) and (4.9), for ε enough small, we have

$$\operatorname{dist}(\varphi_\varepsilon, \mathscr{A}_0) < \delta, \quad \forall\, \varphi_\varepsilon \in \mathscr{A}_\varepsilon,$$

and thus we conclude the upper semicontinuity of the family of attractors at $\varepsilon = 0$. □

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