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EXISTENCE OF SOLUTIONS TO NONLINEAR LEGENDRE BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider nonlinearly perturbed Legendre differential equations subject to the usual boundary conditions. For such problems we establish sufficient conditions for the existence of solutions and in some cases we provide a qualitative description of solutions depending on a parameter. The results presented depend on the size and limiting behavior of the nonlinearities.

1. Introduction

In this paper, we discuss the solvability of boundary value problems which arise as nonlinear perturbations of the classical Legendre differential equation subject to the standard boundary conditions. The framework we present enables us to establish the existence of solutions to boundary value problems under a variety of conditions. Each approach takes advantage of the general linear Sturm-Liouville theory, in particular existing knowledge regarding the spectrum of the Legendre Sturm-Liouville operator. In Section 2.1, we provide a general framework that enables us to discuss the nonlinear boundary value problem as an operator equation of the form $Lx = F(x)$ and we establish conditions for the existence of solutions in the case where the linear part L is invertible. In Sections 2.2 and 2.3, we don't make this invertibility assumption and results we obtain are based on the projection scheme commonly referred to as the Lyapunov-Schmidt procedure. In Section 2.2, we use fixed-point theorems to provide sufficient conditions for the solvability of the boundary value problem that depend on the limiting behavior of the nonlinearity. In Section 2.3, the same projection scheme along with the implicit function theorem is used to establish the existence and qualitative properties of solutions to weakly nonlinear problems.

Approaches similar to the ones appearing in this paper have been used in a variety of settings in the study of nonlinear boundary value problems. For the use of arguments similar to those in Section 2.1 in the continuous and discrete cases, the reader is referred to [2], [10], [13], [17], [18] and [19]. For the general theory of projection methods in nonlinear boundary value problems we suggest [21]. For the use of projection methods

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similar to those in Subsections 2.2 and 2.3, see [6], [9], [14], [15], [22], [23] and [24]. For results involving topological degree theory arguments in the analysis of discrete boundary value problems, the reader may consult [4] and [8].

The classical Legendre eigenvalue-eigenfunction problems consists of finding the scalars μ and functions $x : (-1,1) \to \mathbb{R}$ such that

$$\left[\left(1-t^2\right)x'(t)\right]' + \mu x(t) = 0,$$

for all $t \in (-1,1)$, where

$$\lim_{t\to -1^+} x(t), \qquad \lim_{t\to 1^-} x(t),$$

$$\lim_{t\to -1^+} x'(t), \qquad \lim_{t\to 1^-} x'(t)$$

all exist and are finite. It is well-known that nontrivial solutions of this problem exist if and only if $\mu = k(k+1)$, where k is a nonnegative integer. If $\mu = k(k+1)$, the only solutions are the constant multiples of the k^{th} Legendre polynomial. In this paper, we consider a nonlinear perturbation of the differential equation subject to the same boundary conditions. That is, the existence of finite limits of $x(t)$ and $x'(t)$ at 1 and -1.

2. Differential equations

2.1. The case of invertible L

Even though in this paper we are mainly interested in the cases where the parameter μ in the equation below is an eigenvalue of the associated linear Legendre equation, we devote this first section to the case where $\mu \neq k(k+1)$ for any nonnegative integer k. We consider boundary value problems on $(-1,1)$ of the form,

$$\left[\left(1-t^2\right)x'(t)\right]' + \mu x(t) = f(x(t)) \tag{1}$$

subject to the condition that the following limits exist and are finite

$$\begin{aligned} &\lim_{t\to -1^+} x(t), \qquad \lim_{t\to 1^-} x(t), \\ &\lim_{t\to -1^+} x'(t) \qquad \lim_{t\to 1^-} x'(t). \end{aligned} \tag{2}$$

Throughout this paper, we assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $\mathscr{L}^2$ denote the space of functions $\mathscr{L}^2 = (\mathscr{L}^2[-1,1], \|\cdot\|_2)$, X be defined as the subspace of functions in $\mathscr{L}^2$ where the limits appearing in (2) exist and are finite and

$$D(L) = \left\{x \in X : x' \text{ is absolutely continuous and } x'' \in \mathscr{L}^2\right\}.$$

In this section, we assume that $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz. This implies that $f\circ x\in L^2$, for all $x\in L^2$. We seek conditions under which we can guarantee the existence of a solution to the boundary value problem (1)-(2).

We now present some basic results regarding a closely related linear boundary value problem. If $\mu\neq k(k+1)$, for all k, the equation

$$\left[\left(1-t^2\right)x'(t)\right]'+\mu x(t)=h(t)$$

has exactly one solution satisfying the condition that the following limits exist and are finite

$$\lim_{t\to-1^+} x(t), \qquad \lim_{t\to 1^-} x(t),$$

$$\lim_{t\to-1^+} x'(t) \qquad \lim_{t\to 1^-} x'(t).$$

Define the map $L:D(L)\to\mathscr{L}^2$ by

$$[Lx](t)=\left[\left(1-t^2\right)x'(t)\right]'+\mu x(t).$$

Clearly, if $\mu\neq k(k+1)$, for all k, then L is a bijection from $D(L)$ onto L^2.

Let P_k denote the k^{th}-degree Legendre polynomial and $p(t)=(1-t^2)$. From general Sturm-Liouville theory, the equation $(px')'+\lambda x=0$, subject to the condition that the limits in (2) exist and are finite, has countably many simple eigenvalues $\lambda_k=k(k+1)$ with corresponding eigenfunctions P_k, for $k\geqslant 0$. It is also well-known that L is self-adjoint and that the graph of L is closed. Further, the unique solution $x_h\in D(L)$ to $Lx=h$ guaranteed above can be represented by the eigenfunction expansion,

$$x_h=\sum_{k=0}^{\infty}\frac{(k+\frac{1}{2})\langle h,P_k\rangle}{[\mu-k(k+1)]}P_k,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard $\mathscr{L}^2$ inner product. From this is follows that L^{-1} is continuous and that

$$\|L^{-1}\|\leqslant\left(\sum_{k=0}^{\infty}\left|\frac{1}{(\mu-k(k+1))^2(k+\frac{1}{2})}\right|\right)^{1/2}.$$

This information, as well as more on the general theory of Legendre polynomials and the Legendre differential equation can be found in [3].

The following corollary establishes the continuity of L^{-1} by giving a bound on its operator norm that will be useful later. Before presenting the next corollary, we first must introduce some notation. Let $\mathscr{C}$ denote the space of continuous functions on $[-1,1]$ and $\|\cdot\|_\infty$ denote the supremum norm. That is, for a continuous function x on $[-1,1]$

$$\|x\|_\infty=\sup_{t\in[-1,1]}|x(t)|.$$

COROLLARY 1. *There exists $K > 0$ such that for all $h \in Im(L) \subset \mathscr{L}^2$, the unique solution x_h to the equation $Lx = h$ satisfies*

$$\|x_h\|_\infty \leqslant K\|h\|$$

and

$$\|x_h'\|_\infty \leqslant K\|h\|.$$

Proof. Define the map $\hat{L} : \hat{D}(L) \to Im(L)$ by

$$\left[\hat{L}x\right](t) = \left[\left(1-t^2\right)x'(t)\right]' + \mu x(t),$$

where $\hat{D}(L)$ consists of the same set of functions as $D(L)$, but is endowed with the norm $\|\cdot\|_{H^2}$ given by

$$\|z\|_{H^2} = \|z\|_\infty + \|z'\|_\infty + \|z''\|.$$

Note that the map $\hat{L}$ is a continuous, linear bijection onto $\mathscr{L}^2$, and that $\hat{D}(L)$ and $Im(L)$ are Banach spaces. Therefore, by a consequence of the open mapping theorem, $\hat{L}^{-1}$ is continuous. This means there exists a $K > 0$ such for any $h \in \mathscr{L}^2$ the unique solution x_h to $Lx = h$ satisfies

$$K\|h\| \geqslant \|x_h\|_{H^2} \geqslant \|x_h\|_\infty$$

and

$$K\|h\| \geqslant \|x_h\|_{H^2} \geqslant \|x_h'\|_\infty$$

as required. □

LEMMA 1. *The map $L^{-1} : Im(L) \to \mathscr{L}^2$ is compact.*

Proof. Consider the map $\tilde{L} : \tilde{D}(L) \to Im(L)$ defined by

$$\left[\tilde{L}x\right](t) = \left[\left(1-t^2\right)x'(t)\right]' + \mu x(t),$$

where $\tilde{D}(L)$ consists of the same set of functions as $D(L)$ but endowed with the norm $\|\cdot\|_\infty$. Note that $\tilde{L}$ is invertible due to the fact that L is invertible. We wish to show that $\tilde{L}$ is compact using the Arzela-Ascoli theorem. Let $M > 0$ and define S to be the set $S = \{z \in Im(L) : \|z\| \leqslant M\}$. Let $h \in S$ and observe that

$$\left\|\tilde{L}^{-1}h\right\|_\infty \leqslant K\|h\| \leqslant KM.$$

Therefore, the $\tilde{L}^{-1}(S)$ is a uniformly bounded set of functions in $\mathscr{C}$. We now wish to show that this set is equicontinuous. Let $h \in S$ and let $\varepsilon > 0$. By the previous corollary along with the mean value theorem, for any $h \in \mathscr{L}^2$ $\tilde{L}^{-1}h$ is Lipschitz on S with constant KM. Let $\delta = \varepsilon/KM$ and $|t_1 - t_2| < \delta$. Then we have that

$$|\tilde{L}^{-1}h(t_1) - \tilde{L}^{-1}h(t_2)| \leqslant KM|t_1 - t_2| < \varepsilon.$$

Therefore, $\tilde{L}^{-1}(S)$ is an equicontinuous set of functions in $\mathscr{C}$. By the Arzelá-Ascoli theorem, $\tilde{L}^{-1} : Im(L) \to \bar{D}(L)$ is compact. Therefore, it follows that $L^{-1} : Im(L) \to D(L)$ is a compact operator. $\square$

We now discuss the issue of whether we can guarantee a solution to the nonlinear boundary value problem

$$\left[\left(1-t^2\right)x'(t)\right]' + \mu x(t) = f(x(t)),$$

where x satisfies the condition that limits in (2) exist and are finite. Define $F : \mathscr{L}^2 \to \mathscr{L}^2$ by

$$F(x) = f \circ x.$$

It is evident that the boundary value problem (1)-(2) is equivalent to the operator equation $Lx = F(x)$.

THEOREM 1. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is Lipschitz and that* $\mu \neq k(k+1)$, *for all nonnegative integers* k. *Then if*

$$\lim_{|s|\to\infty} \frac{|f(s)|}{|s|} = 0,$$

there exists a solution to the boundary value problem

$$\left[\left(1-t^2\right)x'(t)\right]' + \mu x(t) = f(x(t))$$

subject to the condition that the limits in (2) *exist and are finite.*

The proof of this theorem is a standard application of Schauder's fixed point theorem applied to the operator $L^{-1}F$. We omit the details. Note that results in this subsection depended heavily on L having an inverse, which is only the case if we assume $\mu \neq k(k+1)$, for any $k \in \mathbb{N}$. The following subsections analyze situations where L is not invertible.

2.2. The case of non-invertible L

We will now assume that $\mu = k(k+1)$, for some $k \in \{0,1,2,\ldots\}$. As a consequence of the general Sturm-Liouville theory outlined in the previous section, $\mu = k(k+1)$ implies that the kernel of L is one-dimensional and spanned by P_k. Further, as stated in [11], we have that $h \in \operatorname{Im}(L)$ if and only if

$$\langle h, P_k \rangle = 0.$$

Therefore, it follows that $\mathrm{Im}\,(L)$=$[\mathrm{Ker}\,(L)]^{\perp}$. In this section we will assume that $\lim_{s\to\infty} f(s)$ and $\lim_{s\to-\infty} f(s)$ exist and are finite. We denote these values by

$$f(\infty) \equiv \lim_{s\to\infty} f(s) \qquad \text{and} \qquad f(-\infty) \equiv \lim_{s\to-\infty} f(s).$$

We employ the Lyapunov-Schmidt procedure. For the readers convenience, we now outline the basic elements of this process. For more details, the reader may consult [5].

First define $U : \mathscr{L}^2 \to \mathscr{L}^2$ by

$$[Ux](t) = \left(k+\frac{1}{2}\right)\langle x, P_k\rangle P_k(t).$$

Note that U is a projection onto $\mathrm{Ker}\,(L) = span\{P_k\}$. Define the projection $E : \mathscr{L}^2 \to \mathscr{L}^2$ onto $[\mathrm{Ker}\,(L)]^{\perp}$ =$\mathrm{Im}\,(L)$ by $E = I - U$. Note that the map L restricted to $D(L)\cap \mathrm{Im}\,(L)$ is a bijection onto $\mathrm{Im}\,(L)$ =$\mathrm{Im}\,(E)$. Therefore, it follows that there exists a linear map M :$\mathrm{Im}\,(E) \to D(L)\cap\mathrm{Im}\,(L)$ satisfying $LMh = h$, for all $h \in\mathrm{Im}\,(L)$ and $MLx = Ex = (I-U)x$, for all $x \in D(L)$. In fact, we can represent this map M with the eigenfunction expansion

$$[Mh](t) = \sum_{l\neq k} \frac{(l+\frac{1}{2})\langle h, P_l\rangle}{[\mu - l(l+1)]} P_l(t).$$

Note that $M : Im(L) \to Im(L)\cap D(L)$ is a compact operator as a consequence of the argument appearing in Lemma 1 along with the fact that $Im(L)$ is a closed subspace of $\mathscr{L}^2$. Using these projections, we analyze the operator equation $Lx = F(x)$ in the following way:

$$Lx = F(x) \iff \begin{cases} E(Lx - F(x)) = 0 \\ \qquad \text{and} \\ (I-E)(Lx - F(x)) = 0 \end{cases}$$

$$\iff \begin{cases} (I-U)x - MEF(x) = 0 \\ \qquad \text{and} \\ F(x) \in Im(L) \end{cases}$$

$$\iff \begin{cases} x = Ux + MEF(x) \\ \qquad \text{and} \\ \int_{-1}^{1} f(x(t))P_k(t)dt = 0 \end{cases}$$

$$\iff \begin{cases} x = \alpha P_k + w(x) \\ \qquad \text{and} \\ \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt = 0, \end{cases}$$

where $w(x) = MEF(x)$.

Define the constants J_1 and J_2 as follows:

$$J_1 = f(\infty)\int_{\{t:P_k(t)>0\}} P_k(t)dt + f(-\infty)\int_{\{t:P_k(t)<0\}} P_k(t)dt,$$
$$J_2 = f(\infty)\int_{\{t:P_k(t)<0\}} P_k(t)dt + f(-\infty)\int_{\{t:P_k(t)>0\}} P_k(t)dt.$$

Note that if $k = 0$, then $J_1 = g(\infty)$ and $J_2 = g(-\infty)$. If $k \geqslant 1$, then

$$J_1 = \left(\int_{\{t:P_k(t)>0\}} P_k(t)dt\right)[f(\infty) - f(-\infty)],$$
$$J_2 = \left(\int_{\{t:P_k(t)>0\}} P_k(t)dt\right)[f(-\infty) - f(\infty)].$$

THEOREM 2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(-\infty)$ and $f(\infty)$ exist and are finite. Then we can guarantee a solution to the boundary value problem* (1)-(2) *in either of the following cases:*

(i) if $k = 0$ and $f(-\infty)f(\infty) < 0$,

(ii) if $k \geqslant 1$ and $f(-\infty) \neq f(\infty)$.

Proof. We begin by noting that

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt$$
$$= \int_{\{t:P_k(t)<0\}} f[\alpha P_k(t) + w(x(t))]P_k(t)dt + \int_{\{t:P_k(t)>0\}} f[\alpha P_k(t) + w(x(t))]P_k(t)dt.$$

Since w is bounded, we have by the Lebesgue dominated convergence theorem that

$$\lim_{\alpha\to\infty}\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt$$
$$= f(\infty)\int_{\{t:P_k(t)>0\}} P_k(t)dt + f(-\infty)\int_{\{t:P_k(t)<0\}} P_k(t)dt = J_1$$

and

$$\lim_{\alpha\to-\infty}\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt$$
$$= f(\infty)\int_{\{t:P_k(t)<0\}} P_k(t)dt + f(-\infty)\int_{\{t:P_k(t)>0\}} P_k(t)dt = J_2.$$

Condition iii) implies that $J_1J_2 < 0$ and we proceed by supposing without loss of generality that $J_2 < 0 < J_1$.

Therefore there exists $\alpha_0 > 0$ such that for $\alpha \geqslant \alpha_0$

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt > 0 \tag{3}$$

and for $\alpha \leqslant -\alpha_0$

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt < 0. \tag{4}$$

Note that M is a compact linear map from $Im(L)$ onto $D(L) \cap Im(L)$ and E is a projection, so w is a nonlinear compact mapping.
Define $H_1 : \mathscr{L}^2 \times \mathbb{R} \to \mathscr{L}^2$ by

$$H_1(x, \alpha) = \alpha P_k + w(x)$$

and $H_2 : \mathscr{L}^2 \times \mathbb{R} \to \mathbb{R}$ by

$$H_2(x, \alpha) = \alpha - \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt.$$

Let $H : \mathscr{L}^2 \times \mathbb{R} \to \mathscr{L}^2 \times \mathbb{R}$ be defined by

$$H(x, \alpha) = (H_1(x, \alpha), H_2(x, \alpha)).$$

Guaranteeing a fixed point for H is equivalent to guaranteeing a solution to (1)-(2). We endow the space $\mathscr{L}^2 \times \mathbb{R}$ with the norm

$$\|(x, \alpha)\| = \max\{\|x\|, |\alpha|\}.$$

Define

$$r = \sup_{t \in \mathbb{R}} |f(t)|.$$

The existence of r is guaranteed by the continuity of $f : \mathbb{R} \to \mathbb{R}$ along with the fact that $f(\infty)$ and $f(-\infty)$ exist and are finite. Choose $\alpha_0 > r$ so that (3) and (4) are satisfied and let $\delta = \alpha_0 + r$. As stated in [3], $|P_k(t)| \leqslant 1$, for all $t \in [-1, 1]$. We know that f and ME are bounded, so there exists $b_1 > 0$ such that for any $x \in \mathscr{L}^2$, $\alpha \in \mathbb{R}$

$$\|H_1(x, \alpha)\| \leqslant b_1.$$

Let $\mathscr{B}$ be the set

$$\mathscr{B} = \left\{(x, \alpha) \in \mathscr{L}^2 \times \mathbb{R} : \|x\| \leqslant b_1, |\alpha| \leqslant \delta\right\}.$$

Clearly $\|H_1(x, \alpha)\| \leqslant b_1$ for all $(x, \alpha) \in \mathscr{B}$ by construction, so it suffices to show that $\|H_2(x, \alpha)\| \leqslant \delta$ for all $(x, \alpha) \in \mathscr{B}$ in order to show that $H(\mathscr{B}) \subset \mathscr{B}$.
Suppose that $\alpha \in [\alpha_0, \delta]$. Then

$$\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt > 0$$

and therefore $H_2(x,\alpha) < \alpha \leqslant \delta$. Further, since $\left|\int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt\right| \leqslant r$ it follows that

$$\alpha - \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt \geqslant \alpha_0 - r \geqslant 0.$$

Therefore, if $\alpha \in [\alpha_0, \delta]$, then $|H_2(x,\alpha)| \in [0,\delta]$. Suppose that $\alpha \in [0,\alpha_0)$. Then

$$|H_2(x,\alpha)| = \left|\alpha - \int_{-1}^{1} f[\alpha P_k(t) + w(x(t))]P_k(t)dt\right| \leqslant \alpha_0 + r = \delta.$$

Therefore, if $(x,\alpha) \in \mathscr{B}$ and $\alpha \in [0,\delta]$, then $|H_2(x,\alpha)| \leqslant \delta$.

A symmetric argument can be used to show that if $(x,\alpha) \in \mathscr{B}$ and $\alpha \in [-\delta, 0]$, then $|H_2(x,\alpha)| \leqslant \delta$. Therefore, $H(\mathscr{B}) \subset \mathscr{B}$. Since $H : \mathscr{L}^2 \times \mathbb{R} \to \mathscr{L}^2 \times \mathbb{R}$ is compact (following from the compactness of M) and $\mathscr{B}$ is closed, bounded and convex, it follows that H is guaranteed a fixed point by Schauder's fixed point theorem. □

2.3. The case of weak nonlinearities

In this subsection, assume that our nonlinearity is of the form $\varepsilon f(x(t))$, where ε is a real parameter and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. That is, we now examine boundary value problems of the form

$$\left[\left(1-t^2\right)x'(t)\right]' + \mu x(t) = \varepsilon f(x(t))$$

subject to the condition that the limits appearing in (2) exist and are finite. Due to the fact that we will impose differentiability conditions on the function-valued operator representing our nonlinearity, we consider operators defined on the space of continuous functions. Again let $\mathscr{C}$ denote the space of continuous functions on $[-1,1]$ endowed with the supremum norm and let

$$\mathscr{D} = \left(C^2[-1,1], \|\cdot\|_\infty\right) \subset \mathscr{C},$$

where $C^2[-1,1]$ denotes the set of twice continuously differentiable functions on $[-1,1]$. In this section, denote $\mathscr{L} : \mathscr{D} \to \mathscr{C}$ by

$$[\mathscr{L}x](t) = \left[\left(1-t^2\right)x'(t)\right]' + \mu x(t)$$

and $F : \mathscr{C} \times \mathbb{R} \to \mathscr{C}$ by

$$[F(x,\varepsilon)](t) = \varepsilon f(x(t)).$$

Suppose again that $\mu = k(k+1)$.

In this section, for $x \in \mathscr{C}$ and $l \in \mathbb{N}$ we denote

$$x_l = \left[\left(l + \frac{1}{2}\right)\int_{-1}^{1} P_l(t)x(t)dt\right].$$

Define the projections $U:\mathscr{C}\to\mathscr{C}$ by

$$[Ux](t) = x_k P_k(t)$$

and $E:\mathscr{C}\to\mathscr{C}$ by $E = I - U$. Note that the map $\mathscr{L}$ restricted to $\mathscr{D}\cap \operatorname{Im}(L)$ is a bijection onto $\operatorname{Im}(L) = \operatorname{Im}(E)$. Therefore, it follows that there exists a linear map $M:\operatorname{Im}(E)\to\mathscr{D}\cap \operatorname{Im}(L)$ satisfying

$$\mathscr{L}Mh = h,$$

for all $h\in \operatorname{Im}(L)$ and

$$M\mathscr{L}x = Ex = (I-U)x,$$

for all $x\in\mathscr{D}$. Note that M is simply

$$\left[\mathscr{L}|_{\mathscr{D}\cap\operatorname{Im}(L)}\right]^{-1}$$

and observe further that M is continuous. We note that solving

$$\mathscr{L}x = F(x,\varepsilon)$$

is equivalent to solving the system

$$\begin{cases} (I-U)x - MEF(x,\varepsilon) = 0 \\ \qquad\text{and} \\ U(f\circ x) = 0. \end{cases}$$

Define the map $G:\mathscr{D}\times\mathbb{R}\to \operatorname{Im}(L)\times \operatorname{Ker}(L)$ by

$$G(x,\varepsilon) = \begin{bmatrix} (I-U)x - MEF(x,\varepsilon) \\ U(f\circ x) \end{bmatrix}.$$

It is well known that F is continuously differentiable with respect to x and for any $x\in\mathscr{C}$, $\varepsilon\in\mathbb{R}$

$$\left(\frac{\partial F}{\partial x}(x,\varepsilon)h\right)(t) = \varepsilon f'(x(t))h(t).$$

From that it follows that

$$\frac{\partial G}{\partial x}(x,\varepsilon)$$

exists for all $(x,\varepsilon)\in\mathscr{C}\times\mathbb{R}$ and is given by

$$\frac{\partial G}{\partial x}(x,\varepsilon)w = \begin{bmatrix} [(I-U) - \varepsilon ME(f'\circ x)]w \\ U(f'\circ x)w \end{bmatrix}.$$

Let $\bar{x} = \alpha P_k$, for $\alpha \in \mathbb{R}$. For $(\bar{x}, 0)$ and $w \in \mathscr{C}$:

$$\frac{\partial G}{\partial x}(\bar{x},0)w = \begin{bmatrix} (I-U)w \\ U(f' \circ \bar{x})w \end{bmatrix}.$$

Since $F \in C^1$ and M is continuous, it follows that $G \in C^1$. For $w \in \mathscr{C}$, we can decompose w as $w = u + v$, where

$$\begin{aligned} u &= w_k P_k, \\ v &= w - w_k P_k. \end{aligned}$$

With this in mind, we can rewrite the previous expression as

$$\frac{\partial G}{\partial x}(\bar{x},0)(u+v) = \begin{bmatrix} v \\ U(f' \circ \bar{x})(u+v) \end{bmatrix}.$$

Define the maps $H_1 : \mathrm{Ker}(L) \to \mathbb{R}$ by

$$H_1(u) = \int_{-1}^{1} P_k(t) f(u(t))dt,$$

$H_2 : \mathbb{R} \to \mathrm{Ker}(L)$ by $H_2(\alpha) = \alpha P_k$ and finally $H : \mathbb{R} \to \mathbb{R}$ by $H = H_1 \circ H_2$. That is,

$$H(\alpha) = \int_{-1}^{1} P_k(t) f(\alpha P_k(t))dt.$$

Therefore for any number in $\mathbb{R}$, $H' :\ \mathrm{Ker}(L) \to \mathbb{R}$ exists and for $\beta \in \mathbb{R}$

$$[H'(\alpha)](\beta) = \int_{-1}^{1} P_k(t)[f'(\alpha P_k(t))](\beta P_k(t))dt.$$

We are now ready to give conditions for the solvability of our boundary value problems examined this section.

THEOREM 3. *Suppose that there exists $\alpha_0 \in \mathbb{R}$ such that $H(\alpha_0) = 0$ and $H'(\alpha_0) \neq 0$. Then there exists and open neighborhood $I \subset \mathbb{R}$ of 0 such that for any $\varepsilon \in I$ there exists a solution to*

$$\left[\left(1-t^2\right)x'(t)\right]' + \mu x(t) = \varepsilon f(x(t))$$

satisfying the condition that the limits appearing in (2) exist and are finite.

Proof. Recall that $G \in C^1$ and let $\bar{x} = \alpha_0 P_k$. Then $(I-U)\bar{x} - MEF(\bar{x},0) = 0$ and

$$UF(\bar{x}) = \int_{-1}^{1} P_k(t) f(\alpha_0 P_k(t))dt = H(\alpha_0 P_k(t)) = 0.$$

Therefore $G(\bar{x},0) = 0$. We now wish to show that $\frac{\partial G}{\partial x}(\bar{x},0)$ is a bijection from $\mathscr{C}$ onto $\operatorname{Im}(L) \times \operatorname{Ker}(L)$. Since $\frac{\partial G}{\partial x}(\bar{x},0)$ is linear, in order to show this map is injective it suffices to show that it has a trivial kernel. Suppose that $\frac{\partial G}{\partial x}(\bar{x},0)(u+v) = 0$. Then

$$0 = v$$

and so

$$0 = U(f' \circ \bar{x})u = \left[\int_{-1}^{1} P_k(t)[f'(\alpha_0 P_k(t))]u(t)dt \right],$$

implying that $u = 0$, due to our assumption that $H'(\alpha_0) \neq 0$. Note that since $H'(\alpha_0)$ is a nonzero linear map from $\mathbb{R} \to \mathbb{R}$, then it is a bijection from $\mathbb{R}$ onto $\mathbb{R}$. This implies that the map $U(f' \circ \bar{x})$ restricted to $\operatorname{Ker}(L)$ is a bijection onto $\operatorname{Ker}(L)$. Given $h_1 \in \operatorname{Im}(L)$ and $h_2 \in \operatorname{Ker}(L)$, we have that

$$\frac{\partial G}{\partial x}(\bar{x},0)\left(h_1 + \hat{h}_2\right) = (h_1, h_2),$$

where $\hat{h}_2$ is the unique element of $\operatorname{Ker}(L)$ that maps to h_2 under $U(f' \circ \bar{x})$. So $\frac{\partial G}{\partial x}(\bar{x},0)$ is surjective and therefore a bijection from $\mathscr{C}$ onto $\operatorname{Im}(L) \times \operatorname{Ker}(L)$. By the implicit function theorem [12], there exists a neighborhood $V_0 \subset \mathbb{R}$ of 0 on which there exists a continuous function $\phi : V_0 \to \mathscr{D}$ satisfying

$$G(\phi(\varepsilon), \varepsilon) = 0,$$

for all $\varepsilon \in V_0$. Denoting $\phi(\varepsilon) = x_\varepsilon$ we have that

$$0 = G(\phi(\varepsilon), \varepsilon) = G(x_\varepsilon, \varepsilon) = \mathscr{L} x_\varepsilon - F(x_\varepsilon, \varepsilon).$$

In other words, for any $\varepsilon \in V_0$ we can guarantee a solution to

$$[(1-t^2)x'(t)]' + \mu x(t) = \varepsilon f(x(t))$$

satisfying the condition that the limits in (2) exist and are finite. $\square$

REMARK 1. Let x_ε denote the solution in $\mathscr{D}$ guaranteed by the implicit function theorem to

$$\left[\left(1-t^2\right)x'(t)\right]' + \mu x(t) = \varepsilon f(x(t)).$$

Note that

$$\lim_{\varepsilon \to 0} x_\varepsilon = \bar{x},$$

where this limit is in the sense of uniform convergence. That is, solutions guaranteed by the above theorem are ones that emanate from a certain solution to the linear homogeneous problem.

EXAMPLE 1. Consider the boundary value problem

$$\left[\left(1-t^2\right)x'(t)\right]' = \varepsilon f(x(t))$$

on $(-1,1)$ subject to the condition that the limits in (2) exist and are finite.

Suppose that there exists a number α_0 such that $f(\alpha_0)=0$ and $f'(\alpha_0)\neq 0$. Then, since the constant Legendre polynomial is $P_0(t)=1$, for $\alpha\in\mathbb{R}$

$$\int_{-1}^{1} P_0(t)f(\alpha P_0(t))dt = \int_{-1}^{1} f(\alpha)dt,$$

so then $\int_{-1}^{1} P_0(t)f(\alpha_0 P_0(t))dt = 0$. However, provided $\beta\neq 0$,

$$\int_{-1}^{1} P_0(t)[f'(\alpha_0 P_0(t))](\beta P_0(t))dt = \beta\int_{-1}^{1} f'(\alpha_0)dt,$$

so then $\int_{-1}^{1} P_0(t)[f'(\alpha_0 P_0(t))](\beta P_0(t))dt \neq 0$.

EXAMPLE 2. Consider the boundary value problem

$$\left[\left(1-t^2\right)x'(t)\right]' + 2x(t) = \varepsilon f(x(t))$$

subject to the condition that the limits in (2) exist and are finite. The constant Legendre polynomial is $P_1(t)=t$, so the condition in Theorem 3 is satisfied provided there exists a number α_0 satisfying

$$\int_{-1}^{1} t f(\alpha_0 t)dt = 0$$

and $f(\alpha_0)\neq f(-\alpha_0)$.

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