

A NON GREEN'S FUNCTION APPROACH TO FRACTIONAL LYAPUNOV-TYPE INEQUALITIES WITH APPLICATIONS TO MULTIVARIATE DOMAINS

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Abstract. We derive Lyapunov-type inequalities for certain fractional differential equations of order α , where $1 < \alpha \leq 2$ or $2 < \alpha \leq 3$. The methods used within rely on considering the maximum value of a nontrivial solution in a given interval as opposed to traditional methods which utilize the Green's function. This particular method provides versatility and can be applied to other fractional boundary value problems where the Green's function is inaccessible. Furthermore, we demonstrate how the inequalities may be extended to fractional multivariate equations in both the left and right-fractional cases.

1. Introduction

For the second-order linear differential equation

$$x'' + q(t)x = 0 \tag{1.1}$$

with $q \in C([a, b], \mathbb{R})$, the following result is known as the Lyapunov inequality:

THEOREM 1.1. (See [22, 3]) *Assume eq. (1.1) has a nontrivial solution $x(t)$ satisfying $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. Then*

$$\int_a^b |q(t)| dt > \frac{4}{b-a}. \tag{1.2}$$

The Lyapunov inequality has been used as an important tool in oscillation, disconjugacy, control theory, eigenvalue problems, and other areas of differential equations. Due to its importance in applications, theorem 1.1 has been extended in many directions by various authors. The reader is directed to [2, 4, 5, 6, 8, 12, 13, 14, 21, 25, 26, 27, 30, 31, 32, 33, 34] for further reading on extensions of the Lyapunov inequality.

Recently, a search for Lyapunov-type inequalities has begun in the study of fractional differential equations. Fractional differential equations have gained attention for

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their applications to many engineering and scientific disciplines, notably in the mathematical modeling of systems and processes in the fields of physical, mechanics, chemistry, and aerodynamics. Although relevant definitions are provided in the next section, we briefly draw the reader's attention to one result relating to Lyapunov-type inequalities for fractional differential equations. Ferreira [10] first obtained a Lyapunov-type inequality with pointwise boundary conditions (BCs). In particular, he considered the Riemann-Liouville fractional differential equation

$$D_{a^+}^{\alpha} x + q(t)x = 0, \quad (1.3)$$

where $q \in C([a, b], \mathbb{R})$ and $1 < \alpha \leq 2$ and obtained the following result:

THEOREM 1.2. *Assume eq. (1.3) has a nontrivial solution $x(t)$ satisfying $x(a) = x(b) = 0$. Then*

$$\int_a^b |q(t)| dt > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1},$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ is the gamma function.

It is easy to see that, for $\alpha = 2$, the result in theorem 1.2 leads to the classical Lyapunov inequality (1.2).

Following the Ferreira's seminal paper [10], the traditional method for obtaining Lyapunov-type inequalities for a fractional boundary value problem (BVP) has been to maximize the corresponding fractional Green's function. The interested reader is invited to review [7, 9, 16, 19, 17, 24, 28] for more Lyapunov-type inequalities involving the Riemann-Liouville and Caputo fractional derivatives with several types of BCs. In fact, this technique involving the Green's function has been employed by virtually all authors, including those cited, finding Lyapunov-type inequalities for fractional BVPs. In this article, however, we bypass the Green's function method by considering the maximum value of a nontrivial solution in a given interval. This new approach provides more versatility and can be applied in the future to explore the Lyapunov-type inequalities for other types of fractional BVPs where finding and analyzing the Green's function is not ideal.

There has also been a lot of work on Lyapunov-type inequalities including integer and fractional order on multivariate domains or for partial differential equations by many authors. To name a few, we refer the reader to [1, 15, 23, 18]. In particular, Anastassiou [1] obtained Lyapunov-type inequalities for various multivariate equations on special domains in \mathbb{R}^n . For $N \geq 2$, denote

$$B(0, R) := \{x \in \mathbb{R}^N : |x| < R\} \text{ for } R > 0,$$

and let A be an open spherical shell in \mathbb{R}^N centered at the origin, i.e., $A := B(0, b) \setminus \overline{B(0, a)}$ for $0 < a < b$. Consider the following equation

$$\frac{\partial^2 y(x)}{\partial r^2} + q(x)y(x) = 0 \quad (1.4)$$

with the BCs

$$y(\partial B(0, a)) = y(\partial B(0, b)) = 0 \tag{1.5}$$

where $q \in C(\bar{A})$ and the derivatives with respect to r are directional derivatives in the radial direction.

THEOREM 1.3. *Assume eq. (1.4) has a nontrivial solution $y(x)$ satisfying (1.5) and $y(x) \neq 0$ on A . Then*

$$\int_A |q(x)| dx > \frac{8\pi^{N/2} a^{N-1}}{\Gamma(N/2)(b-a)}. \tag{1.6}$$

Note that inequality (1.6) holds only when the domain A is radially symmetric and contains a circular hole of radius $a > 0$ inside. Moreover, it becomes less sharp when a is small and provides no information when $a = 0$, i.e., when the hole shrinks to a point. In fact, this is the case in every multivariate Lyapunov-type inequalities found in [1]. In section 5, we discuss several multivariate Lyapunov-type inequalities where we correct this type of issues.

The article is outlined as follows. In section 2, we present necessary definitions and preliminary results regarding Reimann-Liouville fractional differential equations. In sections 3 and 4, we establish Lyapunov-type inequalities for univariate Riemann-Liouville fractional differential equations of order $1 < \alpha \leq 2$ and $2 < \alpha \leq 3$ together with various pointwise and mixed BCs. All results in sections 3 and 4 use the non-Green’s function method. Lastly, in section 5, we will further develop Lyapunov-type inequalities for multivariate fractional BVPs on domains in \mathbb{R}^N which are not necessarily radially symmetric.

2. Background materials and preliminaries

For the convenience of the reader, we present the necessary definitions and lemmas from fractional calculus theory in the sense of Riemann-Liouville. These results can be found in the monograph [20]. For additional reading, see [11, 29].

DEFINITION 2.1. *The left and right fractional integrals of order $\alpha > 0$ of a function $x(t)$ is defined as*

$$\left(I_{a^+}^\alpha x\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds, \quad t > a,$$

and

$$\left(I_b^- \alpha x\right)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} x(s) ds, \quad t < b,$$

respectively. Here $\Gamma(\alpha)$ is the gamma function.

DEFINITION 2.2. *The left and right fractional derivative of order $\alpha > 0$ of a function $x(t)$ is defined as*

$$\begin{aligned} (D_{a^+}^\alpha x)(t) &:= \left(\frac{d}{dt}\right)^n (I_{a^+}^{n-\alpha} x)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} x(s) ds, \quad t > a, \end{aligned} \tag{2.1}$$

and

$$(D_{b^-}^\alpha x)(t) := \left(-\frac{d}{dt}\right)^n (I_{b^-}^{n-\alpha} x)(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} x(s) ds, \quad t < b,$$

respectively. Here $n = \lfloor \alpha \rfloor + 1$ with $\lfloor \alpha \rfloor$ the integer part of α and $\Gamma(\alpha)$ is the gamma function.

In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$(D_{a^+}^n x)(t) = x^{(n)}(t) \text{ and } (D_{b^-}^n x)(t) = (-1)^n x^{(n)}(t).$$

It may be directly verified from definitions 2.1 and 2.2 that for $\alpha_1 > -1$ and $\alpha_2 \geq 0$,

$$I_{a^+}^{\alpha_2} (t-a)^{\alpha_1} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+1+\alpha_2)} (t-a)^{\alpha_2+\alpha_1},$$

and

$$D_{a^+}^{\alpha_2} (t-a)^{\alpha_1} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+1-\alpha_2)} (t-a)^{\alpha_1-\alpha_2}.$$

Additionally,

$$I_{b^-}^{\alpha_2} (b-t)^{\alpha_1} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+1+\alpha_2)} (b-t)^{\alpha_2+\alpha_1},$$

and

$$D_{a^+}^{\alpha_2} (b-t)^{\alpha_1} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+1-\alpha_2)} (b-t)^{\alpha_1-\alpha_2}.$$

Now, we recall a few well known properties in Riemann-Liouville fractional derivatives and integrals. Let $x \in L[a, b]$ and $\alpha_1, \alpha_2 > 0$. Then the following properties are referred to as the ‘‘semigroup property’’ for the fractional integral.

$$I_{a^+}^{\alpha_1} I_{a^+}^{\alpha_2} x(t) = I_{a^+}^{\alpha_1+\alpha_2} x(t) \quad \text{and} \quad I_{b^-}^{\alpha_1} I_{b^-}^{\alpha_2} x(t) = I_{b^-}^{\alpha_1+\alpha_2} x(t).$$

Let $\alpha \geq 0$, $k \in \mathbb{N}$ and $D = d/dt$. If $D_{a^+}^\alpha x(t)$, $D_{a^+}^{\alpha+k} x(t)$, $D_{b^-}^\alpha x(t)$, and $D_{b^-}^{\alpha+k} x(t)$ exist, then

$$D^k (D_{a^+}^\alpha x(t)) = D_{a^+}^{\alpha+k} x(t) \quad \text{and} \quad D^k (D_{b^-}^\alpha x(t)) = D_{b^-}^{\alpha+k} x(t). \tag{2.2}$$

The following assertions show that the fractional differentiation is an operation inverse to fractional integration. Let $x \in L[a, b]$ and $\alpha > 0$. Then

$$D_{a^+}^\alpha I_{a^+}^\alpha x(t) = x(t) \quad \text{and} \quad D_{b^-}^\alpha I_{b^-}^\alpha x(t) = x(t).$$

The following are the composition relations between fractional differentiation and fractional integration operators. As earlier, $x \in L[a, b]$, and $\alpha > 0$. Then

$$D_{a^+}^{\alpha_1} I_{a^+}^{\alpha_2} x(t) = I_{a^+}^{\alpha_2 - \alpha_1} x(t) \quad \text{and} \quad D_{b^-}^{\alpha_1} I_{b^-}^{\alpha_2} x(t) = I_{b^-}^{\alpha_2 - \alpha_1} x(t).$$

In particular, when $\alpha_1 = k \in \mathbb{N}$ and $\alpha_2 > k$, then

$$D_{a^+}^k I_{a^+}^{\alpha_2} x(t) = I_{a^+}^{\alpha_2 - k} x(t) \quad \text{and} \quad D_{b^-}^k I_{b^-}^{\alpha_2} x(t) = I_{b^-}^{\alpha_2 - k} x(t). \tag{2.3}$$

Finally, let $x \in L[a, b]$, $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. Then

$$I_{a^+}^\alpha D_{a^+}^\alpha x(t) = x(t) + \sum_{i=1}^n c_i (t - a)^{\alpha - i} \tag{2.4}$$

with $c_i \in \mathbb{R}$ for $1 \leq i \leq n$.

3. Results for $1 < \alpha \leq 2$

In this section we consider the linear left-fractional differential equation

$$D_{a^+}^\alpha x + q(t)x = 0, \tag{3.1}$$

and the linear right-fractional differential equation

$$D_{b^-}^\alpha x + q(t)x = 0, \tag{3.2}$$

where $q \in C([a, b], \mathbb{R})$ and $1 < \alpha \leq 2$.

We first present a Lyapunov-type inequality for eq. (3.1).

THEOREM 3.1. *Assume eq. (3.1) has a nontrivial solution $x(t)$ satisfying the BCs*

$$x(a) = D_{a^+}^{\alpha-1} x(c) = 0 \tag{3.3}$$

for $c > a$. Then

$$\int_a^c |q(s)| ds > \frac{\Gamma(\alpha)}{(c - a)^{\alpha-1}}. \tag{3.4}$$

Proof. We assume that $x(t)$ satisfies (3.3). Since $x(t)$ is nontrivial and continuous, there exists a $d \in (a, c]$ such that $m = \max_{a \leq t \leq c} |x(t)| = |x(d)|$. From (2.2), we see that

$$D_{a^+}^\alpha x(t) = (D_{a^+}^{\alpha-1} x(t))'.$$

Hence (3.1) becomes

$$(D_{a^+}^{\alpha-1}x)' + q(t)x = 0. \tag{3.5}$$

For $t \in [a, c]$, by integrating (3.5) from t to c and utilizing the fact that $D_{a^+}^{\alpha-1}x(c) = 0$, we have

$$D_{a^+}^{\alpha-1}x(t) = \int_t^c q(s)x(s)ds.$$

Since $|x(t)| \leq m$ and $|x(t)| \neq m$ for $t \in [a, c]$, it follows that

$$\left| D_{a^+}^{\alpha-1}x(t) \right| = \left| \int_t^c q(s)x(s)ds \right| < m \int_a^c |q(s)|ds. \tag{3.6}$$

Note that $0 < \alpha - 1 \leq 1$. From (2.4), we have

$$I_{a^+}^{\alpha-1}(D_{a^+}^{\alpha-1}x(t)) = x(t) + c_1(t-a)^{\alpha-2}. \tag{3.7}$$

Now $x(a) = 0$ implies $c_1 = 0$. Letting $c_1 = 0$ in (3.7), we have

$$x(t) = I_{a^+}^{\alpha-1}(D_{a^+}^{\alpha-1}x(t)).$$

Taking the absolute value on both sides and using (3.6) we see that

$$|x(t)| = |I_{a^+}^{\alpha-1}(D_{a^+}^{\alpha-1}x(t))| \leq I_{a^+}^{\alpha-1}|D_{a^+}^{\alpha-1}x(t)| < \frac{m(t-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^c |q(s)|ds$$

for $t \in [a, c]$. In particular, for $t = d$ we have

$$m = |x(d)| < \frac{m(d-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^c |q(s)|ds \leq \frac{m(c-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^c |q(s)|ds.$$

Canceling m from both sides and rearranging terms, we see that (3.4) holds. \square

Now we present a Lyapunov-type inequality for eq. (3.2). The proof is essentially the same as that of theorem 3.1. Therefore, we omit the proof.

THEOREM 3.2. *Assume eq. (3.2) has a nontrivial solution $x(t)$ satisfying the BCs*

$$D_b^{\alpha-1}x(c) = x(b) = 0 \tag{3.8}$$

for $c < b$. Then

$$\int_c^b |q(s)|ds > \frac{\Gamma(\alpha)}{(b-c)^{\alpha-1}}. \tag{3.9}$$

REMARK 3.1. Observe in the case when $\alpha = 2$, eq. (3.1) and eq. (3.2) reduces to the same differential equation as eq. (1.1), which is used in Lyapunov’s classical inequality of theorem 1.1. Similarly, the corresponding BCs (3.3) and (3.8) become the right focal BCs, $x(a) = x'(c) = 0$ and left focal BCs, $x'(c) = x(b) = 0$. Hence inequality (3.4) simplifies to

$$\int_a^c |q(s)|ds > \frac{1}{c-a}$$

and inequality (3.9) simplifies to

$$\int_c^b |q(s)| ds > \frac{1}{b-c}.$$

Assuming $x(t)$ is a nontrivial solution of (1.1) satisfying both right and left focal BCs, it follows immediately that

$$\int_a^b |q(s)| ds = \int_a^c |q(s)| ds + \int_c^b |q(s)| ds > \frac{1}{c-a} + \frac{1}{b-c} > \frac{4}{b-a},$$

i.e., the classical Lyapunov inequality (1.2).

Now we present a fractional Lyapunov-type inequality for eq. (3.1) with left focal BCs.

THEOREM 3.3. *Assume eq. (3.1) has a nontrivial solution $x(t)$ satisfying the BCs*

$$x'(a) = x(b) = 0 \tag{3.10}$$

for $b > a$. Then

$$\int_a^b |q(s)| ds > \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}. \tag{3.11}$$

Proof. We assume that $x(t)$ satisfies (3.10). Since $1 < \alpha \leq 2$, we may reduce eq. (3.1) to an equivalent integral equation

$$x(t) = -I_{a^+}^\alpha (q(t)x(t)) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2},$$

where we have used (2.4). Differentiating both sides with respect to t and using (2.3) we see that

$$x'(t) = -I_{a^+}^{\alpha-1} (q(t)x(t)) + c_1(\alpha-1)(t-a)^{\alpha-2} + c_2(\alpha-2)(t-a)^{\alpha-3},$$

Now $x'(a) = 0$ implies $c_1 = c_2 = 0$. Hence

$$-x'(t) = I_{a^+}^{\alpha-1} (q(t)x(t)).$$

Define $m := \max_{a \leq t \leq b} |x(t)|$. Recall that $x(b) = 0$. Then for $a \leq t \leq b$ we have

$$x(t) = \int_t^b -x'(s) ds = \int_t^b I_{a^+}^{\alpha-1} (q(s)x(s)) ds.$$

Taking the absolute value on both sides, it follows that

$$\begin{aligned}
 |x(t)| &= \left| \int_t^b I_{a^+}^{\alpha-1} q(s)x(s)ds \right| \leq \left| \int_a^b I_{a^+}^{\alpha-1} q(s)x(s)ds \right| \\
 &= \frac{1}{\Gamma(\alpha-1)} \left| \int_a^b \int_a^s (s-\tau)^{\alpha-2} q(\tau)x(\tau)d\tau ds \right| \\
 &= \frac{1}{\Gamma(\alpha-1)} \left| \int_a^b \left(\int_\tau^b (s-\tau)^{\alpha-2} ds \right) q(\tau)x(\tau)d\tau \right| \\
 &= \frac{1}{(\alpha-1)\Gamma(\alpha-1)} \left| \int_a^b (b-\tau)^{\alpha-1} q(\tau)x(\tau)d\tau \right| \leq \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b |q(\tau)||x(\tau)|d\tau.
 \end{aligned}$$

Since $|x(t)| \leq m$ and $|x(t)| \not\equiv m$ for $t \in [a, b]$, it follows that

$$m < \frac{m(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b |q(\tau)|d\tau.$$

Canceling m from both sides and rearranging terms, we see that (3.11) holds. \square

Similarly, we present a Lyapunov-type inequality for eq. (3.2) with right focal BC. The proof is essentially the same as that of theorem 3.3. Therefore, we omit the proof.

THEOREM 3.4. Assume eq. (3.2) has a nontrivial solution $x(t)$ satisfying the BCs

$$x(a) = x'(b) = 0$$

for $a < b$. Then

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}.$$

Now, we present a Lyapunov-type inequality for eq. (3.1) with mixed BCs.

THEOREM 3.5. Assume eq. (3.1) has a nontrivial solution $x(t)$ satisfying the BCs

$$x(a) = x'(a) = x(b) = 0. \tag{3.12}$$

Then

$$\int_a^b |q(s)|ds > \frac{2\Gamma(\alpha)}{(b-a)^{\alpha-1}}. \tag{3.13}$$

Proof. We assume that $x(t)$ satisfies (3.12). Since $1 < \alpha \leq 2$, it follows from (2.4) that

$$I_{a^+}^\alpha D_{a^+}^\alpha x(t) = x(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2}.$$

Clearly $x(a) = 0$ implies $c_2 = 0$. Hence

$$I_{a^+}^\alpha D_{a^+}^\alpha x(t) = x(t) + c_1(t-a)^{\alpha-1}.$$

Differentiating both sides with respect to t and using (2.3) we have

$$I_{a^+}^{\alpha-1} D_{a^+}^\alpha x(t) = x'(t) + c_1(\alpha - 1)(t - a)^{\alpha-2}.$$

Now $x'(a) = 0$ implies $c_1 = 0$. Hence

$$x(t) = I_{a^+}^\alpha D_{a^+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds. \tag{3.14}$$

Since $x(t)$ is nontrivial and continuous, there exists a $d \in [a, b]$ such that $m = \max_{a \leq t \leq b} |x(t)| = |x(d)|$. Now letting $t = d$ and taking the absolute value on both sides of (3.14) we see that

$$m = |x(d)| = \frac{1}{\Gamma(\alpha)} \left| \int_a^d (d - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds \right| \leq \frac{1}{\Gamma(\alpha)} \int_a^d (d - s)^{\alpha-1} |D_{a^+}^\alpha x(s)| ds.$$

Since $1 < \alpha \leq 2$, it follows that

$$m \leq \frac{1}{\Gamma(\alpha)} \int_a^d (b - s)^{\alpha-1} |D_{a^+}^\alpha x(s)| ds. \tag{3.15}$$

Recall that $x(b) = 0$. Hence letting $t = b$ in (3.14) we have

$$\begin{aligned} 0 &= \frac{1}{\Gamma(\alpha)} \int_a^b (b - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^d (b - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds + \frac{1}{\Gamma(\alpha)} \int_d^b (b - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_a^d (d - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds + \frac{1}{\Gamma(\alpha)} \int_d^b (b - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds \\ &= m + \frac{1}{\Gamma(\alpha)} \int_d^b (b - s)^{\alpha-1} D_{a^+}^\alpha x(s) ds. \end{aligned}$$

It follows that

$$m \leq \frac{1}{\Gamma(\alpha)} \int_d^b (b - s)^{\alpha-1} |D_{a^+}^\alpha x(s)| ds. \tag{3.16}$$

Adding (3.15) and (3.16) we have

$$2m \leq \frac{1}{\Gamma(\alpha)} \int_a^b (b - s)^{\alpha-1} |D_{a^+}^\alpha x(s)| ds. \tag{3.17}$$

Note that $|x(t)| \leq m$ and $|x(t)| \not\equiv m$ for $t \in [a, b]$. Then for $t \in [a, b]$ it follows from eq. (3.1), that

$$|D_{a^+}^\alpha x| = |-q(t)x(t)| \leq |q(t)||x(t)| < m|q(t)|. \tag{3.18}$$

Using (3.18) in (3.17) we have

$$2m < \frac{m}{\Gamma(\alpha)} \int_a^b (b - s)^{\alpha-1} |q(s)| ds \leq \frac{m(b - a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b |q(s)| ds.$$

Canceling m from both sides and rearranging terms, we see that (3.13) holds. \square

Finally, we present a Lyapunov-type inequality for eq. (3.2). The proof is essentially the same as that of theorem 3.5. Therefore, we omit the proof.

THEOREM 3.6. *Assume eq. (3.2) has a nontrivial solution $x(t)$ satisfying the BCs*

$$x(a) = x(b) = x'(b) = 0.$$

Then

$$\int_a^b |q(s)| ds > \frac{2\Gamma(\alpha)}{(b-a)^{\alpha-1}}.$$

REMARK 3.2. Observe in the case when $\alpha = 2$, BVP (3.1), (3.12) becomes

$$x'' + q(t)x = 0, \quad x(a) = x'(a) = x(b) = 0.$$

In this case, the inequality (3.13) simplifies to

$$\int_a^b |q(s)| ds > \frac{2}{b-a}.$$

A similar comment can be made for theorem 3.6. We omit the details here.

4. Results for $2 < \alpha \leq 3$

In this section we consider the linear left-fractional differential equation

$$D_{a^+}^\alpha x + q(t)x = 0, \tag{4.1}$$

and the linear right-fractional differential equation

$$D_b^\alpha x + q(t)x = 0, \tag{4.2}$$

where $q \in C([a, b], \mathbb{R})$ and $2 < \alpha \leq 3$. We first present a Lyapunov-type inequality for eq. (4.1).

THEOREM 4.1. *Assume eq. (4.1) has a nontrivial solution $x(t)$ satisfying the BCs*

$$x(a) = x'(a) = x(b) = 0 \text{ and } D_{a^+}^{\alpha-1}x(\xi) = 0 \tag{4.3}$$

for some $\xi \in [a, b]$. Then

$$\int_a^b |q(s)| ds > \frac{2(2\alpha - 3)^{1/2}\Gamma(\alpha - 1)}{(b-a)^{\alpha-1}}.$$

Proof. We assume that $x(t)$ satisfies (4.3). From (2.2), we see that

$$D_{a^+}^\alpha x = (D_{a^+}^{\alpha-1} x)'$$

Hence eq. (4.1) becomes

$$(D_{a^+}^{\alpha-1} x)' + q(t)x = 0. \tag{4.4}$$

Integrating (4.4) from ξ to t and utilizing the fact that $D_{a^+}^{\alpha-1} x(\xi) = 0$ we have

$$D_{a^+}^{\alpha-1} x + \int_{\xi}^t q(s)x(s)ds = 0. \tag{4.5}$$

We denote $h(t) = \int_{\xi}^t q(s)x(s)ds$. Now consider the following BVP

$$D_{a^+}^{\alpha-1} x = -h(t), \quad x(a) = x'(a) = x(b) = 0. \tag{4.6}$$

Since $x(t)$ is nontrivial and continuous, there exists a $d \in [a, b]$ such that $m = \max_{a \leq t \leq b} |x(t)| = |x(d)|$. Also note that $1 < \alpha - 1 \leq 2$. Then applying the same technique to BVP (4.6) as shown in the proof of theorem 3.5 we see that (3.17) holds with α replaced by $\alpha - 1$, i.e.,

$$2m \leq \frac{1}{\Gamma(\alpha - 1)} \int_a^b (b - s)^{\alpha-2} |D_{a^+}^{\alpha-1} x(s)| ds.$$

By Cauchy-Schwartz inequality, we have

$$2m\Gamma(\alpha - 1) \leq \left(\int_a^b (b - s)^{2\alpha-4} ds \right)^{1/2} \left(\int_a^b |D_{a^+}^{\alpha-1} x(s)|^2 ds \right)^{1/2}.$$

Simplifying the first integral on the right and squaring and rearranging terms we see that

$$\frac{4m^2(2\alpha - 3)\Gamma^2(\alpha - 1)}{(b - a)^{2\alpha-3}} \leq \int_a^b |D_{a^+}^{\alpha-1} x(s)|^2 ds. \tag{4.7}$$

It follows from (4.5) that

$$|D_{a^+}^{\alpha-1} x| = \left| - \int_{\xi}^t q(s)x(s)ds \right| \leq \int_a^b |q(s)||x(s)|ds.$$

Note that $|x(t)| \leq m$ and $|x(t)| \not\equiv m$ for $t \in [a, b]$. Hence

$$|D_{a^+}^{\alpha-1} x| < m \int_a^b |q(s)|ds. \tag{4.8}$$

Using (4.8) in (4.7) we have

$$\frac{4m^2(2\alpha - 3)\Gamma^2(\alpha - 1)}{(b - a)^{2\alpha-3}} \leq m^2(b - a) \left(\int_a^b |q(s)|ds \right)^2.$$

Canceling m from both sides and rearranging terms we see that the conclusion holds immediately. \square

Now, we present a Lyapunov-type inequality for eq. (4.2). The proof is essentially the same as that of theorem 4.1. Therefore, we omit the proof.

THEOREM 4.2. Assume eq. (4.2) has a nontrivial solution $x(t)$ satisfying the BCs

$$x(a) = x(b) = x'(b) = 0 \text{ and } D_b^{\alpha-1}x(\xi) = 0$$

for some $\xi \in [a, b]$. Then

$$\int_a^b |q(s)| ds > \frac{2(2\alpha - 3)^{1/2} \Gamma(\alpha - 1)}{(b - a)^{\alpha - 1}}.$$

REMARK 4.1. Observe in the case when $\alpha = 3$, BVP (4.1), (4.3) becomes

$$x''' + q(t)x = 0, \quad x(a) = x'(a) = x''(\xi) = x(b) = 0$$

for some $\xi \in [a, b]$. In this case, the inequality (4.3) simplifies to

$$\int_a^b |q(s)| ds > \frac{4\sqrt{3}}{(b - a)^2}.$$

A similar comment can be made for theorem 4.2. We omit the details here.

5. Multivariate Lyapunov-type inequalities

In the last section, we show how the Lyapunov-type inequalities in sections 3 and 4 can be extended to fractional multivariate equations. To avoid redundancy, we only give the extension for the left-fractional differential equation. First we introduce some notation.

For $N \geq 2$, we denote

$$S^{N-1} := \{u \in \mathbb{R}^N : |u| = 1\}$$

as the unit sphere in \mathbb{R}^N . It is well known that the surface area of S^{N-1} is

$$\int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)},$$

where Γ stands for the gamma function as given in section 2. Note that every $u \in \mathbb{R}^N \setminus \{0\}$ has a unique representation of the form $u = r\omega$ with $|u| = r$ for some $r > 0$ and $\omega \in S^{N-1}$.

Assume that $a, b, c \in C(S^{N-1}, \mathbb{R})$ and $0 < a(\omega) < c(\omega) < b(\omega)$ for all $\omega \in S^{N-1}$. We define a doubly connected region A in \mathbb{R}^N as

$$A := \{u = r\omega : r \in (a(\omega), b(\omega)), \omega \in S^{N-1}\},$$

together with its subregions

$$A_1 := \{u = r\omega : r \in (a(\omega), c(\omega)), \omega \in S^{N-1}\}$$

and

$$A_2 := \{u = r\omega : r \in (c(\omega), b(\omega)), \omega \in S^{N-1}\}.$$

Clearly, $A = A_1 \cup A_2$. Let the corresponding boundaries be denoted by

$$B_a = \{u = r\omega : r = a(\omega), \omega \in S^{N-1}\},$$

$$B_b = \{u = r\omega : r = b(\omega), \omega \in S^{N-1}\},$$

and

$$B_c = \{u = r\omega : r = c(\omega), \omega \in S^{N-1}\}.$$

Figure 1 gives a graphical interpretation for the region A .

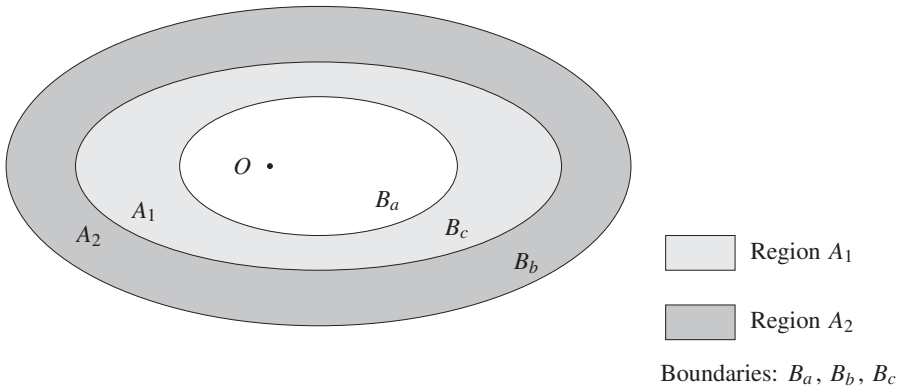


Figure 1: Region A

Let $\omega \in S^{N-1}$ be fixed. For any $\gamma > 0$ and $u = r\omega$ with $r > a(\omega)$, we denote by $(\mathcal{D}_r^\gamma y)(u)$ the γ -th order Riemann-Liouville directional derivative of $y(u)$ in the radial direction at $a(\omega)$, i.e.,

$$\left(\mathcal{D}_r^\gamma y\right)(u) := \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial r^n} \int_{a(\omega)}^r (r-s)^{n-\gamma-1} y(s\omega) ds, \tag{5.1}$$

where $n = \lfloor \gamma \rfloor + 1$ and Γ is the gamma function. Now, on the region A , we consider the equation

$$\left(\mathcal{D}_r^\alpha y\right)(u) + q(u)y = 0, \quad 1 < \alpha \leq 2, \tag{5.2}$$

where $q \in C(\bar{A})$, together with one of the following BC

$$y(B_a) = 0 \quad \text{and} \quad \left(\mathcal{D}_r^{\alpha-1} y\right)(B_c) = 0; \tag{5.3}$$

$$\frac{\partial y(B_a)}{\partial r} = 0 \quad \text{and} \quad y(B_b) = 0; \tag{5.4}$$

or

$$y(B_a) = y(B_b) = 0 \quad \text{and} \quad \frac{\partial y(B_a)}{\partial r} = 0. \tag{5.5}$$

We first present a Lyapunov-type inequality for BVP (5.2), (5.3).

THEOREM 5.1. *Assume eq. (5.2) has a nontrivial solution $y(u)$ satisfying (5.3) on A_1 . Then*

$$\int_{A_1} |u|^{1-N} |q(u)| du > \frac{2\pi^{N/2}\Gamma(\alpha)}{\Gamma(N/2)(c-a)^{\alpha-1}}, \tag{5.6}$$

where $a = \min\{a(\omega) : \omega \in S^{N-1}\}$ and $c = \max\{c(\omega) : \omega \in S^{N-1}\}$.

Proof. For a fixed $\omega \in S^{N-1}$, we denote $z(r) := y(r\omega)$. By comparing (5.1) with (2.1) we see that $(\mathcal{D}_r^\gamma y)(u) = (D_{a(\omega)+z}^\gamma)(r)$ for $\gamma > 0$. Since $y(u)$ is a nontrivial solution of eq. (5.2) satisfying BC (5.3), we have that for $\omega \in S^{N-1}$, $z(r)$ is a nontrivial solution of the equation

$$(D_{a(\omega)+z}^\alpha)(r) + q(r\omega)z = 0, \quad 1 < \alpha \leq 2$$

satisfying the BC

$$z(a(\omega)) = 0 \quad \text{and} \quad (D_{a(\omega)+z}^{\alpha-1})(c(\omega)) = 0.$$

Thus by theorem 3.1,

$$\int_{a(\omega)}^{c(\omega)} |q(r\omega)| dr > \frac{\Gamma(\alpha)}{(c(\omega) - a(\omega))^{\alpha-1}}. \tag{5.7}$$

Recall that $r = |u|$ and for any $\Omega \subset \mathbb{R}^N$ and $f \in C(\Omega)$,

$$\int_{\Omega} f(u) du = \int_{\Omega} r^{N-1} f(r\omega) dr d\omega.$$

Hence

$$\int_{A_1} |u|^{1-N} |q(u)| du = \int_{S^{N-1}} \left(\int_{a(\omega)}^{c(\omega)} |q(r\omega)| dr \right) d\omega.$$

Integrating both sides of (5.7) with respect to ω on S^{N-1} and noting that $0 < a \leq a(\omega) < c(\omega) \leq c$, we obtain

$$\begin{aligned} \int_{A_1} |u|^{1-N} |q(u)| du &= \int_{S^{N-1}} \left(\int_{a(\omega)}^{c(\omega)} |q(r\omega)| dr \right) d\omega > \Gamma(\alpha) \int_{S^{N-1}} \frac{d\omega}{(c(\omega) - a(\omega))^{\alpha-1}} \\ &\geq \frac{\Gamma(\alpha)}{(c-a)^{\alpha-1}} \int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}\Gamma(\alpha)}{\Gamma(N/2)(c-a)^{\alpha-1}}, \end{aligned}$$

i.e., (5.6) holds. \square

Now we present a Lyapunov-type inequality for BVP (5.2), (5.4) and BVP (5.2), (5.5). The proofs are essentially the same as the proof of theorem 5.1. We omit the details here.

THEOREM 5.2. Assume eq. (5.2) has a nontrivial solution $y(u)$ satisfying (5.4) on A . Then

$$\int_A |u|^{1-N} |q(u)| du > \frac{2\pi^{N/2}\Gamma(\alpha)}{\Gamma(N/2)(b-a)^{\alpha-1}},$$

where $a = \min\{a(\omega) : \omega \in S^{N-1}\}$ and $b = \max\{b(\omega) : \omega \in S^{N-1}\}$.

THEOREM 5.3. Assume eq. (5.2) has a nontrivial solution $y(u)$ satisfying (5.5) on A . Then

$$\int_A |u|^{1-N} |q(u)| du > \frac{4\pi^{N/2}\Gamma(\alpha)}{\Gamma(N/2)(b-a)^{\alpha-1}},$$

where $a = \min\{a(\omega) : \omega \in S^{N-1}\}$ and $b = \max\{b(\omega) : \omega \in S^{N-1}\}$.

REMARK 5.1. Here we discuss some special cases of theorem 5.1. Similar arguments will follow for theorems 5.2 and 5.3. We leave the details to the interested reader. Let $\alpha = 2$. Then BVP (5.2), (5.3) becomes

$$\frac{\partial^2 y(u)}{\partial r^2} + q(u)y(u) = 0, \quad y(B_a) = \frac{\partial y(B_c)}{\partial r} = 0$$

and the inequality (5.6) becomes

$$\int_{A_1} |u|^{1-N} |q(u)| du > \frac{2\pi^{N/2}}{\Gamma(N/2)(c-a)}.$$

Furthermore, letting $a(\omega) = 0$ for $\omega \in S^{N-1}$, i.e., the hole shrinks to a point, we have the nontrivial inequality

$$\int_{A_1} |u|^{1-N} |q(u)| du > \frac{2\pi^{N/2}}{c\Gamma(N/2)}.$$

Since $|u| = r \geq a > 0$, it follows that

$$\int_{A_1} |q(u)| du > \frac{2\pi^{N/2}a^{N-1}}{c\Gamma(N/2)}.$$

Finally, on the region A , we consider the equation

$$\left(\mathcal{D}_r^\alpha y\right)(u) + q(u)y = 0, \quad 2 < \alpha \leq 3, \tag{5.8}$$

where $q \in C(\bar{A})$, together with the following BC

$$y(B_a) = \frac{\partial y(B_a)}{\partial r} = y(B_a) = 0 \quad \text{and} \quad \left(\mathcal{D}_r^{\alpha-1} y\right)(B_\xi) = 0, \tag{5.9}$$

where $a, b, \xi \in C(S^{N-1}, \mathbb{R})$ and $0 < a(\omega) \leq \xi(\omega) \leq b(\omega)$ for all $\omega \in S^{N-1}$. Additionally, let

$$B_\xi = \{u = r\omega : r = \xi(\omega), \omega \in S^{N-1}\}$$

and the region A and the boundaries B_a and B_b be defined as before. We now present a Lyapunov-type inequality for BVP (5.8), (5.9).

THEOREM 5.4. Assume eq. (5.8) has a nontrivial solution $y(u)$ satisfying (5.9) on A . Then

$$\int_A |u|^{1-N} |q(u)| du > \frac{4\pi^{N/2} (2\alpha - 3)^{1/2} \Gamma(\alpha - 1)}{\Gamma(N/2) (b - a)^{\alpha - 1}},$$

where $a = \min\{a(\omega) : \omega \in S^{N-1}\}$ and $b = \max\{b(\omega) : \omega \in S^{N-1}\}$.

The proof is essentially the same as the proof of theorem 5.1. We leave the details to the interested reader.

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