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EIGENVALUE CRITERIA FOR EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR $\alpha-$ORDER FRACTIONAL DIFFERENTIAL EQUATIONS, $(2<\alpha<3)$, ON THE HALF–LINE

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Abstract. This article concerns nonexistence and existence of positive solutions to the fractional differential equation

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, & 0\leqslant t<\infty, \\ u(0)=D^{\alpha-2}u(0)=\lim_{t\to\infty}D^{\alpha-1}u(t)=0, \end{cases}$$

where $\alpha\in(2,3)$, D^{α} is the standard Riemann-Liouville derivative and $f:\mathbb{R}^{+}\times\mathbb{R}^{+}\to\mathbb{R}^{+}$ is a continuous function. The main results obtained here, are under eigenvalue criteria.

1. Introduction and main results

Because that fractional differential equations are considered as alternative models to nonlinear differential equations, study of existence of positive solutions to boundary value problems associated with fractional differential equations has become a very important area of applied mathematics over the last few decades. Such a subject has been discussed in many recent papers; see, for example [7], [8], [10], [11], [12], [13], [14], [17], [18], [19] and references therein.

We are concerned in this paper with nonexistence and existence of positive solutions to the fractional boundary value problem (fbvp for short),

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, & 0\leqslant t<\infty, \\ u(0)=D^{\alpha-2}u(0)=\lim_{t\to\infty}D^{\alpha-1}u(t)=0, \end{cases} \tag{1.1}$$

where $\alpha\in(2,3)$, D^{α} is the standard Riemann-Liouville derivative and $f:\mathbb{R}^{+}\times\mathbb{R}^{+}\to\mathbb{R}^{+}$ is a continuous function.

Motivated by the works in [1], [4], [5], [6] and [20], we want to establish nonexistence and existence results to the fbvp (1.1) under eigenvalue criteria.

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Set

$$\mathcal{Q}_\alpha = \left\{ q \in C\left(\mathbb{R}^+, \mathbb{R}\right) : q(s) > 0 \text{ a.e. } s > 0 \text{ and } \int_0^{+\infty} q(s)\,(1+s)^{\alpha-1}\,ds < \infty \right\}.$$

A continuous function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is said to be $\mathcal{Q}_\alpha$-Caratheodory if for all $r > 0$ there is $\psi_r \in \mathcal{Q}_\alpha$ such that

$$\left| g\left(t, (1+t)^{\alpha-1} u\right) \right| \leqslant \psi_r(t)\,(1+t)^{\alpha-1}, \text{ for all } t \in \mathbb{R}^+ \text{ and } u \in [-r, r].$$

A continuous function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that

$$\left| (1+t)^{1-\alpha} g(t,u) \right| \leqslant a(t) + b(t)\,|u|^\rho, \text{ for all } t, u \in \mathbb{R}^+,$$

where $\rho \in (0, +\infty)$ and $a, \widetilde{b} \in \mathcal{Q}_\alpha \in C(\mathbb{R}^+)$ with $\widetilde{b}(s) = (1+s)^{(\alpha-1)(\rho-1)} b(s)$, is a typical $\mathcal{Q}_\alpha$-Caratheodory function.

Consider for q in $\mathcal{Q}_\alpha$ the linear fractional boundary value problem

$$\begin{cases} D^\alpha u(t) + \mu q(t) u(t) = 0, \text{ a.e. } t \in (0, +\infty), \\ u(0) = D^{\alpha-2} u(0) = 0, \ \lim_{t\to\infty} D^{\alpha-1} u(t) = 0, \end{cases} \tag{1.2}$$

where μ is a real parameter.

PROPOSITION 1. *For all functions q in $\mathcal{Q}_\alpha$ the fbvp (1.2) admits a unique positive eigenvalue $\mu_\alpha(q)$.*

PROPOSITION 2. *Assume that the nonlinearity f is a $\mathcal{Q}_\alpha$-Caratheodory function and there exists $q \in \mathcal{Q}_\alpha$ such that one of the following Hypotheses (1.3) and (1.4)*

$$\mu_\alpha(q) < 1 \text{ and } f(t,u) \geqslant q(t)\,u, \text{ for all } t, u \geqslant 0, \tag{1.3}$$

$$\mu_\alpha(q) > 1 \text{ and } f(t,u) \leqslant q(t)\,u, \text{ for all } t, u \geqslant 0 \tag{1.4}$$

holds true. Then the fbvp (1.1) has no positive solutions.

The existence result for positive solutions to the fbvp (1.1) needs to introduce the following additional notations. Set for a $\mathcal{Q}_\alpha$-Caratheodory function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $q \in \mathcal{Q}_\alpha$ and $\nu = 0, +\infty$

$$g_\nu^+(q) = \limsup_{u\to\nu} \left(\max_{t\geqslant 0} \frac{g(t, (1+t)^{\alpha-1} u)}{(1+t)^{\alpha-1} q(t) u} \right),$$

$$g_\nu^-(q) = \liminf_{u\to\nu} \left(\min_{t\geqslant 0} \frac{g(t, (1+t)^{\alpha-1} u)}{(1+t)^{\alpha-1} q(t) u} \right).$$

THEOREM 1. *Assume that the nonlinearity f is $\mathcal{Q}_\alpha$-Caratheodory function and there exist two functions q_0, q_∞ in $\mathcal{Q}_\alpha$ such that one of the following hypotheses (1.5) or (1.6) holds true:*

$$\frac{f^+_{+\infty}(q_\infty)}{\mu_\alpha(q_\infty)} < 1 < \frac{f^-_0(q_0)}{\mu_\alpha(q_0)} \tag{1.5}$$

or

$$\frac{f^+_0(q_0)}{\mu_\alpha(q_0)} < 1 < \frac{f^-_{+\infty}(q_0)}{\mu_\alpha(q_\infty)}. \tag{1.6}$$

Then the fbvp (1.1) admits a positive solution.

Consider now the particular case of the fbvp (1.1), where the nonlinearity f takes the form:

$$f(t,u) = m(t)(1+t)^{\alpha-1} h\left(t, \frac{u}{(1+t)^{\alpha-1}}\right).$$

Namely, we consider the fbvp

$$\begin{cases} D^\alpha u(t) + m(t)(1+t)^{\alpha-1} h\left(t, \frac{u(t)}{(1+t)^{\alpha-1}}\right) = 0, \ t > 0, \\ u(0) = D^{\alpha-2}u(0) = 0, \ \lim_{t\to\infty} D^{\alpha-1}u(t) = 0, \end{cases} \tag{1.7}$$

where $m \in \mathcal{Q}_\alpha$ and $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$\begin{cases} \text{for all } r > 0 \text{ there is } M_r > 0 \text{ such that} \\ h(t,u) \leqslant M_r, \text{ for all } t \geqslant 0 \text{ and } u \in [0,r]. \end{cases} \tag{1.8}$$

Set $\nu = 0, +\infty$

$$h^+_\nu = \limsup_{u\to\nu}\left(\max_{t\geqslant 0} \frac{h(t,u)}{u}\right),$$
$$h^-_\nu = \liminf_{u\to\nu}\left(\min_{t\geqslant 0} \frac{h(t,u)}{u}\right).$$

We obtain from Proposition 2 and Theorem 1 the following corollaries.

COROLLARY 1. *Assume that that hypothesis (1.8) holds true. If either*

$$h(t,u) < \eta u \text{ with } 0 < \eta < \mu_\alpha(m)$$

or

$$h(t,u) > \eta u \text{ with } \eta > \mu_\alpha(m),$$

then the fbvp (1.7) admits no positive solution.

COROLLARY 2. *Assume that hypothesis (1.8) holds true. If either*

$$h^+_{+\infty} < \mu_\alpha(m) < h^-_0$$

or

$$h^+_0 < \mu_\alpha(m) < h^-_{+\infty},$$

then the fbvp (1.7) admits a positive solution.

At the end of this section, we consider the particular case of the fbvp (1.1), where $f(t,u)=p(t)\,(1+t)^{\alpha-1}\,u^{\rho}$. Namely, we consider the case of the fbvp

$$\begin{cases} D^{\alpha}u(t)+p(t)u^{\rho}=0,\ t>0,\\ u(0)=D^{\alpha-2}u(0)=0,\ \lim_{t\to\infty}D^{\alpha-1}u(t)=0, \end{cases} \tag{1.9}$$

where $\rho>0$ and $p\in C(\mathbb{R}^{+},\mathbb{R}^{+})$.

We obtain from Corollary 2 the following existence result.

COROLLARY 3. *Assume that $m\in\mathcal{Q}_{\alpha}$ where $m(t)=p(t)\,(1+t)^{(\alpha-1)(\rho-1)}$. Then for all $\rho\in(0,1)\cup(1,+\infty)$ the fbvp (1.9) admits a positive solution.*

Proof. We have

$$p(t)u^{\rho}=p(t)\,(1+t)^{\rho(\alpha-1)}\left(\frac{u}{(1+t)^{(\alpha-1)}}\right)^{\rho}=m(t)\,(1+t)^{\alpha-1}h\left(\frac{u}{(1+t)^{(\alpha-1)}}\right),$$

where $h(x)=x^{\rho}$ and

$$\begin{cases} h_{+\infty}^{+}=0<\mu_{\alpha}(m)<h_{0}^{-}=+\infty,\ \text{if}\ \rho\in(0,1),\\ h_{0}^{+}=0<\mu_{\alpha}(m)<h_{+\infty}^{-}=+\infty,\ \text{if}\ \rho\in(1,+\infty). \end{cases}$$

Thus, existence of a positive solution for the fbvp (1.9) is obtained from Corollary 2. □

2. Abstract background

Let X be a real Banach space, the standard notations $\mathcal{L}(X)$ and $r(L)$ refer respectively to the set of all linear bounded self-mapping defined on X and the spectral radius of an operator L in $\mathcal{L}(X)$. Let K be a cone in X, that is K is a nonempty closed convex subset of X such that $K\cap(-K)=\{0_E\}$ and $tK\subset K$ for all $t\geqslant 0$. Hereafter, $\preceq$ denotes the partial order induced by the cone K on the Banach space X. We write for all $x,y\in X:\ x\preceq y$ (or $y\succeq x$) if $y-x\in K$ and $x\prec y$ (or $y\succ x$) if $y-x\in K\setminus\{0_E\}$.

DEFINITION 1. Let L be a compact operator in $\mathcal{L}(X)$. L is said to be

i) positive, if $L(K)\subset K$,

ii) strongly positive, if $int(K)\neq\emptyset$ and $L(K\setminus\{0_X\})\subset int(K)$,

iii) lower bounded on the cone K, if

$$\inf\{\|Lu\|:u\in K\cap\partial B(0_E,1)\}>0.$$

In all what follows, $\mathcal{L}_K(X)$ denotes the subset of all positive compact operators in $\mathcal{L}(X)$ and we set for all $L\in\mathcal{L}_K(X)$

$$\Lambda_L=\{\theta\geqslant 0:\exists u\succ 0_X\ \text{such that}\ Lu\succeq\theta u\},$$
$$\Gamma_L=\{\theta\geqslant 0:\exists u\succ 0_X\ \text{such that}\ Lu\preceq\theta u\}.$$

DEFINITION 2. Let L be an operator in $\mathscr{L}_K(X)$ and $\mu > 0$. The operator L is said to have the strongly index-jump property (SIJP for short) at μ if

$$r(L) = \sup \Lambda_L = \inf \Gamma_L.$$

PROPOSITION 3. (Proposition 3.16 in [2]) *Let L be an operator in $\mathscr{L}_K(X)$. If L is strongly positive, then L has the SIJP at $r(L)$.*

THEOREM 2. (Theorem 3.23 in [2]) *Let L be an operator in $\mathscr{L}_K(X)$ and assume that there is an increasing sequence (L_n) of operators in $\mathscr{L}_K(X)$ such that $L_n \to L$ in operator norm and for all integers $n \geqslant 1$, L_n has the SIJP at μ_n. Then L has the SIJP at $\mu = \lim \mu_n = \sup \mu_n$.*

REMARK 1. We have from Proposition 3.14 and Proposition 3.15 in [3] that if $L \in \mathscr{L}_K(X)$ has the SIJP at μ, then μ is the unique positive eigenvalue of L.

REMARK 2. It is easy to see that if $L \in \mathscr{L}_K(X)$ has the SIJP at μ and $L(K) \subset P \subset K$, where P is a cone in E, then $L \in \mathscr{L}_P(X)$ has the SIJP at μ.

In this work, the problem of existence and nonexistence of positive solutions for the fbvp (1.1) will be converted to that of existence and nonexistence of fixed point for a completely continuous mapping defined on a cone of an appropriate functional space. This why we need the following two abstract results. Let $T : K \to K$ be a completely continuous mapping. The following proposition provides under eigenvalue criteria a nonexistence result for fixed point to the mapping T.

PROPOSITION 4. *Assume that there exists $L \in \mathscr{L}_K(X)$ having the SIJP at μ such that one of the following conditions (2.1) and (2.2) holds true,*

$$\mu > 1 \text{ and } Tu \succeq Lu, \text{ for all } u \in K, \tag{2.1}$$

$$\mu < 1 \text{ and } Tu \preceq Lu, \text{ for all } u \in K. \tag{2.2}$$

Then T has no positive fixed point.

Proof. We present the proof in the case of (2.1) holds, the other case is checked similarly. To the contrary, suppose there exists $u \succ 0_X$ such that $Tu = u$. In this case we have that $u = Tu \succeq Lu$, $1 \in \Lambda_L$ and $\mu = \inf \Lambda_L \leqslant 1$.This contradicts $\mu > 1$ in hypothesis (2.1). □

The following theorem is an adapted version of Theorem 3.25 in [2]. It provides under eigenvalue criteria an existence result for fixed point to the mapping T.

THEOREM 3. *Assume that there exist two operators L_1, L_2 in $\mathscr{L}_K(X)$ and two functions $F_1, F_2 : K \to K$ such that*

$$\begin{cases} L_1 \text{ is lower bounded on } K, \\ L_1 \text{ has the SIJP at } r(L_1), \\ 0 < r(L_2) < 1 < r(L_1) \text{ and} \\ L_1 u - F_1 u \preceq Tu \preceq L_2 u + F_2 u, \text{ for all } u \in K. \end{cases}$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to 0 \tag{2.3}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to \infty, \tag{2.4}$$

then T has a positive fixed point.

3. Riemann-Liouville fractional derivative

Now, let us recall some basic facts related to the theory of fractional differential equations. Let β be a positive real number, the Riemann-Liouville fractional integral of order β of a function $f:(0,+\infty)\to\mathbb{R}$ is defined by

$$I_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds,$$

where $\Gamma(\beta)$ is the gamma function, provided that the right side is pointwise defined on $(0,+\infty)$. For example, we have for any real $\sigma > -1$, $I_{0^+}^{\beta} t^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+1)} t^{\sigma+\beta}$.

The Riemann-Liouville fractional derivative of order β, of a continuous function $f:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds,$$

where $n = [\beta]+1$, $[\beta]$ denotes the integer part of the number β, provided that the right side is pointwise defined on $\mathbb{R}^+$.

As a basic example, we quote for $\sigma > -1$, $D_{0^+}^{\beta} t^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\beta+1)} t^{\sigma-\beta}$. Thus, if $u \in C(0,+\infty) \cap \mathbb{L}^1(0,+\infty)$, then the fractional differential equation $D_{0^+}^{\beta} u(t) = 0$ has $u(t) = \sum_{i=1}^{i=[\beta]+1} c_i t^{\beta-i}$, $c_i \in \mathbb{R}$, as unique solution and if u has a fractional derivative of order β in $C(0,+\infty) \cap \mathbb{L}^1(0,+\infty)$, then

$$I_{0^+}^{\beta} D_{0^+}^{\beta} u(t) = u(t) + \sum_{i=1}^{i=[\beta]+1} c_i t^{\beta-i},\ c_i \in \mathbb{R}. \tag{3.1}$$

For a detailed presentation on fractional differential calculs, see [15] or [16].

4. Fixed point formulation

Now, we introduce some spaces and operators needed for the proof of the main results of this paper. Throughout, we let E and F be the linear spaces defined by

$$E = \left\{ u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to\infty} \frac{u(t)}{t^{\alpha-1}} = 0 \in \mathbb{R} \right\},$$

equipped with the norm $\|\cdot\|_E$ where for all $u \in E$, $\|u\|_E = \sup_{t>0} \frac{|u(t)|}{(1+t)^{\alpha-1}}$, E becomes a Banach space.

In all what follows E^+ denotes the cone of nonnegative functions in E and the subset P of E defined by

$$P = \{u \in E : u(t) \geqslant \gamma(t)\,\|u\|_E\,,\ \text{for all } t \geqslant 0\},$$

where

$$\gamma(t) = \min\left(1, t^{\alpha-1}\right),$$

is a cone in E.

Let $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t < \infty, \\ t^{\alpha-1}, & 0 \leqslant t \leqslant s < \infty. \end{cases}$$

LEMMA 1. *The functions G and $\frac{\partial G}{\partial t}$ are continuous and have the following properties:*

$$G(0,s) = 0, \text{ for all } s \geqslant 0, \tag{4.1}$$

$$0 < G(t,s) \leqslant \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \text{ for all } t,s \geqslant 0, \tag{4.2}$$

$$\lim_{t\to 0} \frac{G(t,s)}{t^{\alpha-1}} = \frac{1}{\Gamma(\alpha)},\ \lim_{t\to+\infty} \frac{G(t,s)}{t^{\alpha-1}} = 0, \text{ for all } s \geqslant 0, \tag{4.3}$$

$$G(t,s) \geqslant \gamma(t)\frac{G(\tau,s)}{(1+\tau)^{\alpha-1}}, \text{ for all } t,\tau,s \geqslant 0, \tag{4.4}$$

$$\frac{\partial G}{\partial t}(t,s) > 0, \text{ for all } t,s > 0. \tag{4.5}$$

Proof. Properties (4.1), (4.2), (4.3) and (4.5) are easy to check, let us prove property (4.4). Set for $\eta > 0$ and $s \in (0,\eta)$, $\varphi_\eta(s) = \eta^{\alpha-1} - (\eta - s)^{\alpha-1}$. The function φ_η has the following properties: for all $s \in (0,\eta)$,

$$\varphi_\eta'(s) > 0 \text{ and } \left(\frac{\varphi_\eta(s)}{s^{\alpha-1}}\right)' < 0, \text{ for all } s \in (0,\eta),$$

$$\lim_{s\to 0} \frac{\varphi_\eta(s)}{s^{\alpha-1}} = +\infty \text{ and } \lim_{s\to\eta} \frac{\varphi_\eta(s)}{s^{\alpha-1}} = 1,$$

$$(1+\tau)^{\alpha-1} \geqslant \varphi_\eta(s) \geqslant s^{\alpha-1}, \text{ for all } s \in (0,\eta), \tag{4.6}$$

$$\text{if } \eta < \xi \text{ then } \varphi_\eta(s) < \varphi_\xi(s), \text{ for all } s \in (0,\eta).$$

Therefore, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \varphi_t(s), \text{ if } 0 \leqslant s \leqslant t < \infty, \\ \varphi_t(t), \text{ if } 0 \leqslant t \leqslant s < \infty \end{cases} \tag{4.7}$$

and we obtain from the above properties of the function φ_η that for all $t, \tau, s > 0$,

$$\frac{G(t,s)}{G(\tau,s)} = \begin{cases} \frac{\varphi_t(t)}{\varphi_\tau(s)} \geqslant \frac{t^{\alpha-1}}{\varphi_\tau(\tau)} \geqslant \frac{t^{\alpha-1}}{(1+\tau)^{\alpha-1}} \geqslant \frac{\gamma(t)}{(1+\tau)^{\alpha-1}}, \text{ if } t \leqslant s \leqslant \tau, \\ \frac{\varphi_t(s)}{\varphi_\tau(\tau)} \geqslant \left(\frac{s}{\tau}\right)^{\alpha-1} \geqslant 1 \geqslant \frac{1}{(1+\tau)^{\alpha-1}} \geqslant \frac{\gamma(t)}{(1+\tau)^{\alpha-1}}, \text{ if } \tau \leqslant s \leqslant t, \\ \frac{\varphi_t(t)}{\varphi_\tau(\tau)} \geqslant \frac{t^{\alpha-1}}{(\tau+1)^{\alpha-1}} \geqslant \frac{\gamma(t)}{(1+\tau)^{\alpha-1}}, \text{ if } \tau, t \leqslant s, \\ \frac{\varphi_t(s)}{\varphi_\tau(s)} \geqslant \frac{\varphi_\tau(s)}{\varphi_\tau(s)} = 1 \geqslant \frac{1}{(1+\tau)^{\alpha-1}} \geqslant \frac{\gamma(t)}{(1+\tau)^{\alpha-1}}, \text{ if } s \leqslant \tau \leqslant t, \\ \frac{\varphi_t(s)}{\varphi_\tau(s)} = \frac{\frac{\varphi_t(s)}{s^{\alpha-1}}}{\frac{\varphi_\tau(s)}{s^{\alpha-1}}} \geqslant \frac{1}{\frac{\varphi_\tau(s)}{t^{\alpha-1}}} \geqslant \frac{t^{\alpha-1}}{(1+\tau)^{\alpha-1}} \geqslant \frac{\gamma(t)}{(1+\tau)^{\alpha-1}}, \text{ if } s \leqslant t \leqslant \tau, \end{cases}$$

proving property (4.4) of the function G. □

LEMMA 2. *For all functions h in $C(\mathbb{R}^+,\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R}^+)$, $u(t) = \int_0^{+\infty} G(t,s)h(s)ds$ is the unique solution to the fbvp:*

$$\begin{cases} D^\alpha u(t) + h(t) = 0 \text{ in } (0,+\infty), \\ u(0) = D^{\alpha-2}u(0) = \lim_{t\to+\infty} D^{\alpha-1}u(t) = 0. \end{cases}$$

Proof. Applying the operator I^α, we obtain from (3.1) that u is a solution to $D^\alpha u(t) + h(t) = 0$ if and only if

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds, \tag{4.8}$$

where c_1, c_2 and c_3 are real constants. Consequently, the boundary condition $u(0) = 0$ and (4.8) lead to $c_3 = 0$,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$$

and

$$D^{\alpha-2}u(t) = -\int_0^t (t-s)h(s)ds + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1). \tag{4.9}$$

The boundary condition $D^{\alpha-2}u(0) = 0$ and (4.9) lead to $c_2 = 0$,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_1 t^{\alpha-1}$$

and

$$D^{\alpha-1}u(t) = -\int_0^t h(s)ds + c_1\Gamma(\alpha). \tag{4.10}$$

At the end, the boundary condition $D^{\alpha-1}u(+\infty)=0$ and (4.10) lead to $c_1=\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}h(s)ds$ and

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\infty}h(s)ds=\int_0^{+\infty}G(t,s)h(s)ds. \quad \square$$

In order to prove the compactness of operators, we make use of the following Lemma.

LEMMA 3. *([9])A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) M is bounded in E,

(b) the functions belonging to $\left\{u : u(t)=\dfrac{x(t)}{(1+t)^{\alpha-1}},\ x\in M\right\}$ are locally equicontinuous on $[0,+\infty)$, that is, equicontinuous on every compact interval of $\mathbb{R}^+$ and

(c) the functions belonging to $\left\{u : u(t)=\dfrac{x(t)}{(1+t)^{\alpha-1}},\ x\in M\right\}$ are equiconvergent at $+\infty$, that is, given $\varepsilon>0$, there corresponds $T(\varepsilon)>0$ such that $|x(t)-x(+\infty)|<\varepsilon$, for any $t\geqslant T(\varepsilon)$ and $x\in M$.

LEMMA 4. *Let $g:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be a $\mathscr{Q}_\alpha$-Caratheodory function. The operator $T_g:E\to E$ with $T_gu(t)=\int_0^{+\infty}G(t,s)g(s,u(s))ds$, is well defined and is completely continuous. Moreover, if $g(t,u)\geqslant 0$, for all $t,u\geqslant 0$, then $T_g(E^+)\subset P$ and $u\in E$ is a fixed point of T_g if and only if u is a solution to*

$$\begin{cases} D^\alpha u(t)+g(t,u(t))=0, & 0<t<\infty,\\ u(0)=D^{\alpha-2}u(0)=\lim_{t\to\infty}D^{\alpha-1}u(t)=0. \end{cases}$$

Proof. The fact that $T_gu\in E$ for all $u\in E$ follows from the following estimates (4.11), (4.15) and (4.16). Let $\Omega\subset\overline{B}(0_E,R)$ be a subset of E and let $\psi_R\in\mathscr{Q}_\alpha$ such that

$$g(t,(1+t)^{\alpha-1}u)\leqslant\psi_R(t)\,(1+t)^{\alpha-1}\text{, for all } t\geqslant 0 \text{ and } u\in[-R,R].$$

We have then for all $u\in\Omega$

$$\begin{aligned}\frac{|Tu(t)|}{(1+t)^{\alpha-1}}&\leqslant\int_0^{+\infty}\frac{G(t,s)}{(1+t)^{\alpha-1}}\left|g\left(s,(1+s)^{\alpha-1}\frac{u(s)}{(1+s)^{\alpha-1}}\right)\right|ds\\ &\leqslant\int_0^{+\infty}\psi_R(s)\,(1+s)^{\alpha-1}ds. \end{aligned}\tag{4.11}$$

This shows that the operator T is bounded on Ω.

Let $[\xi,\eta]$ be an interval of $\mathbb{R}^+$. We have for all $u\in\Omega$ and all $t_1,t_2\in[\xi,\eta]$ with $0<t_2-t_1<1$,

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{(1+t_2)^{\alpha-1}}-\frac{Tu(t_1)}{(1+t_1)^{\alpha-1}}\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\left(\frac{t_2-s}{1+t_2}\right)^{\alpha-1}-\left(\frac{t_1-s}{1+t_1}\right)^{\alpha-1}\right|\psi_R(s)\,(1+s)^{\alpha-1}\,ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\frac{t_2-s}{1+t_2}\right)^{\alpha-1}\psi_R(s)\,(1+s)^{\alpha-1}\,ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\left|\left(\frac{t_2}{1+t_2}\right)^{\alpha-1}-\left(\frac{t_1}{1+t_1}\right)^{\alpha-1}\right|\int_0^{+\infty}\psi_R(s)\,(1+s)^{\alpha-1}\,ds.
\end{aligned}\tag{4.12}$$

We have by the mean value theorem:

$$\begin{aligned}
\left|\left(\frac{t_2-s}{1+t_2}\right)^{\alpha-1}-\left(\frac{t_1-s}{1+t_1}\right)^{\alpha-1}\right| &\leqslant(\alpha-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-2}\left|\frac{t_2-s}{1+t_2}-\frac{t_1-s}{1+t_1}\right| \\
&\leqslant(\alpha-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-2}\frac{(t_2-t_1)\,(1+s)}{(1+t_2)\,(1+t_1)} \\
&\leqslant(\alpha-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-2}(t_2-t_1)
\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}
\left|\left(\tfrac{t_2}{1+t_2}\right)^{\alpha-1}-\left(\tfrac{t_1}{1+t_1}\right)^{\alpha-1}\right| &\leqslant(\alpha-1)\left(\tfrac{\eta}{1+\eta}\right)^{\alpha-2}\left(\tfrac{t_2}{1+t_2}-\tfrac{t_1}{1+t_1}\right) \\
&\leqslant(\alpha-1)\left(\tfrac{\eta}{1+\eta}\right)^{\alpha-2}(t_2-t_1).
\end{aligned}\tag{4.14}$$

Set $|\psi_R|_\alpha=\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\psi_R(s)\,(1+s)^{\alpha-1}\,ds$, inserting (4.13) and (4.14) in (4.12) we obtain:

$$\begin{aligned}
\left|\frac{Tu(t_2)}{(1+t_2)^{\alpha-1}}-\frac{Tu(t_1)}{(1+t_1)^{\alpha-1}}\right| &\leqslant 2\frac{(\alpha-1)}{\Gamma(\alpha)}\left(\frac{\eta}{1+\eta}\right)^{\alpha-2}|\psi_R|_\alpha\,(t_2-t_1)+\left(\frac{t_2-t_1}{1+t_2}\right)^{\alpha-1}|\psi_R|_\alpha \\
&\leqslant\frac{1}{\Gamma(\alpha)}\left(2\,(\alpha-1)\left(\frac{\eta}{1+\eta}\right)^{\alpha-2}+1\right)|\psi_R|_\alpha\,(t_2-t_1)
\end{aligned}\tag{4.15}$$

Proving that $T_g(\Omega)$ is equicontinuous.

We have for any u in Ω and $t\geqslant 0$

$$\left|\frac{Tu(t)}{(1+t)^{\alpha-1}}\right|\leqslant\int_0^{+\infty}\frac{G(t,s)}{(1+t)^{\alpha-1}}\,|g(s,u(s))|\,ds$$

$$\leqslant \int_0^{+\infty} \frac{G(t,s)}{(1+t)^{\alpha-1}} \psi_R(s)(1+s)^{\alpha-1} ds = H(t). \tag{4.16}$$

Since $\psi_R \in \mathscr{Q}_\alpha$, the property (4.3) of the function G and the dominated convergence theorem lead to $\lim_{t\to\infty} H(t) = 0$ and then $T_g(\Omega)$ is equiconvergent.

Now, let us prove that the operator T_g is continuous on Ω. Let u be a function in Ω and $(u_n) \subset \Omega$ is such that $\lim u_n = u$. Because of

$$\begin{aligned}
\left\|T_g u_n - T_g u\right\|_E &\leqslant \sup_{t\geqslant 0} \int_0^{+\infty} \frac{G(t,s)}{(1+t)^{\alpha-1}} |g(s,u_n(s)) - g(s,u(s))| ds \\
&\leqslant \sup_{t\geqslant 0} \frac{t^{\alpha-1}}{(1+t)^{\alpha-1}} \int_0^{+\infty} |g(s,u_n(s)) - g(s,u(s))| ds \\
&\leqslant \int_0^{+\infty} |g(s,u_n(s)) - g(s,u(s))| ds,
\end{aligned}$$

$$|g(s,u_n(s)) - g(s,u(s))| \leqslant 2\psi_R(s),$$

and

$$\lim |g(s,u_n(s)) - g(s,u(s))| = 0, \text{ for all } s \geqslant 0,$$

we have by the Lebesgue dominated convergence theorem $\lim \left\|T_g u_n - T_g u\right\|_E = 0$. Proving the continuity of T_g. Thus, $T_g(\Omega)$ satisfies all conditions of Lemma 3 and the mapping T_g is completely continuous.

At the end, assume that $g(t,u) \geqslant 0$ for all $t,u \geqslant 0$, we obtain from property (4.4) of the function G, that for any $u \in E^+$ and $t,\tau \geqslant 0$

$$T_g u(t) = \int_0^{+\infty} G(t,s) g(s,u(s)) ds \geqslant \int_0^{+\infty} \gamma(t) \frac{G(\tau,s)}{(1+\tau)^{\alpha-1}} g(s,u(s)) ds$$

leading to

$$T_g u(t) \geqslant \gamma(t) \sup_{\tau\geqslant 0} \int_0^{+\infty} \frac{G(\tau,s)}{(1+\tau)^{\alpha-1}} g(s,u(s)) ds \geqslant \gamma(t) \left\|T_g u\right\|.$$

This proves that $T_g(E^+) \subset P$ and in particular $T_g(P) \subset P$. The fact that fixed point of T_g are solutions to the fbvp in Lemma 4 follows from Lemma 2. □

We obtain from Lemma 4 the following formulation of the fbvp (1.1).

COROLLARY 4. *A function $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a positive solution to the fbvp (1.1) if and only if $u = T_f u$, where $T_f : P \to P$ is completely continuous and $T_f u(t) = \int_0^{+\infty} G(t,s) f(s,u(s)) ds$ for all $u \in P$.*

5. Proofs of main results

5.1. Auxilliary result

The main result of this subsection (Theorem 4) and its proof need to introduce additional notations. With a function q in $\mathscr{Q}_\alpha$ and $T > 0$ are associated the linear operators L_q in $\mathscr{L}(E)$, L_q^F, $L_{q,T}^F$ in $\mathscr{L}(F)$ and $K_{q,T}^F$ in $\mathscr{L}(F_T)$ defined by

$$\begin{aligned}
L_q u(t) &= \int_0^{+\infty} G(t,s)q(s)u(s)ds, \text{ for all } u \in E,\\
L_q^F u &= L_q u, \text{ for all } u \in F,\\
L_{q,T}^F u(t) &= \int_0^{T} G_T(t,s)q(s)u(s)ds, \text{ for all } u \in F,\\
K_{q,T}^F u(t) &= \int_0^{T} G_T(t,s)q(s)u(s)ds, \text{ for all } u \in F_T,
\end{aligned}$$

where for $T > 0$ $G_T : \mathbb{R}^+ \times [0,T] \to \mathbb{R}^+$ be such that

$$G_T(t,s) = \begin{cases} G(t,s), \text{ if } t,s \in [0,T], \\ G(T,s), \text{ if } t \geqslant T \end{cases}$$

and F_T, F_T^1 are the Banach spaces defined by

$$\begin{aligned}
F_T &= \left\{ u \in C[0,T] : \lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} = l \in \mathbb{R} \right\}\\
F_T^1 &= \left\{ u \in F_T : \frac{u(t)}{t^{\alpha-1}} \in C^1[0,T] \right\}
\end{aligned}$$

equipped respectively with the norms

$$\begin{aligned}
\|u\|_{F_T} &= \sup_{t\in[0,T]} \frac{|u(t)|}{t^{\alpha-1}}, \text{ for all } u \in F_T,\\
\|u\|_{F_T^1} &= \|u\|_{F_T} + \sup_{t\in[0,T]} \left| \left(\frac{u(t)}{t^{\alpha-1}} \right)' \right|, \text{ for all } u \in F_T^1.
\end{aligned}$$

Set for $T > 0$

$$S_T = \left\{ u \in F_T : u(t) > 0, \text{ for all } t \in (0,T] \text{ and } \lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} > 0 \right\}.$$

LEMMA 5. *The set S_T is open in the Banach space F_T and $S \subset F_T^+$.*

Proof. We have $F_T \setminus S_T = F_1 \cup F_2$, where

$$\begin{aligned}
F_1 &= \{u \in F_T : u(t_*) < 0 \text{ for some } t_* \in (0,T]\}\\
F_2 &= \left\{ u \in F_T : \lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} \leqslant 0 \right\}.
\end{aligned}$$

It is clear that F_2 is a closed set in F_T, so let $(u_n) \subset F_1$ be a sequence converging to u in F_T and $(t_n) \subset (0,T]$ with $u_n(t_n) \leqslant 0$ and $\lim t_n = \overline{t} \in [0,T]$. We distinguish then two cases:

Case 1. $\bar{t} \in (0,T]$. In this case we have $u(\bar{t}) = \lim_{n\to+\infty} u_n(t_n) \leqslant 0$ and $u \in F_1$.

Case 2. $\bar{t} = 0$. In this case we have $u \in F_2$. By the contrary suppose that $u \notin F_2$ and $\lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} = l > 0$. Thus, there is $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geqslant n_0$

$$-\frac{l}{4} < \frac{u_n(t)}{t^{\alpha-1}} - \frac{u(t)}{t^{\alpha-1}} < \frac{l}{4}, \text{ for all } t \in (0,T]$$

and

$$\frac{3l}{4} < \frac{u(t)}{t^{\alpha-1}} < \frac{5l}{4}, \text{ for all } t \in (0,\delta].$$

Let n_1 be such that $t_n \in (0,\delta]$, for all $n \geqslant n_1$. We have then for all $n \geqslant \max(n_0,n_1)$ the contradiction

$$0 \geqslant u_n(t_n) = t_n^{\alpha-1}\frac{u_n(t_n)}{t_n^{\alpha-1}} = t_n^{\alpha-1}\left(\left(\frac{u_n(t_n)}{t_n^{\alpha-1}} - \frac{u(t_n)}{t_n^{\alpha-1}}\right) + \left(\frac{u(t_n)}{t_n^{\alpha-1}}\right)\right) \geqslant t_n^{\alpha-1}\left(-\frac{l}{4} + \frac{3l}{4}\right)$$
$$= t_n^{\alpha-1}\frac{l}{2} > 0.$$

This ends the proof. □

LEMMA 6. *For all functions q in $\mathcal{Q}_\alpha$ and all $T > 0$, the operator $L_{q,T}^F$ has the SIJP at $r(L_{q,T}^F)$.*

Proof. Observe that $L_{q,T}^F$ has the SIJP at $r(L_{q,T}^F)$ if and only if $K_{q,T}^F$ has the SIJP at $r(K_{q,T}^F) = r(L_{q,T}^F)$. To this aim, we will prove that the operator $K_{q,T}^F$ is strongly positive and we conclude then by Proposition 4 that it has the SIJP at $r(K_{q,T}^F)$.

Let us prove first that $K_{q,T}^F$ is compact. We have for all $u \in F_T$

$$\left|\left(\frac{K_{q,T}^F u(t)}{t^{\alpha-1}}\right)'\right| = \frac{\alpha-1}{\Gamma(\alpha)}\left|\int_0^t \frac{s}{t^2}\left(1-\frac{s}{t}\right)^{\alpha-2} q(s)u(s)ds\right|$$
$$\leqslant \frac{\alpha-1}{\Gamma(\alpha)}\left(\int_0^t \frac{s}{t^2}\left(1-\frac{s}{t}\right)^{\alpha-2} s^{\alpha-1}ds\right)\|q\|_\infty \|u\|_{F_T}.$$

Leading to $\lim_{t\to 0}\left(\frac{K_{q,T}^F u(t)}{t^{\alpha-1}}\right)' = 0$ and $L_{q,T}^F u \in F_T^1$. Thus, the operator $\widetilde{K}_{q,T}^F : F_T \to F_T^1$, where $\widetilde{K}_{q,T}^F u(t) = K_{q,T}^F u(t)$, for all $u \in F_T$ and all $t \in [0,T]$, is well defined and $K_{q,T}^F \in \mathscr{L}\left(F_T, F_T^1\right)$. Since the embeding j of F_T^1 in F_T is compact and $K_{q,T}^F = j \circ \widetilde{K}_{q,T}^F$, we have that $K_{q,T}^F$ is compact.

Now, since for all $u \in F_T^+$ with $u \neq 0$

$$K_{q,T}^F u(t) = \int_0^T G_T(t,s)q(s)u(s)ds > 0, \text{ for all } t \in (0,T]$$

and (by dominated convergence theorem)

$$\lim_{t\to 0}\frac{K^F_{q,T}u(t)}{t^{\alpha-1}}=\lim_{t\to 0}\int_0^T\frac{G_T(t,s)}{t^{\alpha-1}}q(s)u(s)ds=\int_0^T q(s)u(s)ds>0,$$

we have $K^F_{q,T}\left(F^+_T\setminus\{0\}\right)\subset S_T\subset F^+_T$ and the operator $K^F_{q,T}$ is strongly positive. This ends the proof. □

THEOREM 4. *For all functions q in $\mathcal{Q}_\alpha$ the operator L_q has the SIJP at $r(L_q)$ and is lower bounded on the cone P.*

Proof. First let us prove that L^F_q has the SIJP at $r\left(L^F_q\right)$. This will be obtained from Theorem 2 whence we prove that $L^F_q=\lim_{T\to\infty}L^F_{q,T}$ in operator norm and $T\to L^F_{q,T}$ is increasing. We have for all $u\in F$ with $\|u\|_F=1$,

$$\begin{aligned}\left|\frac{L^F_q u(t)-L^F_{q,T}u(t)}{t^{\alpha-1}}\right| &= \left|\int_0^{+\infty}\frac{G(t,s)}{t^{\alpha-1}}q(s)u(s)ds-\int_0^T\frac{G_T(t,s)}{t^{\alpha-1}}q(s)u(s)ds\right|\\ &\leqslant \left|\int_0^T\frac{G(t,s)-G_T(t,s)}{t^{\alpha-1}}q(s)u(s)ds\right|+\left|\int_T^{+\infty}\frac{G(t,s)}{t^{\alpha-1}}q(s)u(s)ds\right|\\ &\leqslant \left|\int_0^T\frac{G(t,s)-G_T(t,s)}{t^{\alpha-1}}q(s)s^{\alpha-1}ds\right|+\int_T^{+\infty}\frac{G(t,s)}{t^{\alpha-1}}q(s)s^{\alpha-1}ds\\ &\leqslant \int_T^{+\infty}q(s)s^{\alpha-1}ds+\left|\int_0^T\frac{G(t,s)-G_T(t,s)}{t^{\alpha-1}}q(s)s^{\alpha-1}ds\right|.\end{aligned}$$

Since $G_T(t,s)=G(t,s)$, for $t,s\leqslant T$, we have

$$\left|\frac{L^F_q u(t)-L^F_{q,T}u(t)}{t^{\alpha-1}}\right|\leqslant\int_T^{+\infty}q(s)s^{\alpha-1}ds,\ \text{for all } t\leqslant T$$

and since $\frac{\partial}{\partial t}\left(\frac{G(t,s)}{t^{\alpha-1}}\right)<0$, for $s\in(0,t)$, we have in the case of $t\geqslant T$,

$$\begin{aligned}\left|\frac{L^F_q u(t)-L^F_{q,T}u(t)}{t^{\alpha-1}}\right| &\leqslant \int_T^{+\infty}q(s)s^{\alpha-1}ds+\int_0^T\frac{G(t,s)}{t^{\alpha-1}}q(s)s^{\alpha-1}ds+\int_0^T\frac{G_T(t,s)}{t^{\alpha-1}}q(s)s^{\alpha-1}ds\\ &\leqslant \int_T^{+\infty}q(s)s^{\alpha-1}ds+\int_0^T\frac{G(t,s)}{t^{\alpha-1}}q(s)s^{\alpha-1}ds+\int_0^T\frac{G(T,s)}{T^{\alpha-1}}q(s)s^{\alpha-1}ds\\ &\leqslant \int_T^{+\infty}q(s)s^{\alpha-1}ds+2\int_0^T\frac{G(T,s)}{T^{\alpha-1}}q(s)s^{\alpha-1}ds.\end{aligned}$$

The above estimates lead to

$$\begin{aligned}\left\|L^F_q-L^F_{q,T}\right\| &= \sup_{\|u\|_F=1}\left(\sup_{t>0}\left|\frac{L^F_q u(t)-L^F_{q,t}u(t)}{t^{\alpha-1}}\right|\right)\\ &\leqslant \int_T^{+\infty}q(s)s^{\alpha-1}ds+2\int_0^T\frac{G(T,s)}{T^{\alpha-1}}q(s)s^{\alpha-1}ds\to 0,\ \text{as } T\to\infty.\end{aligned}$$

Then by means of the dominated convergence theorem, we conclude that $\lim_{T\to+\infty}\left\|L_q^F - L_{q,T}^F\right\| = 0$ and $L_{q,T}^F$ converge to L_q^F in operator norm.

For $T_1 < T_2$ and $u \in F^+$, we have

$$L_{q,T_2}^F u(t) - L_{q,T_1}^F u(t) = \int_0^{T_2} G_{T_2}(t,s)q(s)u(s)ds - \int_0^{T_1} G_{T_1}(t,s)q(s)u(s)ds$$
$$= \int_0^{T_1} \left(G_{T_2}(t,s) - G_{T_1}(t,s)\right) q(s)u(s)ds + \int_{T_1}^{T_2} G_{T_2}(t,s)q(s)u(s)ds.$$

Because of

$$G_{T_2}(t,s) - G_{T_1}(t,s) = \begin{cases} 0, \text{ if } t \leqslant T_1, \\ \varphi_t(s) - \varphi_{T_1}(s), \text{ if } T_1 \leqslant t \leqslant T_2, \\ \varphi_{T_2}(s) - \varphi_{T_1}(s), \text{ if } T_2 \leqslant t \end{cases} \geqslant 0$$

we have $L_{q,T_2}^F \geqslant L_{q,T_1}^F$.

At this stage, we are able to prove that L_q has the SIJP at $r(L_q)$. We have $\Lambda_{L_q^F} \subset \Lambda_{L_q}$ and $\Gamma_{L_q^F} \subset \Gamma_{L_q}$. So, let us prove that $\Lambda_{L_q^F} = \Lambda_{L_q}$ and $\Gamma_{L_q^F} = \Gamma_{L_q}$. To this aim, let $\lambda \geqslant 0$ and $u \in E^+ \setminus \{0\}$ be such that $L_q u \succeq \lambda u$. We have $U = L_q u \in F^+ \setminus \{0\}$, $L_q^F U = L_q L_q u \succeq \lambda L_q u = \lambda U$ and $\lambda \in \Lambda_{L_q^F}$. This proves that $\Lambda_{L_q^F} = \Lambda_{L_q}$. In similar way, we also obtain that $\Gamma_{L_q^F} = \Gamma_{L_q}$. Thus, we have that

$$r(L_q^F) = \sup\left(\Lambda_{L_q^F}\right) = \sup\left(\Lambda_{L_q}\right) = \inf\left(\Gamma_{L_q^F}\right) = \inf\left(\Gamma_{L_q}\right)$$

and the operator L_q has the SIJP at $r\left(L_q^F\right)$. Furthermore, since the cone E^+ is total in E and Remark 1 claims that $r\left(L_q^F\right)$ is the unique positive eigenvalue of L_q, we have $r(L_q) = r\left(L_q^F\right)$ and L_q has the SIJP at $r(L_q)$.

It remains to show that the operator L_q is lower bounded on the cone P. We have from property (4.4) of the function G

$$L_q u(t) = \int_0^{+\infty} G(t,s)q(s)u(s)ds \geqslant \left(\int_0^{+\infty} G(t,s)q(s)\gamma(s)(1+s)^{\alpha-1}ds\right)\|u\|_E$$

leading to

$$\|L_q u\| = \sup_{t\geqslant 0}\frac{L_q u(t)}{(1+t)^{\alpha-1}} \geqslant \sup_{t\geqslant 0}\left(\int_0^{+\infty}\frac{G(t,s)}{(1+t)^{\alpha-1}}q(s)\gamma(s)(1+s)^{\alpha-1}ds\right)\|u\|_E.$$

This completes the proof. □

5.2. Proof of Propostion 1

We have from Lemma 4 that μ is a positive eigenvalue of the linear eigenvalue problem (1.2) if and only if μ^{-1} is a positive eigenvalue of the compact operator L_q. Since Theorem 4 claims that L_q has the SIJP at $r(L_q)$, we have from Remark 1 that $r(L_q)$ is the unique positive eigenvalue of L_q. Therefore, we have that $\mu = 1/r(L_q)$ is the unique positive eigenvalue of the linear eigenvalue problem (1.2).

5.3. Proof of Proposition 2

Assume that hypothesis (1.3) holds true (the case where (1.4) holds is checked similarly). We have then from Theorem 4 the operator L_q has the SIJP at $r(L_q)$,

$$r(L_q) = \frac{1}{\mu_\alpha(q)} > 1$$

and for all $u \in P$

$$T_f u(t) = \int_0^{+\infty} G(t,s) f(s,u(s)) ds \geqslant \int_0^{+\infty} G(t,s) q(s) u(s) ds = L_q u(t).$$

Thus, hypothesis (2.1) holds and Proposition 4 claims that the operator T_f has no fixed point. At end, we conclude by Corollary 4 that the fbvp (1.1) has no positive solution.

5.4. Proof of Theorem 1

Assume that hypothesis (1.5) holds true (the case where (1.6) holds is checked similarly). Then we obtain from $f^+_{+\infty}(q_\infty) < \mu_1(q_\infty)$ that for $\varepsilon \in \left(0, \mu_\alpha(q_\infty) - f^+_{+\infty}(q_\infty)\right)$ there is R large such that $f(t,u) \leqslant (\mu_\alpha(q_\infty) - \varepsilon) q_\infty(t) (1+t)^{\alpha-1} u$, for all $t \geqslant 0$ and $u \geqslant R$. Since the nonlinearity f is $\mathcal{Q}_\alpha$-Caratheodory, there is $\psi_R \in \mathcal{Q}_\alpha$ such that

$$f(t,u) \leqslant (\mu_\alpha(q_\infty) - \varepsilon) q_\infty(t) (1+t)^{\alpha-1} u + \psi_R(t) (1+t)^{\alpha-1}, \text{ for all } t,u \geqslant 0. \quad (5.1)$$

Also, we have from $f_0^-(q_0) > \mu_\alpha(q_0)$ that for $\varepsilon \in \left(0, f_0^-(q_0) - \mu_\alpha(q_\infty)\right)$ there is $r > 0$ such that $f(t,u) \geqslant (\mu_\alpha(q_\infty) + \varepsilon) q_0(t) (1+t)^{\alpha-1} u$, for all $t \geqslant 0$ and $u \in [0,r]$. Thus we have

$$f(t,u) \geqslant (\mu_\alpha(q_0) + \varepsilon) q_0(t) (1+t)^{\alpha-1} u - \widetilde{f}(t,u), \text{ for all } t,u \geqslant 0, \quad (5.2)$$

where $\widetilde{f}(t,u) = \sup\left(0, (\mu_\alpha(q_\infty) + \varepsilon) q_0(t) (1+t)^{\alpha-1} u - f(t,u)\right)$. Therefore, we obtain from (5.1) and (5.2) that

$$L_{q_0} u - F_0 u \leqslant T_f u \leqslant L_{q_\infty} u + F_\infty u, \text{ for all } u \in P,$$

where

$$F_0 u(t) = \int_0^{+\infty} G(t,s) \widetilde{f}(t,u(s)) ds,$$
$$F_\infty u(t) = \int_0^{+\infty} G(t,s) \psi_R(s) (1+s)^{\alpha-1} ds,$$
$$r(L_{q_0}) = \frac{(\mu_\alpha(q_0) + \varepsilon)}{\mu_\alpha(q_0)} > 1 > \frac{(\mu_\alpha(q_\infty) - \varepsilon)}{\mu_\alpha(q_\infty)} = r(L_{q_\infty}).$$

We conclude from Theorem 4, Theorem 3 and Corollary 4 that the fbvp (1.1) admits a positive solution.

REMARK 3. We have from Property (4.5) of the function G that if u is a solution to the fbvp (1.1) then u is increasing and $\lim_{t\to+\infty} u(t)$ exists. Thus, it is natural to ask about the boundedness of the solution u. A sufficient condition on the nonlinearity f to have an unbounded solution is

$$\lim_{t\to+\infty}\int_1^t s^{\alpha-1} f(s,\theta)ds = +\infty \text{ uniformely for } \theta \text{ in compact intervals of } (0,+\infty). \tag{5.3}$$

Indeed, if u is a solution of the fbvp (1.1), then we obtain from (4.6) and (4.7) that for all $t \geqslant 1$,

$$\begin{aligned} u(t) &= \int_0^{+\infty} G(t,s)f(s,u(s))ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t G(t,s)f(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_t^{+\infty} G(t,s)f(s,u(s))ds \\ &\geqslant \frac{1}{\Gamma(\alpha)}\int_0^t G(t,s)f(s,u(s))ds \geqslant \frac{1}{\Gamma(\alpha)}\int_1^t s^{\alpha-1}f(s,u(s))ds. \end{aligned}$$

Thus, if $\lim_{t\to+\infty} u(t) = l \in \mathbb{R}^+$ we have then the contradiction

$$l = \lim_{t\to+\infty} u(t) \geqslant \frac{1}{\Gamma(\alpha)} \lim_{t\to+\infty}\int_1^t s^{\alpha-1} f(s,u(s))ds = +\infty.$$

In the particular case where $f(t,u) = m(t)u^{\rho}$, where $\rho > 0$ and $m \in C(\mathbb{R}^+,\mathbb{R}^+)$, the condition (5.3) is equivalent to

$$\int_0^{+\infty} s^{\alpha-1} m(s)ds = +\infty.$$

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