Differential
Equations
& Applications
Volume 11, Number 4 (2019), 481–494
doi:10.7153/dea-2019-11-23

FINAL STATE PROBLEM FOR THE NONLOCAL NONLINEAR SCHRÖDINGER EQUATION WITH DISSIPATIVE NONLINEARITY

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(*Communicated by P. I. Naumkin*)

Abstract. We consider the asymptotic behavior of solutions to the nonlocal nonlinear Schrödinger equation with dissipative nonlinearity. We prove that there exists a solution which has different behavior from that of the typical cubic nonlinear Schrödinger equation.

1. Introduction

We consider the nonlocal nonlinear Schrödinger (NLS) equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda u(x)^2 \overline{u(-x)}, \tag{1.1}$$

where $\lambda \in \mathbb{C}$. Ablowitz and Musslimani [1] proved that (1.1) with $\lambda \in \mathbb{R}$ is a complete integrable model, and hence has infinitely many conserved quantities. For example, if $\lambda \in \mathbb{R}$ and u is a solution to (1.1), then

$$\int_{\mathbb{R}} u(t,x)\overline{u(t,-x)}dx, \quad \int_{\mathbb{R}} \left\{ \partial_x u(t,x)\overline{\partial_x u(t,-x)} - \lambda u(t,x)^2 \overline{u(t,-x)}^2 \right\} dx$$

are independent of t. They found solutions to (1.1) which are different to that of the typical cubic NLS equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u. \tag{1.2}$$

Our aim in this paper is to investigate the asymptotic behavior of solutions to (1.1) which differs from that of (1.2) when $\operatorname{Im}\lambda \neq 0$.

Since L^2-norm of solutions to (1.2) with $\lambda \in \mathbb{R}$ is conserved, the global-in-time well-posedness for (1.2) follows form the local well-posedness in [19]. However, while the same argument as in [19] implies the local well-posedness for (1.7), the conserved quantities of (1.1) do not work well to extend the local solution even if $\lambda \in \mathbb{R}$.

Mathematics subject classification (2010): 35Q55.
Keywords and phrases: Nonlinear Schrödinger equation, asymptotic behavior, critical nonlinearity, dissipative nonlinearity.
This work was supported by JSPS KAKENHI Grant numbers JP16K17624 and JP19K14578.
The authors would like to thank the anonymous referee for his/her useful comments.
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The long-time behavior of (1.2) has been studied by several researchers (see [15, 5, 2, 7, 16, 6, 9, 12, 11] and references therein). It is known that the solution to NLS equation scatters when the degree of nonlinearity is bigger than three. In addition, there is no nontrivial solution to (1.2) is asymptotically free. In this sense, the cubic nonlinearity of the NLS equation in the one dimension is critical. Ozawa [15] constructed the modified wave operator of (1.2) from a ball in $H^{0,2}(\mathbb{R})$ to $L^2(\mathbb{R})$ when $\lambda \in \mathbb{R}$, where $H^{s,m}(\mathbb{R})$ denotes the weighted Sobolev space equipped with the norm

$$\|f\|_{H^{s,m}} := \left\| \langle x \rangle^m \left(1-\partial_x^2\right)^{\frac{s}{2}} f \right\|_{L^2},$$

for $s,m \in \mathbb{R}$. More precisely, he proved that for $\lambda \in \mathbb{R}$ and given small $u_+ \in H^{0,2}(\mathbb{R})$, there exist $T>0$ and a unique solution $u \in C([T,\infty);L^2(\mathbb{R}))$ to (1.2) satisfying

$$\left\| u(t,x) - e^{i\frac{|x|^2}{2t} - i\frac{\pi}{4}} t^{-\frac{1}{2}} \widehat{u}_+ \left(\frac{x}{t}\right) e^{-i\lambda |\widehat{u}_+(\frac{x}{t})|^2 \log t} \right\|_{L^2} \to 0,$$

as $t \to \infty$. Hayashi and Naumkin [6] showed the existence of the modified wave operator from a ball in $H^{0,\alpha}(\mathbb{R})$ to $H^{0,\beta}(\mathbb{R})$, for $1/2 < \beta < \alpha < 1$. Moreover, Shimomura [17] proved that the solution to (1.2) with $\operatorname{Im}\lambda < 0$ decays faster by logarithmic order than the linear solutions (see also [18, 8]). Namely, the solution u to (1.2) behaves like the following as $t \to \infty$:

$$e^{i\frac{|x|^2}{2t} - i\frac{\pi}{4}} t^{-\frac{1}{2}} \widehat{u}_+ \left(\frac{x}{t}\right) W\left(t, \frac{x}{t}\right)^{-\frac{1}{2}} e^{\frac{i}{2} \frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda} \log W\left(t, \frac{x}{t}\right)}, \tag{1.3}$$

where $W(t,\xi) := 1 - 2(\operatorname{Im}\lambda)|\widehat{u}_+(\xi)|^2 \log t$.

In the study of spinor Bose-Einstein condensates, the following system of cubic NLS equations is considered ([10]):

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\partial_x^2 u_1 = \overline{u_1} u_2^2, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\partial_x^2 u_2 = u_1^2 \overline{u_2}, \end{cases} \tag{1.4}$$

where m_1 and m_2 are positive constants. We focus only on the mass resonance case, namely $m_1 = m_2$, because the systems are asymptotically free in the remaining cases (see, for example, [4]). Then, (1.4) has the mass conservation law, i.e., $\|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2$ is independent of t. The second author [20] showed that mass transition phenomenon occurs for (1.4) under the mass resonance condition (see [3, 4, 14] for the two dimensional case).

The long-time behavior of systems of NLS equations heavily depends on structure of the nonlinearity. In fact, the second author [20] also observed the mass transition phenomenon to the following NLS system:

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2}\partial_x^2 u_1 = |u_1|^2 u_2, \\ i\partial_t u_2 + \dfrac{1}{2}\partial_x^2 u_2 = |u_2|^2 u_1. \end{cases} \tag{1.5}$$

On the other hand, Nakamura et al. [13] constructed the modified wave operator of the following system:

$$\begin{cases} i\partial_t u_1 + \frac{1}{2}\partial_x^2 u_1 = u_1^2\overline{u_2} + |u_1|^2 u_2, \\ i\partial_t u_2 + \frac{1}{2}\partial_x^2 u_2 = u_2^2\overline{u_1} + |u_2|^2 u_1. \end{cases} \tag{1.6}$$

In particular, the asymptotic behavior of (1.6) is given by

$$e^{i\frac{|x|^2}{2t} - i\frac{\pi}{4}} t^{-\frac{1}{2}} \widehat{u}_{j+}\left(\frac{x}{t}\right) e^{-2i\mathrm{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)\left(\frac{x}{t}\right)\log t},$$

for $j = 1,2$ (see Remark 2.2 below). Note that more general cases are treated in [13].

By setting $u_1(t,x) = u(t,x)$ and $u_2(t,x) = u(t,-x)$, (1.1) is written as the following system of cubic NLS equations:

$$\begin{cases} i\partial_t u_1 + \frac{1}{2}\partial_x^2 u_1 = \lambda u_1^2\overline{u_2}, \\ i\partial_t u_2 + \frac{1}{2}\partial_x^2 u_2 = \lambda u_2^2\overline{u_1}. \end{cases} \tag{1.7}$$

The structure of (1.7) is different from those of (1.4), (1.5), and (1.6). Indeed, $\|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2$ is not conserved even if (u_1,u_2) is a solution to (1.7) and $\lambda \in \mathbb{R}$, while (1.7) is mass resonance.

A formal calculation shows that

$$\partial_t \int_{\mathbb{R}} u_1(t,x)\overline{u_2(t,x)}dx = 2(\mathrm{Im}\,\lambda)\int_{\mathbb{R}} \left(u_1(t,x)\overline{u_2(t,x)}\right)^2 dx$$

provided for solutions (u_1,u_2) to (1.7). Hence, $\int_{\mathbb{R}} u_1(t,x)\overline{u_2(t,x)}dx$ is conserved if $\mathrm{Im}\,\lambda = 0$. Moreover, if $u_1(t,x)\overline{u_2(t,x)}$ is real-valued, $\mathrm{Im}\,\lambda < 0$ implies a dissipative nature:

$$\int_{\mathbb{R}} u_1(t,x)\overline{u_2(t,x)}dx < \int_{\mathbb{R}} u_1(0,x)\overline{u_2(0,x)}dx.$$

Let u_{1+}, u_{2+} be given final states. To ensure the dissipative nature, we impose the following assumption:

ASSUMPTION 1. There exists a constant $\eta \geqslant 1$ such that

$$\frac{1}{\eta}|\widehat{u}_{2+}(\xi)| \leqslant |\widehat{u}_{1+}(\xi)| \leqslant \eta|\widehat{u}_{2+}(\xi)|, \tag{A-1}$$

$$(\mathrm{Im}\,\lambda)\,\mathrm{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi) \leqslant \eta\left|\mathrm{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right| \tag{A-2}$$

hold for any $\xi \in \mathbb{R}$.

We choose asymptotic behavior (w_1,w_2) as follows:

$$w_j(t,x) := e^{i\frac{|x|^2}{2t} - i\frac{\pi}{4}} t^{-\frac{1}{2}} \varphi_j\left(t,\frac{x}{t}\right), \tag{1.8}$$

where φ_1 and φ_2 are defined by

$$\varphi_1(t,\xi) := \widehat{u}_{1+}(\xi)R(t,\xi)^{-\frac{1}{2}}e^{-\frac{1}{2}\frac{\mathrm{Re}\lambda}{\mathrm{Im}\lambda}A(t,\xi)}e^{\frac{i}{2}\left(\frac{\mathrm{Re}\lambda}{\mathrm{Im}\lambda}\log R(t,\xi)-A(t,\xi)\right)}, \tag{1.9}$$

$$\varphi_2(t,\xi) := \widehat{u}_{2+}(\xi)R(t,\xi)^{-\frac{1}{2}}e^{\frac{1}{2}\frac{\mathrm{Re}\lambda}{\mathrm{Im}\lambda}A(t,\xi)}e^{\frac{i}{2}\left(\frac{\mathrm{Re}\lambda}{\mathrm{Im}\lambda}\log R(t,\xi)+A(t,\xi)\right)}, \tag{1.10}$$

$$R(t,\xi) := \left|1-2(\mathrm{Im}\lambda)\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t\right|,$$

$$A(t,\xi) := \mathrm{Arg}\left(1-2(\mathrm{Im}\lambda)\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t\right).$$

Here, we define the angle of $z\in\mathbb{C}$ by $\mathrm{Arg}\, z$, namely $\mathrm{Arg}\, z := \theta$ for $z=|z|e^{i\theta}$ and $-\pi<\theta\leqslant\pi$. We note that (A-2) in Assumption 1 implies that $R(t,\xi)\neq 0$. See Lemma 3.1 below. Moreover, (A-2) yields that $A(t,\xi)\in(-\pi,\pi)$, for $t\geqslant 1$. Indeed, if $\mathrm{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)=0$, it follows from (A-2) that $(\mathrm{Im}\lambda)\mathrm{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\leqslant 0$, namely $A(t,\xi)=0$.

We are now in position to state our main result.

THEOREM 1.1. *Let* $u_{1+},u_{2+}\in H^{0,1}(\mathbb{R})$. *Assume that* $\mathrm{Im}\lambda\neq 0$ *and Assumption 1 holds. Then, there exists* $T>0$ *such that the system* (1.7) *admits a (unique) solution* $(u_1,u_2)\in C([T,\infty);L^2(\mathbb{R})^2)$ *satisfying*

$$\|u_j(t)-w_j(t)\|_{L^2} = o\left(t^{-\frac{1}{4}}\right),$$

as $t\to\infty$, *for* $j=1,2$.

In proving Theorem 1.1, the key point is choice of the asymptotic profile (1.9) and (1.10). Roughly speaking, because w_1 and w_2 satisfy $\|w_j(t)\|_{L^\infty}=O\left(t^{-1/2}\right)$ and

$$\begin{cases} i\partial_t w_1+\dfrac{1}{2}\partial_x^2 w_1 = \lambda w_1^2\overline{w_2}+o\left(t^{-1}\right), \\ i\partial_t w_2+\dfrac{1}{2}\partial_x^2 w_2 = \lambda w_2^2\overline{w_1}+o\left(t^{-1}\right), \end{cases}$$

the difference u_j-w_j decays faster than that of the linear solutions. Accordingly, we can apply the contraction mapping theorem as in [15] (see also [6, 20]).

In the proof of Theorem 1.1, we show that $\varphi_j(t,x)$ defined in (1.9) and (1.10) has logarithmic decay because of Assumption 1. See (3.6) below. Thanks to this dissipative feature, the smallness of the data u_{j+} is not needed.

We mention some remarks on Theorem 1.1. First, although we can solve the corresponding profile equation to (1.7) even if $\lambda\in\mathbb{R}$, the expected asymptotic behavior may decay slower than the linear solutions (see Remark 2.1 below). Second, the uniqueness holds only in $X_{b,T}$ defined by (3.1) below, which is a subspace of $C([T,\infty);L^2(\mathbb{R})^2)$. Third, in general, the asymptotic behavior obtained in Theorem 1.1 is different from that of the typical NLS equation (see (1.3)), because of the presence of $A(t,\xi)$ in (1.9) and (1.10).

REMARK 1.1. By setting $u_{1+}(x) = u_{2+}(-x)$ in Theorem 1.1, we can obtain the asymptotic behavior of (1.1). More precisely, let $u_+ \in H^{0,1}(\mathbb{R})$ and assume that Assumption 1 holds with $u_{1+}(x) = u_+(x)$ and $u_{2+}(x) = u_+(-x)$. Then, there exists a (unique) solution u to (1.1) with

$$\left\| u(t,x) - e^{i\frac{|x|^2}{2t} - i\frac{\pi}{4}} t^{-\frac{1}{2}} \varphi\left(t, \frac{x}{t}\right) \right\|_{L^2} = o\left(t^{-\frac{1}{4}}\right),$$

where φ is defined by

$$\begin{aligned}
\varphi(t,\xi) &:= \widehat{u}_+(\xi) R_0(t,\xi)^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda} A_0(t,\xi)} e^{\frac{i}{2}\left(\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\log R_0(t,\xi) - A_0(t,\xi)\right)},\\
R_0(t,\xi) &:= \left| 1 - 2(\operatorname{Im}\lambda)\left(\widehat{u}_+(\xi)\overline{\widehat{u}_+(-\xi)}\right)\log t \right|,\\
A_0(t,\xi) &:= \operatorname{Arg}\left(1 - 2(\operatorname{Im}\lambda)\left(\widehat{u}_+(\xi)\overline{\widehat{u}_+(-\xi)}\right)\log t\right).
\end{aligned}$$

Here, we give some examples for Assumption 1. For simplicity, let $\operatorname{Im}\lambda = -1$ and

$$\widehat{u}_{1+}(\xi) = f(\xi) + ig(\xi), \quad \widehat{u}_{2+}(\xi) = f(-\xi) + ig(-\xi)$$

for real-valued f, g. Then,

$$\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi) = f(\xi)f(-\xi) + g(\xi)g(-\xi) + i(-f(\xi)g(-\xi) + f(-\xi)g(\xi)). \quad (1.11)$$

Note that $u_+ := u_{1+}$ is also a final state of the nonlocal NLS equation (1.1), because $u_{1+}(x) = u_{2+}(-x)$. If both of f and g are even, (A-1) and (A-2) hold, while this case is the same as the final state problem for the typical NLS equation (1.2).

Let κ be a positive even Schwartz function, e.g., $\kappa(\xi) = e^{-\xi^2}$.

(i) $f(\xi) = (\sinh\xi)e^{-\xi^2}$ and $g(\xi) = (\cosh\xi)e^{-\xi^2}$ satisfy

$$\text{R.H.S. of (1.11)} = (1 - i\sinh 2\xi)e^{-2\xi^2}.$$

Hence, (A-2) holds. We also note that (A-1) follows from $|\widehat{u}_{1+}(\xi)| = |\widehat{u}_{2+}(\xi)| = \sqrt{\cosh 2\xi}\, e^{-\xi^2}$.

(ii) Set $f(\xi) = 2\xi\kappa(\xi)$ and $g(\xi) = |\xi|\kappa(\xi)$. Then, $|\widehat{u}_{1+}(\xi)| = |\widehat{u}_{2+}(\xi)| = \sqrt{5}|\xi|\kappa(\xi)$ and

$$\text{R.H.S. of (1.11)} = (-3\xi^2 - 2i\xi|\xi|)\kappa(\xi)^2.$$

Hence, (A-1) and (A-2) hold.

(iii) Let $f(\xi) = 2(\sin\xi)\kappa(\xi)$ and $g(\xi) = \kappa(\xi)$. Then, we have $|\widehat{u}_{1+}(\xi)| \sim |\widehat{u}_{2+}(\xi)| \sim \kappa(\xi)$ and

$$\text{R.H.S. of (1.11)} = \left(1 - 4\sin^2\xi - 4i\sin\xi\right)\kappa(\xi)^2.$$

If $\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi) < 0$, we have $\left|\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right| > 2\kappa(\xi)^2$. Accordingly, (A-1) and (A-2) follow.

(iv) Let

$$f(\xi) = g(\xi) = \begin{cases} \xi^2 \kappa(\xi), & \text{if } \xi \geqslant 0, \\ \xi \kappa(\xi), & \text{if } \xi < 0. \end{cases}$$

Then, $\widehat{u}_{1+}(\xi) = \xi \widehat{u}_{2+}(\xi)$ and

$$\text{R.H.S. of (1.11)} = 2\xi^3 \kappa(\xi)^2,$$

for $\xi \geqslant 0$. Hence, (A-1) fails, while (A-2) holds.

(v) Let $f(\xi) = (\sin \xi)\kappa(\xi)$ and $g(\xi) = (\cos \xi)\kappa(\xi)$. Then, we have $|\widehat{u}_{1+}(\xi)| = |\widehat{u}_{2+}(\xi)| = \kappa(\xi)$ and

$$\text{R.H.S. of (1.11)} = (\cos 2\xi - i \sin 2\xi)\kappa(\xi)^2.$$

The condition (A-2) fails, while (A-1) holds.

REMARK 1.2. Since we are interested in the asymptotic behavior of the nonlocal NLS equation which differs from that of the typical NLS equation, the result and the proofs are done in the setting of $H^{0,1}$ data. However, with some extra work as in [6], we may replace $H^{0,1}(\mathbb{R})$ by $H^{0,s}(\mathbb{R})$ for $s > 1/2$.

This paper is organized as follows. In Section 2, we solve a corresponding system of ordinary differential equations to (1.7), which determines the asymptotic profile. In Section 3, we prove our main result Theorem 1.1.

We summarize the notation used throughout this paper. We denote the Fouorier transform of f by $\mathscr{F} f$ or $\widehat{f}$, which is defined by

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

In estimates, we use the notation $A \lesssim B$ to mean $A \leqslant CB$ for some constant $C > 0$. We define $A \ll B$ to mean $A \leqslant B/C$.

2. On the profile equation

In this section, we solve the profile equation given by

$$\begin{cases} i\partial_t \varphi_1 = \lambda t^{-1} \varphi_1^2 \overline{\varphi}_2, \\ i\partial_t \varphi_2 = \lambda t^{-1} \overline{\varphi}_1 \varphi_2^2. \end{cases} \tag{2.1}$$

A direct calculation shows that

$$\partial_t (\varphi_1 \overline{\varphi}_2) = 2t^{-1} (\operatorname{Im} \lambda)(\varphi_1 \overline{\varphi}_2)^2,$$

which leads to

$$(\varphi_1 \overline{\varphi}_2)(t) = \frac{\sigma}{1 - 2(\operatorname{Im} \lambda)\sigma \log t},$$

where $\sigma := \varphi_1(1)\overline{\varphi_2(1)} \in \mathbb{C}$. Hence, (2.1) is reduced to

$$\begin{cases} \partial_t \varphi_1 = \dfrac{-i\lambda\sigma}{t(1-2(\operatorname{Im}\lambda)\sigma\log t)}\varphi_1, \\ \partial_t \varphi_2 = \dfrac{-i\lambda\overline{\sigma}}{t(1-2(\operatorname{Im}\lambda)\overline{\sigma}\log t)}\varphi_2. \end{cases} \tag{2.2}$$

When $\operatorname{Im}\lambda \neq 0$, (2.2) has the solution

$$\varphi_1(t) = \varphi_1(1)\exp\left(\frac{i\lambda}{2\operatorname{Im}\lambda}\operatorname{Log}(1-2(\operatorname{Im}\lambda)\sigma\log t)\right),$$
$$\varphi_2(t) = \varphi_2(1)\exp\left(\frac{i\lambda}{2\operatorname{Im}\lambda}\operatorname{Log}(1-2(\operatorname{Im}\lambda)\overline{\sigma}\log t)\right),$$

where $\operatorname{Log} z := \log|z| + i\operatorname{Arg} z$, for $z \in \mathbb{C}$. By setting

$$R(t) := |1-2(\operatorname{Im}\lambda)\sigma\log t|, \quad A(t) := \operatorname{Arg}(1-2(\operatorname{Im}\lambda)\sigma\log t)$$

for simplicity, these solutions are written as follows:

$$\varphi_1(t) = \varphi_1(1)R(t)^{-\frac{1}{2}}e^{-\frac{1}{2}\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}A(t)}e^{\frac{i}{2}\left(\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\log R(t)-A(t)\right)},$$
$$\varphi_2(t) = \varphi_2(1)R(t)^{-\frac{1}{2}}e^{\frac{1}{2}\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}A(t)}e^{\frac{i}{2}\left(\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\log R(t)+A(t)\right)}.$$

Thanks to the presence of $R(t)^{-1/2}$, the nonlinearity has a dissipative nature as in [17].

REMARK 2.1. When $\operatorname{Im}\lambda = 0$, (2.2) with $\lambda \in \mathbb{R}$ leads to

$$\varphi_1(t) = \varphi_1(1)\exp(-i\lambda\sigma\log t) = \varphi_1(1)t^{\lambda\operatorname{Im}\sigma}e^{-i\lambda\operatorname{Re}\sigma\log t},$$
$$\varphi_2(t) = \varphi_2(1)\exp(-i\lambda\overline{\sigma}\log t) = \varphi_2(1)t^{-\lambda\operatorname{Im}\sigma}e^{-i\lambda\operatorname{Re}\sigma\log t}.$$

Hence, we conjecture that the asymptotic behavior of (1.7) with $\operatorname{Im}\lambda = 0$ is determined by

$$\widetilde{w}_1(t,x) := e^{i\frac{|x|^2}{2t}-i\frac{\pi}{4}}t^{-\frac{1}{2}}\widehat{u}_{1+}\left(\frac{x}{t}\right)t^{\lambda\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)\left(\frac{x}{t}\right)}e^{-i\lambda\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)\left(\frac{x}{t}\right)\log t},$$
$$\widetilde{w}_2(t,x) := e^{i\frac{|x|^2}{2t}-i\frac{\pi}{4}}t^{-\frac{1}{2}}\widehat{u}_{2+}\left(\frac{x}{t}\right)t^{-\lambda\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)\left(\frac{x}{t}\right)}e^{-i\lambda\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)\left(\frac{x}{t}\right)\log t}.$$

Indeed, these functions satisfy

$$i\partial_t\widetilde{w}_1 + \frac{1}{2}\partial_x^2\widetilde{w}_1 = \lambda\widetilde{w}_1^2\overline{\widetilde{w}_2} + \frac{1}{2}t^{-\frac{5}{2}}e^{i\frac{x^2}{2t}-\frac{\pi}{4}}\partial_x^2\left(\widehat{u}_{1+}t^{\lambda\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)}e^{-i\lambda\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)\log t}\right)\left(\frac{x}{t}\right),$$
$$i\partial_t\widetilde{w}_2 + \frac{1}{2}\partial_x^2\widetilde{w}_2 = \lambda\widetilde{w}_2^2\overline{\widetilde{w}_1} + \frac{1}{2}t^{-\frac{5}{2}}e^{i\frac{x^2}{2t}-\frac{\pi}{4}}\partial_x^2\left(\widehat{u}_{2+}t^{-\lambda\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)}e^{-i\lambda\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)\log t}\right)\left(\frac{x}{t}\right).$$

We can find that there exists a solution (u_1, u_2) to (1.7) satisfying

$$\|u_j - \widetilde{w}_j\|_{L^2} \to 0, \tag{2.3}$$

as $t\to\infty$, for $j=1,2$, if $\operatorname{Im}\left(\widehat{u_1}\overline{\widehat{u_2}}\right)(\xi)=0$, for $\xi\in\mathbb{R}$, while this asymptotic behavior is the same as that of (1.6). On the other hand, because $\operatorname{Im}(\widehat{u_1}\overline{\widehat{u_2}})\neq 0$ yields that $\|w_j(t)\|_{L^\infty}\lesssim t^{-1/2}$ fails, we could not obtain (2.3). This case is excluded from the consideration in this paper, because we assume $\operatorname{Im}\lambda\neq 0$ in Theorem 1.1.

REMARK 2.2. The corresponding profile equation for (1.6) is given by

$$\begin{cases} i\partial_t\varphi_1=t^{-1}\left(\varphi_1^2\overline{\varphi_2}+|\varphi_1|^2\varphi_2\right),\\ i\partial_t\varphi_2=t^{-1}\left(\overline{\varphi_1}\varphi_2^2+|\varphi_2|^2\varphi_1\right).\end{cases}\tag{2.4}$$

Since $\partial_t(\varphi_1\overline{\varphi_2})=0$, (2.4) is reduced to

$$\begin{cases} \partial_t\varphi_1=-i2t^{-1}(\operatorname{Re}\sigma)\varphi_1,\\ \partial_t\varphi_2=-i2t^{-1}(\operatorname{Re}\sigma)\varphi_2,\end{cases}$$

where $\sigma:=\varphi_1(1)\overline{\varphi_2(1)}\in\mathbb{C}$. Hence, the solution to (2.4) are written as follows:

$$\begin{aligned}\varphi_1(t)&=\varphi_1(1)e^{-2i(\operatorname{Re}\sigma)\log t},\\ \varphi_2(t)&=\varphi_2(1)e^{-2i(\operatorname{Re}\sigma)\log t}.\end{aligned}$$

Hence, the modified scattering for (1.6) follows from essentially the same argument as in [15].

3. Proof of Theorem 1.1

We introduce the following notation:

$$U(t):=e^{\frac{i}{2}t\partial_x^2},\quad M(t):=e^{i\frac{|x|^2}{2t}},\quad D(t)\psi:=e^{-i\frac{\pi}{4}}t^{-\frac12}\psi\left(\frac{x}{t}\right).$$

Then, we note that

$$U(t)=M(t)D(t)\mathscr{F}M(t),\quad w_j(t)=M(t)D(t)\varphi_j,$$

where (w_1,w_2) is defined by (1.8). By (1.7) and (2.1), we have

$$\begin{aligned}
&i\partial_t(\mathscr{F}U(-t)u_1(t)-\varphi_1)=\lambda\mathscr{F}U(-t)\left(u_1^2\overline{u_2}\right)-\frac{\lambda}{t}\varphi_1^2\overline{\varphi_2}\\
&\Leftrightarrow\mathscr{F}U(-t)u_1(t)=\varphi_1+\lambda\int_t^\infty\mathscr{F}U(-\tau)\left(u_1^2\overline{u_2}-\frac{1}{\tau}U(\tau)\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\right)d\tau\\
&\Leftrightarrow u_1(t)=U(t)\mathscr{F}^{-1}\varphi_1+\lambda\int_t^\infty U(t-\tau)\left(u_1^2\overline{u_2}-\frac{1}{\tau}U(\tau)\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\right)d\tau\\
&\Leftrightarrow u_1(t)=M(t)D(t)\varphi_1+\lambda\int_t^\infty U(t-\tau)\left(u_1^2\overline{u_2}-(M(\tau)D(\tau)\varphi_1)^2\overline{M(\tau)D(\tau)\varphi_2}\right)d\tau\\
&\qquad+\lambda\int_t^\infty U(t-\tau)M(\tau)D(\tau)\mathscr{F}(M(\tau)-1)\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\frac{d\tau}{\tau}\\
&\qquad+M(t)D(t)\mathscr{F}(M(t)-1)\mathscr{F}^{-1}\varphi_1.
\end{aligned}$$

Let $J(t)$ be the generator of the Galilean transformation, i.e. $J(t) := x + it\partial_x = U(t)xU(-t)$. We also use the operator $|J(t)|^\beta$ defined by

$$|J(t)|^\beta := U(t)|x|^\beta U(-t) = M(t)\left(-t^2\partial_x^2\right)^{\frac{\beta}{2}} M(-t),$$

for $\beta \geqslant 0$. We define the function space

$$X_{b,T} := \left\{(u_1,u_2) \in C([T,\infty);L^2(\mathbb{R})^2) : \|u_j - w_j\|_{X_{b,T}} < \infty \ (j=1,2)\right\} \tag{3.1}$$

equipped with the norm

$$\|f\|_{X_{b,T}} := \sup_{t\in[T,\infty)} \left(t^{\frac{\beta}{2}+b}\|f\|_{L^2} + t^b \left\||J(t)|^\beta f\right\|_{L^2}\right),$$

for $\beta, b, T > 0$.

Set

$$\frac{1}{2} < \beta < 1, \quad 0 < b < \frac{1-\beta}{4}. \tag{3.2}$$

We will show that the map $\Phi = (\Phi_1, \Phi_2)$ defined by

$$\begin{aligned} \Phi_1(u_1,u_2)(t) = w_1(t) &+ \lambda \int_t^\infty U(t-\tau)\left(u_1^2\overline{u_2} - w_1^2\overline{w_2}\right)(\tau)d\tau \\ &+ \lambda \int_t^\infty U(t-\tau)M(\tau)D(\tau)\mathscr{F}(M(\tau)-1)\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\frac{d\tau}{\tau} \\ &+ M(t)D(t)\mathscr{F}(M(t)-1)\mathscr{F}^{-1}\varphi_1, \\ \Phi_2(u_1,u_2)(t) = w_2(t) &+ \lambda \int_t^\infty U(t-\tau)\left(u_2^2\overline{u_1} - w_2^2\overline{w_1}\right)(\tau)d\tau \\ &+ \lambda \int_t^\infty U(t-\tau)M(\tau)D(\tau)\mathscr{F}(M(\tau)-1)\mathscr{F}^{-1}\left[\varphi_2^2\overline{\varphi_1}\right]\frac{d\tau}{\tau} \\ &+ M(t)D(t)\mathscr{F}(M(t)-1)\mathscr{F}^{-1}\varphi_2 \end{aligned} \tag{3.3}$$

is a contraction mapping on a ball in $X_{b,T}$, for sufficiently large T.

We only consider the estimates for Φ_1, because Φ_2 is similarly handled. We write the second, third, and fourth terms on the right hand side of (3.3) as I_1, I_2, and I_3, respectively:

$$\Phi_1(u_1,u_2)(t) - w_1(t) =: I_1 + I_2 + I_3.$$

Let K be a positive constant. We define

$$B_K(X_{b,T}) := \left\{(u_1,u_2) \in X_{b,T} : \|u_j - w_j\|_{X_{b,T}} < K \ (j=1,2)\right\}.$$

First, we observe the lower bound of $R(t,\xi)$.

LEMMA 3.1. *If* $\operatorname{Im}\lambda \neq 0$ *and* (A-2) *holds, we have*

$$R(t,\xi) \gtrsim 1 + \left|\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right| \log t, \tag{3.4}$$

for $t \geqslant 1$ *and* $\xi \in \mathbb{R}$, *where the implicit constant depends only on* $\operatorname{Im}\lambda$ *and* η.

Proof. When $(\operatorname{Im}\lambda)\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t \leqslant 1/8$, a direct calculation yields that

$$\begin{aligned} R(t,\xi) &= \sqrt{4(\operatorname{Im}\lambda)^2\left|\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|^2(\log t)^2 - 4(\operatorname{Im}\lambda)\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t + 1} \\ &\geqslant \sqrt{4(\operatorname{Im}\lambda)^2\left|\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|^2(\log t)^2 + \frac{1}{2}}. \end{aligned}$$

We note that (A-2) implies that

$$\left|\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right| \leqslant \left(\frac{\eta}{|\operatorname{Im}\lambda|}+1\right)\left|\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|, \tag{3.5}$$

when $(\operatorname{Im}\lambda)\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi) > 0$. Hence, if $(\operatorname{Im}\lambda)\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t \geqslant 1$, it follows from (3.5) that

$$\begin{aligned} R(t,\xi) &= \sqrt{\left(1-2(\operatorname{Im}\lambda)\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t\right)^2 + 4(\operatorname{Im}\lambda)^2\left(\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t\right)^2} \\ &\geqslant \sqrt{1+4(\operatorname{Im}\lambda)^2\left(\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t\right)^2} \\ &\geqslant \sqrt{1+4\frac{(\operatorname{Im}\lambda)^4}{(\eta+|\operatorname{Im}\lambda|)^2}\left|\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|^2(\log t)^2}. \end{aligned}$$

If $1/8 < (\operatorname{Im}\lambda)\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\log t < 1$, by (3.5), we have

$$R(t,\xi) \geqslant 2|\operatorname{Im}\lambda|\left|\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|\log t \geqslant \frac{1}{4}\frac{\left|\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|}{\left|\operatorname{Re}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|} \geqslant \frac{1}{4}\frac{|\operatorname{Im}\lambda|}{\eta+|\operatorname{Im}\lambda|}.$$

Moreover, by (3.5), we get

$$R(t,\xi) \geqslant 2|\operatorname{Im}\lambda|\left|\operatorname{Im}\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|\log t \geqslant 2\frac{(\operatorname{Im}\lambda)^2}{\eta+|\operatorname{Im}\lambda|}\left|\left(\widehat{u}_{1+}\overline{\widehat{u}_{2+}}\right)(\xi)\right|\log t.$$

Therefore, we obtain (3.4). □

By (1.9), (1.10), (3.4), and (A-1), we have

$$\|\varphi_j(t)\|_{L^\infty} \lesssim \left\|\widehat{u}_{j+}R(t)^{-\frac{1}{2}}\right\|_{L^\infty} \lesssim (\log t)^{-\frac{1}{2}}\sqrt{\left\|\frac{\widehat{u}_{1+}}{\widehat{u}_{2+}}\right\|_{L^\infty} + \left\|\frac{\widehat{u}_{2+}}{\widehat{u}_{1+}}\right\|_{L^\infty}} \lesssim (\log t)^{-\frac{1}{2}}. \tag{3.6}$$

Moreover, since

$$|\partial_\xi R(t,\xi)| + |R(t,\xi)\partial_\xi A(t,\xi)| \lesssim (|\partial_\xi \widehat{u}_{1+}(\xi)\widehat{u}_{2+}(\xi)| + |\widehat{u}_{1+}(\xi)\partial_\xi \widehat{u}_{2+}(\xi)|)\log t,$$

from (3.4), we get

$$\|\varphi_1(t)\|_{H^1} \lesssim \|\widehat{u}_{1+}\|_{H^1} + \|\widehat{u}_{1+}\|_{H^1}^2\|\widehat{u}_{2+}\|_{H^1}\log t$$

$$\lesssim \|u_{1+}\|_{H^{0,1}}(1+\|u_{1+}\|_{H^{0,1}}\|u_{2+}\|_{H^{0,1}}\log t),$$

$$\|\varphi_2(t)\|_{H^1} \lesssim \|\widehat{u}_{2+}\|_{H^1} + \|\widehat{u}_{1+}\|_{H^1}\|\widehat{u}_{2+}\|_{H^1}^2 \log t$$
$$\lesssim \|u_{2+}\|_{H^{0,1}}(1+\|u_{1+}\|_{H^{0,1}}\|u_{2+}\|_{H^{0,1}}\log t).$$

Hence, by taking $T = T(\|u_{1+}\|_{H^{0,1}}, \|u_{2+}\|_{H^{0,1}})$ sufficiently large, we have

$$\|\varphi_1(t)\|_{H^1} + \|\varphi_2(t)\|_{H^1} \leqslant (\log t)^2, \tag{3.7}$$

for $t \geqslant T$.

Next, we observe some L^∞-bounds. By (3.6), we have

$$\|w_j(t)\|_{L^\infty} = t^{-\frac{1}{2}}\|\varphi_j(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}(\log t)^{-\frac{1}{2}}, \tag{3.8}$$

for $j=1,2$ and $t \geqslant T$. Since the Gagliardo-Nirenberg type inequality yields that

$$\|f(t)\|_{L^\infty} = \|M(-t)f(t)\|_{L^\infty} \lesssim \|M(-t)f(t)\|_{L^2}^{1-\frac{1}{2\beta}} \left\|(-\partial_x^2)^{\frac{\beta}{2}}(M(-t)f(t))\right\|_{L^2}^{\frac{1}{2\beta}}$$
$$\lesssim \|f(t)\|_{L^2}^{1-\frac{1}{2\beta}} \left\|t^{-\beta}|J(t)|^\beta f(t)\right\|_{L^2}^{\frac{1}{2\beta}},$$

by (3.2), we have

$$\|u_j - w_j\|_{L^\infty} \lesssim K t^{-\frac{1}{4}-\frac{\beta}{2}-b} \leqslant t^{-\frac{1}{2}-b}(\log t)^{-3}, \tag{3.9}$$

for $j=1,2$, $t \geqslant T = T(\|u_{1+}\|_{H^{0,1}}, \|u_{2+}\|_{H^{0,1}}, K) \gg 1$ and $(u_1,u_2) \in B_K(X_{b,T})$.

For the term I_1, we use (3.8) and (3.9) to obtain

$$\begin{aligned}\|I_1\|_{L^2} &\lesssim \int_t^\infty \|(u_1+w_1)(u_1-w_1)\overline{u_2}\|_{L^2}d\tau + \int_t^\infty \|w_1^2(\overline{u_2-w_2})\|_{L^2}d\tau \\ &\lesssim \int_t^\infty (\|u_1\|_{L^\infty}+\|w_1\|_{L^\infty})\|u_1-w_1\|_{L^2}\|u_2\|_{L^\infty}d\tau + \int_t^\infty \|w_1\|_{L^\infty}^2\|u_2-w_2\|_{L^2}d\tau \\ &\lesssim \int_t^\infty \tau^{-\frac{\beta}{2}-b-1}(\log\tau)^{-1}d\tau \lesssim t^{-\frac{\beta}{2}-b}(\log t)^{-1} \leqslant \frac{K}{100}t^{-\frac{\beta}{2}-b},\end{aligned} \tag{3.10}$$

for $t \geqslant T = T(K) \gg 1$.

For the term I_2, (3.6), (3.7), and (3.2) yield that

$$\begin{aligned}\|I_2\|_{L^2} &\lesssim \int_t^\infty \left\|U(t-\tau)M(\tau)D(\tau)\mathscr{F}(M(\tau)-1)\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\right\|_{L^2}\frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \tau^{-\frac{1}{2}}\left\||x|\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\right\|_{L^2}\frac{d\tau}{\tau} \lesssim \int_t^\infty \tau^{-\frac{1}{2}}\|\partial_x(\varphi_1^2\overline{\varphi_2})\|_{L^2}\frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \tau^{-\frac{1}{2}}(\|\partial_x\varphi_1\|_{L^2}\|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty} + \|\varphi_1\|_{L^\infty}^2\|\partial_x\varphi_2\|_{L^2})\frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \tau^{-\frac{3}{2}}\log\tau d\tau \lesssim t^{-\frac{1}{2}}\log t \leqslant \frac{K}{100}t^{-\frac{\beta}{2}-b},\end{aligned} \tag{3.11}$$

for $t \geqslant T = T(K) \gg 1$.

For the term I_3, by (3.7) and (3.2), we have

$$\begin{aligned}\|I_3\|_{L^2} &\lesssim \|M(t)D(t)\mathscr{F}(M(t)-1)\mathscr{F}^{-1}\varphi_1\|_{L^2} \lesssim t^{-\frac{1}{2}}\|\partial_x\varphi_1\|_{L^2} \lesssim t^{-\frac{1}{2}}(\log t)^2 \\ &\leqslant \frac{K}{100}t^{-\frac{\beta}{2}-b},\end{aligned} \tag{3.12}$$

for $t \geqslant T = T(K) \gg 1$.

In what follows, we consider the contribution of the second term in the $X_{b,T}$-norm. By (3.7),

$$\left\||J(t)|^\beta w_j(t)\right\|_{L^2} = \left\|(-\partial_x^2)^{\frac{\beta}{2}}\varphi_j(t)\right\|_{L^2} \lesssim \|\varphi_j(t)\|_{H^1} \lesssim (\log t)^2,$$

which yields that

$$\begin{aligned}\left\||J(t)|^\beta u_j(t)\right\|_{L^2} &\leqslant \left\||J(t)|^\beta w_j(t)\right\|_{L^2} + \left\||J(t)|^\beta (u_j(t)-w_j(t))\right\|_{L^2} \lesssim (\log t)^2 + Kt^{-b} \\ &\lesssim (\log t)^2,\end{aligned}$$

for $t \geqslant T = T(K) \gg 1$.

By the fractional Leibniz rule, (3.8), and (3.9), we have

$$\begin{aligned}\left\||J(t)|^\beta I_1\right\|_{L^2} &\lesssim \int_t^\infty \tau^\beta \left\|(-\partial_x^2)^{\frac{\beta}{2}}\left\{M(-\tau)(u_1+w_1)\overline{M(-\tau)u_2}M(-\tau)(u_1-w_1)\right\}\right\|_{L^2} d\tau \\ &\quad + \int_t^\infty \tau^\beta \left\|(-\partial_x^2)^{\frac{\beta}{2}}\left\{(M(-\tau)w_1)^2\overline{M(-\tau)(u_2-w_2)}\right\}\right\|_{L^2} d\tau \\ &\lesssim \int_t^\infty \left\||J(\tau)|^\beta(u_1-w_1)\right\|_{L^2}\|u_2\|_{L^\infty}(\|u_1\|_{L^\infty}+\|w_1\|_{L^\infty})d\tau \\ &\quad + \int_t^\infty \|u_1-w_1\|_{L^\infty}\left\||J(\tau)|^\beta u_2\right\|_{L^2}(\|u_1\|_{L^\infty}+\|w_1\|_{L^\infty})d\tau \\ &\quad + \int_t^\infty \|u_1-w_1\|_{L^\infty}\|u_2\|_{L^\infty}\left(\left\||J(\tau)|^\beta u_1\right\|_{L^2}+\left\||J(\tau)|^\beta w_1\right\|_{L^2}\right)d\tau \\ &\quad + \int_t^\infty \left\||J(\tau)|^\beta w_1\right\|_{L^2}\|w_1\|_{L^\infty}\|u_2-w_2\|_{L^\infty}d\tau \\ &\quad + \int_t^\infty \|w_1\|_{L^\infty}^2\left\||J(\tau)|^\beta(u_2-w_2)\right\|_{L^2}d\tau \\ &\lesssim K\int_t^\infty \tau^{-1-b}(\log\tau)^{-1}d\tau \\ &\lesssim Kt^{-b}(\log t)^{-1} \leqslant \frac{K}{100}t^{-b},\end{aligned} \tag{3.13}$$

for $t \geqslant T = T(K) \gg 1$.

The contribution of I_2 is estimates as follows: by (3.6), (3.7), and (3.2),

$$\begin{aligned}\left\| |J(t)|^\beta I_2 \right\|_{L^2} &\lesssim \int_t^\infty \left\| |J(t)|^\beta U(t-\tau)M(\tau)D(\tau)\mathscr{F}(M(\tau)-1)\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\right\|_{L^2}\frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \left\| |x|^\beta (1-M(-\tau))\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\right\|_{L^2}\frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \tau^{-\frac{1-\beta}{2}} \left\| |x|\mathscr{F}^{-1}\left[\varphi_1^2\overline{\varphi_2}\right]\right\|_{L^2}\frac{d\tau}{\tau} \lesssim \int_t^\infty \tau^{-\frac{1-\beta}{2}} \left\|\varphi_1^2\overline{\varphi_2}\right\|_{H^1}\frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \tau^{-\frac{1-\beta}{2}} \left(\|\varphi_1\|_{H^1}\|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty} + \|\varphi_1\|_{L^\infty}^2\|\varphi_2\|_{H^1}\right)\frac{d\tau}{\tau} \\ &\lesssim \int_t^\infty \tau^{-1-\frac{1-\beta}{2}} \log\tau\, d\tau \lesssim t^{-\frac{1-\beta}{2}}\log t \leqslant \frac{K}{100}t^{-b},\end{aligned} \tag{3.14}$$

for $t \geqslant T = T(K) \gg 1$. Similarly,

$$\begin{aligned}\left\| |J(t)|^\beta I_3 \right\|_{L^2} &\lesssim \left\| |J(t)|^\beta M(t)D(t)\mathscr{F}(M(t)-1)\mathscr{F}^{-1}[\varphi_1]\right\|_{L^2} \\ &\lesssim \left\| |x|^\beta (1-M(-t))\mathscr{F}^{-1}[\varphi_1]\right\|_{L^2} \lesssim t^{-\frac{1-\beta}{2}}\|\varphi_1\|_{H^1} \lesssim t^{-\frac{1-\beta}{2}}(\log t)^2 \\ &\leqslant \frac{K}{100}t^{-b},\end{aligned} \tag{3.15}$$

for $t \geqslant T = T(K) \gg 1$.

From (3.10)–(3.15), we obtain

$$\|\Phi_1(u_1,u_2) - w_1\|_{X_{b,T}} \leqslant \frac{K}{4}.$$

By a similar manner, we obtain that Φ is a contraction mapping on $B_K(X_{b,T})$, which concludes the proof. □

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(Received February 6, 2019)

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