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POSITIVE SOLUTIONS FOR A SINGULAR COUPLED SYSTEM OF NONLINEAR HIGHER–ORDER FRACTIONAL q–DIFFERENCE BOUNDARY VALUE PROBLEMS WITH TWO PARAMETERS

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Abstract. In this paper, we are concern with the existence of positive solutions for a singular system of nonlinear fractional q-difference equations with coupled integral boundary conditions and two parameters. By using the properties of the Green's function and Guo-Krasnosel'skii fixed point theorem, some existence results of at least one positive solution are obtained. As applications, two examples are presented to illustrate the main results.

1. Introduction

The subject of fractional differential equations has gained considerable popularity and importance due mainly to its fact that fractional differential equations describe many phenomena than the corresponding integer order differential equations in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, etc. For the theory and applications of fractional calculus, reader can see [18, 22]. Many researchers pay more attentions to the existence of positive solutions for a system of nonlinear fractional differential equations with integral and multi-point boundary conditions, see [9, 13, 21, 25, 26, 34] and the references therein. In [35], by applying a nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorems, Yuan et al. considered the multiple positive solutions for four-point coupled boundary value problem for systems of the nonlinear semipositone fractional differential equation. In [14, 16], the authors investigated the existence of positive solutions for nonlinear Riemann-Liouville fractional differential equations with coupled integral boundary conditions, respectively. In [24], Wang et al. studied the existence of a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters. By using the properties of the Green's function and the Guo-Krasnosel'skii fixed point theorem, Henderson and Luca [15] focused on the introduction of positive solutions for boundary value problems for systems of fractional differential equations. By applying the corresponding Green's

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function and Guo-Krasnosel'skii fixed point theorems, the author [30, 31] investigated the positive solutions for nonlinear semipositone Hadamard fractional differential systems with coupled integral and four-point coupled boundary conditions, respectively.

Recently, fractional q-difference equations, regarded as q-analog of fractional differential equations, have been studied by a lot of researchers. In papers [3, 5, 7, 8, 11, 19, 20, 23, 27, 28, 32, 38], the authors studied the existence and uniqueness of solutions or positive solutions for the nonlinear fractional q-difference equations with boundary conditions by using some standard fixed point theorems as well as monotone iterative technique and lower-upper solution method. By applying the properties of the Green function, the upper and lower solutions method and some well-known fixed-point theorems, Yuan and Yang [36, 37] considered the positive solutions to nonlinear boundary value problems for delayed fractional q-difference systems and four-point boundary value problems of fractional q-difference equations with p-Laplacian operator, respectively. In [1] and [2], some important q-fractional inequalities were proved. Those inequalities are necessary for the development of q-fractional systems. In [4], by applying some standard fixed point theorems, Ahmad et al. showed some existence results for sequential q-fractional integro-differential equations with nonlocal four-point boundary conditions. Graef and Kong [10] investigated the existence of positive solutions for boundary value problems with fractional q-derivatives. In [40], Zhou and Liu obtained the uniqueness and existence of solutions for fractional q-difference system with four-point boundary conditions obtained based on the nonlinear alternative of Leray-Schauder type and Banach's fixed point theorem. In [29], by applying the nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorems, the author gave the existence results of nonlinear semipositone fractional q-difference systems with boundary value conditions. By using some well-known fixed point theorems, Yang and Qin [33] investigated the existence of positive solutions for a class of nonlinear Caputo type fractional q-difference equations with integral boundary conditions. By applying a mixed monotone method and Guo-Krasnoselskii fixed point theorem, Zhao and Yang [39] studied the existence and uniqueness results of positive solutions for the singular coupled integral boundary value problem of nonlinear higher-order fractional q-difference equations.

Motivated by the wide applications of coupled boundary value problems and the results mentioned above, we consider the following singular fractional q-difference systems and two parameters

$$D_q^{\alpha_1} u(t) + \lambda_1 f_1(t,u(t),v(t)) = 0,\ \ D_q^{\alpha_2} v(t) + \lambda_2 f_2(t,u(t),v(t)) = 0, \quad t \in (0,1), \tag{1}$$

with the coupled integral boundary value conditions

$$\begin{aligned} &D_q^{j_1} u(0) = D_q^{j_2} v(0) = 0, \quad 0 \leqslant j_i \leqslant n_i - 2, \\ &u(1) = \mu_1 \int_0^1 g_1(s)v(s)d_qs, \quad v(1) = \mu_2 \int_0^1 g_2(s)u(s)d_qs, \end{aligned} \tag{2}$$

where $\mu_i > 0$, $\alpha_i \in (n_i - 1, n_i]$ with $3 \leqslant n_i \in \mathbb{N}$, D^{α_i} is the Riemann-Liouville type fractional q-derivative of fractional order α_i, $i = 1,2$; λ_1, λ_2 are two positive parameters, $f_1, f_2 : (0,1) \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ are two continuous functions and may be singular at $t = 0,1$. In this article, we will obtain some existence results of at least one positive solution for singular coupled boundary value problem (1) and (2) by applying the properties of the Green's function and Guo-Krasnoselskii fixed point theorem. At the end, two examples are given to illustrate our main results.

2. Preliminaries

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of the q-fractional boundary value problem (1). These details can be found in the recent literature; see [6, 17] and references therein.

Let $q \in (0,1)$ and define

$$[a]_q = \frac{q^a - 1}{q-1}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a-b)^n$ with $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ is

$$(a-b)^{(0)} = 1, \quad (a-b)^{(n)} = \prod_{k=0}^{n-1}(a-bq^k), \quad n \in \mathbb{N}_0, \ a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b = 0$, then $a^{(\alpha)} = a^\alpha$. Here we point out that the following equality holds

$$(a-b)^{(\alpha)} = a(a-bq^{\alpha-1})(a-b)^{(\alpha-1)}.$$

The q-gamma function is defined by

$$\Gamma_q(x) = (1-q)^{(x-1)}(1-q)^{1-x}, \ \ x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x \to 0}(D_q f)(x)$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [6]. We now point out five formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$\int_a^b f(s)(D_q g)(s)d_qs = [f(s)g(s)]_{s=a}^{s=b} - \int_a^b (D_q f)(s)g(qs)d_qs \ \ (q\text{−integration by parts}),$$
$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)}, \ \ {}_sD_q(t-s)^{(\alpha)} = -[\alpha]_q(t-qs)^{(\alpha-1)},$$
$$[a(t-s)]^{(\alpha)} = a^\alpha(t-s)^{(\alpha)}, \ \ \left({}_xD_q \int_0^x f(x,t)d_qt\right)(x) = \int_0^x {}_xD_q f(x,t)d_qt + f(qx,x).$$

Denote that if $\alpha > 0$ and $a \leqslant b \leqslant t$, then $(t-a)^{(\alpha)} \geqslant (t-b)^{(\alpha)}$ [7].

DEFINITION 2.1. [18] Let $\alpha \geqslant 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt, \ \ \alpha > 0, \ \ x \in [0,1].$$

DEFINITION 2.2. [6] The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geqslant 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

LEMMA 2.1. [6] *Let* $\alpha, \beta \geqslant 0$ *and* f *be a function defined on* $[0,1]$. *Then the next formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = I_q^{\alpha+\beta} f(x)$;

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

LEMMA 2.2. [8] *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}(D_q^k f)(0).$$

For the sake of simplicity, we always assume that the following assumptions hold.

(H1) $g_1, g_2 : [0,1] \to [0,\infty)$ are two continuous functions and satisfy

$$\nu_1 = \int_0^1 s^{\alpha_2-1} g_1(s) d_q s, \;\; \nu_2 = \int_0^1 s^{\alpha_1-1} g_2(s) d_q s, \;\; \kappa = 1 - \mu_1\mu_2\nu_1\nu_2 > 0.$$

(H2) $f_i : (0,1)\times[0,+\infty)\times[0,+\infty) \to (-\infty,+\infty)$ is continuous and satisfies the following inequality $-q_i(t) \leqslant f_i(t,u,v) \leqslant p_i(t)h_i(t,u,v)$, $(t,u,v) \in (0,1)\times[0,+\infty)^2$, $i = 1,2$, where $h_i \in C([0,1]\times[0,+\infty)^2, [0,+\infty))$, $q_i, p_i \in C((0,1),[0,+\infty))$, and

$$0 < \int_0^1 p_i(s) d_q s < +\infty, \quad 0 < \int_0^1 q_i(s) d_q s < +\infty, \quad i = 1,2.$$

LEMMA 2.3. [39] *Assume that* (**H1**) *holds. Then for $x, y \in C[0,1]$, the boundary value problem*

$$D_q^{\alpha_1} u(t) + x(t) = 0, \quad D_q^{\alpha_2} v(t) + y(t) = 0, \quad t \in (0,1),$$

with the coupled integral boundary value conditions (2) *has a unique integral representation*

$$\begin{aligned} u(t) &= \int_0^1 K_1(t,qs)x(s) d_q s + \int_0^1 H_1(t,qs) y(s) d_q s, \\ v(t) &= \int_0^1 K_2(t,qs) y(s) d_q s + \int_0^1 H_2(t,qs) x(s) d_q s, \end{aligned}$$

where

$$\begin{aligned} K_1(t,s) =& G_1(t,s) + \kappa^{-1}\mu_1\mu_2\nu_1 t^{\alpha_1-1}\int_0^1 g_2(\tau)G_1(\tau,s) d_q\tau, \\ H_1(t,s) =& \kappa^{-1}\mu_1 t^{\alpha_1-1}\int_0^1 g_1(\tau)G_2(\tau,s) d_q\tau, \\ K_2(t,s) =& G_2(t,s) + \kappa^{-1}\mu_1\mu_2\nu_2 t^{\alpha_2-1}\int_0^1 g_1(\tau)G_2(\tau,s) d_q\tau, \\ H_2(t,s) =& \kappa^{-1}\mu_2 t^{\alpha_2-1}\int_0^1 g_2(\tau)G_1(\tau,s) d_q\tau, \\ G_i(t,s) =& \frac{1}{\Gamma_q(\alpha_i)} \begin{cases} t^{\alpha_i-1}(1-s)^{(\alpha_i-1)} - (t-s)^{(\alpha_i-1)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha_i-1}(1-s)^{(\alpha_i-1)}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \quad i = 1,2. \end{aligned}$$

LEMMA 2.4. [39] *The functions* $K_i(t,s)$ *and* $H_i(t,s)$ $(i=1,2)$ *defined by Lemma 2.3 satisfy the following conditions:*

(a) $K_i(t,s)$ *and* $H_i(t,s)$ *are continuous functions on* $(t,s)\in[0,1]\times[0,1]$ *and* $K_i(t,qs)\geqslant 0$ *and* $H_i(t,qs)\geqslant 0$*, for* $(t,s)\in[0,1]^2$*,* $i=1,2$*;*

(b) $\mathsf{p}t^{\alpha_i-1}\varphi_i(qs)\leqslant K_i(t,qs)\leqslant\rho\varphi_i(qs)$*,* $K_i(t,qs)\leqslant\rho t^{\alpha_i-1}$*,* $\mathsf{p}t^{\alpha_1-1}\varphi_2(qs)\leqslant H_1(t,qs)\leqslant\rho\varphi_2(qs)$*,* $\mathsf{p}t^{\alpha_2-1}\varphi_1(qs)\leqslant H_2(t,qs)\leqslant\rho\varphi_1(qs)$ *and* $H_i(t,qs)\leqslant\rho t^{\alpha_i-1}$ *for* $\psi_i(t)=t^{\alpha_i-1}(1-t)$ *and* $\varphi_i(s)=(1-s)^{\alpha_i-1}s$*,* $(t,s)\in[0,1]^2$*,* $i=1,2$*, where* $\mathsf{p}=\min\{\mathsf{p}_1,\mathsf{p}_2,\mathsf{p}_3,\ \mathsf{p}_4\}$*,* $\rho=\max\{\rho_1,\rho_2,\rho_3,\rho_4\}$ *and*

$$\mathsf{p}_1=\frac{q^{\alpha_1-2}\mu_1\mu_2\nu_1}{\kappa\Gamma_q(\alpha_1)}\int_0^1 g_2(\tau)\psi_1(\tau)d_q\tau,\ \ \mathsf{p}_2=\frac{q^{\alpha_2-2}\mu_1}{\kappa\Gamma_q(\alpha_2)}\int_0^1 g_1(\tau)\psi_2(\tau)d_q\tau,$$
$$\mathsf{p}_3=\frac{q^{\alpha_2-2}\mu_1\mu_2\nu_2}{\kappa\Gamma_q(\alpha_2)}\int_0^1 g_1(\tau)\psi_2(\tau)d_q\tau,\ \ \mathsf{p}_4=\frac{q^{\alpha_1-2}\mu_2}{\kappa\Gamma_q(\alpha_1)}\int_0^1 g_2(\tau)\psi_1(\tau)d_q\tau,$$
$$\rho_1=\frac{[\alpha_1-1]_q}{\Gamma_q(\alpha_1)}\left(1+\kappa^{-1}\mu_1\mu_2\nu_1\int_0^1 g_2(\tau)d_q\tau\right),\ \ \rho_2=\frac{\mu_1[\alpha_2-1]_q}{\kappa\Gamma_q(\alpha_2)}\int_0^1 g_1(\tau)d_q\tau,$$
$$\rho_3=\frac{[\alpha_2-1]_q}{\Gamma_q(\alpha_2)}\left(1+\kappa^{-1}\mu_1\mu_2\nu_2\int_0^1 g_1(\tau)d_q\tau\right),\ \ \rho_4=\frac{\mu_2[\alpha_1-1]_q}{\kappa\Gamma_q(\alpha_1)}\int_0^1 g_2(\tau)d_q\tau.$$

REMARK 2.1. [39] From Lemma 2.4, for $t,\tau,s\in[0,1]$, we have

$$K_1(t,qs)\geqslant\omega t^{\alpha_1-1}H_2(\tau,qs),K_2(t,qs)\geqslant\omega t^{\alpha_2-1}H_1(\tau,qs),H_1(t,qs)\geqslant\omega t^{\alpha_1-1}K_2(\tau,qs),$$
$$H_2(t,qs)\geqslant\omega t^{\alpha_2-1}K_1(\tau,qs),K_i(t,qs)\geqslant\omega t^{\alpha_i-1}K_i(\tau,qs),H_i(t,qs)\geqslant\omega t^{\alpha_i-1}H_i(\tau,qs),$$

where $i=1,2$, $\omega=\mathsf{p}/\rho$, p,ρ are defined as Lemma 2.4, $0<\omega<1$.

LEMMA 2.5. *Assume that* (**H1**) *and* (**H2**) *hold, then the coupled integral boundary value problem*

$$-D_q^{\alpha_1}\overline{\omega}_1(t)=\lambda_1 q_1(t),\quad -D_q^{\alpha_2}\overline{\omega}_2(t)=\lambda_2 q_2(t),\quad t\in(0,1),\quad \lambda>0,$$

with the coupled integral boundary value conditions

$$D_q^{j_1}\overline{\omega}_1(0)=D_q^{j_2}\overline{\omega}_2(0)=0,\overline{\omega}_1(1)=\mu_1\int_0^1 g_1(s)\overline{\omega}_2(s)d_qs,\overline{\omega}_2(1)=\mu_2\int_0^1 g_2(s)\overline{\omega}_1(s)d_qs,$$

where $0\leqslant j_i\leqslant n_i-2$*, has an unique integral representation*

$$\begin{cases}\overline{\omega}_1(t)=\lambda_1\displaystyle\int_0^1 K_1(t,qs)q_1(s)d_qs+\lambda_2\int_0^1 H_1(t,qs)q_2(s)d_qs,\\ \overline{\omega}_2(t)=\lambda_2\displaystyle\int_0^1 K_2(t,qs)q_2(s)d_qs+\lambda_1\int_0^1 H_2(t,qs)q_1(s)d_qs,\end{cases}$$

which satisfies

$$\overline{\omega}_i(t)\leqslant\rho t^{\alpha_i-1}\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right),\ \ t\in[0,1],\ \ i=1,2.\qquad(3)$$

Proof. It follows from Lemmas 2.3 and 2.4 and the condition (**H2**) that the proof of Lemma 2.5 is easily proved. □

Let $X = C[0,1]\times C[0,1]$, then X is a Banach space with the norm $\|(u,v)\| = \max\{\|u\|,\|v\|\}$, $\|u\| = \max_{t\in[0,1]}|u(t)|$, $\|v\| = \max_{t\in[0,1]}|v(t)|$. Denote $P = \{(u,v)\in X : u(t)\geqslant \omega t^{\alpha_1-1}\|(u,v)\|, v(t)\geqslant \omega t^{\alpha_2-1}\|(u,v)\|,\ t\in[0,1]\}$, where ω is defined as Remark 2.1. It is easy to see that P is a positive cone in X. It can be easily seen that P is a cone in X. For any real constants r and R with $0<r<R$, define $P_r = \{(u,v)\in P : \|(u,v)\|<r\}$, $P_{[r,R]} = \{(u,v)\in P : r\leqslant\|(u,v)\|\leqslant R\}$.

Next we only consider the following fractional q-difference system with boundary conditions (2):

$$\begin{cases} D_q^{\alpha_1}u(t)+\lambda_1(f_1(t,[u(t)-\overline{\omega}_1(t)]^*,[v(t)-\overline{\omega}_2(t)]^*)+q_1(t))=0,\ t\in(0,1),\ \lambda_1>0,\\ D_q^{\alpha_2}v(t)+\lambda_2(f_2(t,[u(t)-\overline{\omega}_1(t)]^*,[v(t)-\overline{\omega}_2(t)]^*)+q_2(t))=0,\ t\in(0,1),\ \lambda_2>0,\end{cases} \tag{4}$$

where a modified function $[z(t)]^*$ for any $z\in C[0,1]$ by $[z(t)]^*=z(t)$, if $z(t)\geqslant 0$, and $[z(t)]^*=0$, if $z(t)<0$.

LEMMA 2.6. *If $(u,v)\in X$ with $u(t)>\overline{\omega}_1(t)$ and $v(t)>\overline{\omega}_2(t)$, for any $t\in(0,1)$, is a positive solution of the singular system (4) and (2), then $(u-\overline{\omega}_1, v-\overline{\omega}_2)$ is a positive solution of the singular system (1) and (2).*

Proof. In fact, if $(u,v)\in X$ is a positive solution of the singular system (4) such that $u(t)>\overline{\omega}_1(t)$ and $v(t)>\overline{\omega}_2(t)$, for any $t\in(0,1]$, then from (4) and the definition of $[\cdot]^*$, we have the following system with boundary conditions (2):

$$\begin{cases} D_q^{\alpha_1}u(t)+\lambda_1(f_1(t,u(t)-\overline{\omega}_1(t),v(t)-\overline{\omega}_2(t))+q_1(t))=0,\ t\in(0,1),\ \lambda_1>0,\\ D_q^{\alpha_2}v(t)+\lambda_2(f_2(t,u(t)-\overline{\omega}_1(t),v(t)-\overline{\omega}_2(t))+q_2(t))=0,\ t\in(0,1),\ \lambda_2>0.\end{cases} \tag{5}$$

Let $x=u-\overline{\omega}_1$ and $y=v-\overline{\omega}_2$, then $D_q^{\alpha_1}x(t)=D_q^{\alpha_1}u(t)-D_q^{\alpha_1}\overline{\omega}_1(t)$ and $D_q^{\alpha_2}y(t)=D_q^{\alpha_2}v(t)-D_q^{\alpha_2}\overline{\omega}_2(t)$, for $t\in(0,1)$, which imply that $-D_q^{\alpha_1}x(t)=-D_q^{\alpha_1}u(t)-\lambda_1q_1(t)$, $-D_q^{\alpha_2}y(t)=-D_q^{\alpha_2}v(t)-\lambda_2q_2(t)$, $t\in(0,1)$. Thus (5) becomes

$$D_q^{\alpha_1}x(t)+\lambda_1f_1(t,x(t),y(t))=0, D_q^{\alpha_2}y(t)+\lambda_2f_2(t,x(t),y(t))=0, t\in(0,1),\lambda_1,\lambda_2>0,$$

with the coupled integral boundary value conditions

$$D_q^{j_1}x(0)=D_q^{j_2}y(0)=0,\ x(1)=\mu_1\int_0^1 g_1(s)y(s)d_qs,\ y(1)=\mu_2\int_0^1 g_2(s)x(s)d_qs,$$

where $0\leqslant j_i\leqslant n_i-2$, i.e., $(u-\overline{\omega}_1, v-\overline{\omega}_2)$ is a positive solution of the singular system (1) and (2). The proof is completed. □

Employing Lemma 2.3, the singular system (4) can be expressed as

$$\begin{aligned} u(t)=&\lambda_1\int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\\ &+\lambda_2\int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs,\end{aligned}$$

$$\begin{aligned} v(t) =& \lambda_2 \int_0^1 K_2(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs \\ &+ \lambda_1 \int_0^1 H_2(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs, \end{aligned} \tag{6}$$

for $t \in [0,1]$, we always assume that $x = u - \overline{\omega}_1$ and $y = v - \overline{\omega}_2$. By a solution of the singular system (4), we mean a solution of the corresponding system of integral equation (6). Defined an operator $\mathscr{T} : P \to P$ by $\mathscr{T}(u,v) = (\mathscr{T}_1(u,v), \mathscr{T}_2(u,v))$, where operators $\mathscr{T}_i : P \to C[0,1]$ $(i = 1,2)$ are defined by

$$\begin{aligned} \mathscr{T}_1(u,v)(t) =& \lambda_1 \int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs \\ &+ \lambda_2 \int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs, \\ \mathscr{T}_2(u,v)(t) =& \lambda_2 \int_0^1 K_2(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs \\ &+ \lambda_1 \int_0^1 H_2(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs, \end{aligned} \tag{7}$$

for $t \in [0,1]$. Clearly, if $(u,v) \in P$ is a fixed point of $\mathscr{T}$, then (u,v) is a solution of the singular system (1) and (2).

LEMMA 2.7. *Assume that* (**H1**) *and* (**H2**) *hold, then* $\mathscr{T} : P \to P$ *is a completely continuous operator.*

Proof. For any fixed $(u,v) \in P$, there exists a constant $L > 0$ such that $\|(u,v)\| \leqslant L$. Then we have

$$\begin{aligned} &[u(s) - \overline{\omega}_1(s)]^* \leqslant u(s) \leqslant \|u\| \leqslant \|(u,v)\| \leqslant L, \\ &[v(s) - \overline{\omega}_2(s)]^* \leqslant v(s) \leqslant \|v\| \leqslant \|(u,v)\| \leqslant L, \end{aligned}$$

where $s \in [0,1]$. For any $t \in [0,1]$, it follows from (7) and Lemma 2.5 that

$$\begin{aligned} \mathscr{T}_1(u,v)(t) =& \lambda_1 \int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs \\ &+ \lambda_2 \int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs \\ \leqslant& \lambda_1\rho \int_0^1 \varphi_1(qs)(p_1(s)h_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs \\ &+ \lambda_2\rho \int_0^1 \varphi_2(qs)(p_2(s)h_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs \\ \leqslant& \lambda_1\rho \int_0^1 \varphi_1(qs)(Mp_1(s)+q_1(s))d_qs + \lambda_2\rho \int_0^1 \varphi_2(qs)(Mp_2(s)+q_2(s))d_qs \\ \leqslant& \lambda_1 M\rho \int_0^1 (p_1(s)+q_1(s))d_qs + \lambda_2 M\rho \int_0^1 (p_2(s)+q_2(s))d_qs < +\infty, \end{aligned}$$

where $M = \max\left\{\max_{t\in[0,1],u,v\in[0,L]} h_1(t,u,v), \max_{t\in[0,1],u,v\in[0,L]} h_2(t,u,v)\right\}+1$. Similarly, we have

$$\mathscr{T}_2(u,v)(t) \leqslant \lambda_1 M\rho \int_0^1 (p_1(s)+q_1(s))d_qs + \lambda_2 M\rho \int_0^1 (p_2(s)+q_2(s))d_qs < +\infty.$$

Thus $\mathscr{T} : P \to C[0,1]\times C[0,1]$ is well defined.

Next, we show that $\mathscr{T} : P \to P$. For any fixed $(u,v)\in P$ and $t,\tau \in [0,1]$, by Remark 2.1, we obtain

$$\begin{aligned}
\mathscr{T}_1(u,v)(t) =&\lambda_1 \int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\\
&+\lambda_2 \int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\\
\geqslant&\lambda_1 \int_0^1 \omega t^{\alpha_1-1} K_1(\tau,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\\
&+\lambda_2 \int_0^1 \omega t^{\alpha_1-1} H_1(\tau,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\\
=&\omega t^{\alpha_1-1}\Big(\lambda_1 \int_0^1 K_1(\tau,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\\
&+\lambda_2 \int_0^1 H_1(\tau,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\Big) = \omega t^{\alpha_1-1}\mathscr{T}_1(u,v)(\tau),
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{T}_1(u,v)(t) =&\lambda_1 \int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\\
&+\lambda_2 \int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\\
\geqslant&\lambda_1 \int_0^1 \omega t^{\alpha_1-1} H_2(\tau,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\\
&+\lambda_2 \int_0^1 \omega t^{\alpha_1-1} K_2(\tau,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\\
=&\omega t^{\alpha_1-1}\Big(\lambda_1 \int_0^1 K_2(\tau,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\\
&+\lambda_2 \int_0^1 H_2(\tau,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\Big) = \omega t^{\alpha_1-1}\mathscr{T}_2(u,v)(\tau).
\end{aligned}$$

Then, we have

$$\mathscr{T}_1(u,v)(t) \geqslant \omega t^{\alpha_1-1}\|\mathscr{T}_1(u,v)\|, \quad \mathscr{T}_1(u,v)(t) \geqslant \omega t^{\alpha_1-1}\|\mathscr{T}_2(u,v)\|,$$

that is, $\mathscr{T}_1(u,v)(t) \geqslant \omega t^{\alpha_1-1}\|(\mathscr{T}_1(u,v),\mathscr{T}_2(u,v))\|$. In the same way, we can prove that

$$\mathscr{T}_2(u,v)(t) \geqslant \omega t^{\alpha_2-1}\|\mathscr{T}_2(u,v)\|, \quad \mathscr{T}_2(u,v)(t) \geqslant \omega t^{\alpha_2-1}\|\mathscr{T}_1(u,v)\|,$$

that is, $\mathscr{T}_2(u,v)(t) \geqslant \omega t^{\alpha_2-1}\|(\mathscr{T}_1(u,v),\mathscr{T}_2(u,v))\|$. This implies that $\mathscr{T}(P)\subset P$. According to the Ascoli-Arzela theorem, we can easily get that $\mathscr{T}: P\to P$ is completely continuous. The proof is completed. □

In order to obtain the main results in this paper, we will use the following fixed point theorem.

LEMMA 2.8. [12] *Let X be a Banach space, and let $P\subset X$ be a cone in X. Assume Ω_1,Ω_2 are open subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $S: P\to P$ be a completely continuous operator such that, either*

(a) $\|Sw\|\leqslant\|w\|,\ w\in P\cap\partial\Omega_1,\ \|Sw\|\geqslant\|w\|,\ w\in P\cap\partial\Omega_2$, *or*

(b) $\|Sw\|\geqslant\|w\|,\ w\in P\cap\partial\Omega_1,\ \|Sw\|\leqslant\|w\|,\ w\in P\cap\partial\Omega_2$.

Then S has at least one fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. Main results

THEOREM 3.1. *Assume that* **(H1)**, **(H2)** *hold and for any fixed $\lambda_1,\lambda_2\in(0,\infty)$, the following conditions hold:*

(H3) *There exists a constant $r_1>\max\left\{L_1,L_2,\frac{\rho}{\omega}\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\right\}$, such that $h_i(t,u,v)\leqslant(r_1/L_i)-1$ for $(t,u,v)\in[0,1]\times[0,r_1]^2,\ i=1,2$.*

(H4) $0<l_1\leqslant\liminf_{u\to+\infty}\inf_{\substack{t\in[c,d]\subset(0,1)\\ v\in[0,\infty)}}\frac{f_1(t,u,v)}{u}\leqslant\infty$, *or* $0<l_1\leqslant\liminf_{v\to+\infty}\inf_{\substack{t\in[c,d]\subset(0,1)\\ u\in[0,\infty)}}\frac{f_1(t,u,v)}{v}\leqslant\infty$,

where ω is defined as Remark 2.1, ρ is defined as Lemma 2.5, $\gamma=\min\{c^{\alpha_1-1},c^{\alpha_2-1}\}$,

$$L_i=3\left(\lambda_i\rho\int_0^1(p_i(s)+q_i(s))d_qs\right)^{-1},\quad i=1,2,\quad l_1=\frac{3}{2}\left(\lambda_1\rho\omega\gamma^2\int_c^d\varphi_1(qs)d_qs\right)^{-1}.$$

Then the singular coupled boundary value problem (1) and (2) has at least one positive solution $(\bar{u},\bar{v})$. Moreover, $(\bar{u},\bar{v})$ satisfies $\bar{u}(t)\geqslant\bar{l}\,t^{\alpha_1-1}$ and $\bar{v}(t)\geqslant\bar{l}\,t^{\alpha_2-1}$, $t\in[0,1]$, for some positive constant $\bar{l}$.

Proof. For any $(u,v)\in\partial P_{r_1}$ and $s\in[0,1]$, by the definition of $\|\cdot\|$, we know that

$$[u(s)-\overline{\omega}_1(s)]^*\leqslant u(s)\leqslant\|u\|\leqslant\|(u,v)\|\leqslant r_1,$$
$$[v(s)-\overline{\omega}_2(s)]^*\leqslant v(s)\leqslant\|v\|\leqslant\|(u,v)\|\leqslant r_1,$$

where $s \in [0,1]$. For any $(u,v) \in \partial P_{r_1}$, by condition (**H3**) and Lemma 2.5, we get

$$\begin{aligned}
\|\mathscr{T}_1(u,v)\| = &\max_{t\in[0,1]} \left|\lambda_1 \int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs \right. \\
&\left. +\lambda_2 \int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs \right| \\
\leqslant &\max_{t\in[0,1]} \left|\lambda_1 \int_0^1 \rho t^{\alpha_1-1}(p_1(s)h_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs \right. \\
&\left. +\lambda_2 \int_0^1 \rho t^{\alpha_1-1}(p_2(s)h_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs \right| \\
\leqslant &\rho\lambda_1 \int_0^1 \left(p_1(s)\left(\frac{r_1}{L_1}-1\right)+q_1(s)\right)d_qs \\
&+\rho\lambda_2 \int_0^1 \left(p_2(s)\left(\frac{r_1}{L_2}-1\right)+q_2(s)\right)d_qs \\
\leqslant &\frac{r_1}{L_1}\rho\lambda_1 \int_0^1 (p_1(s)+q_2(s))d_qs + \frac{r_1}{L_2}\rho\lambda_2 \int_0^1 (p_2(s)+q_2(s))d_qs \\
= &\frac{2r_1}{3} < r_1 = \|(u,v)\|.
\end{aligned}$$

Similarly, for any $(u,v) \in \partial P_{r_1}$, by condition (**H3**) and Lemma 2.5, we have $\|\mathscr{T}_2(u,v)\| < r_1 = \|(u,v)\|$. Consequently,

$$\|\mathscr{T}(u,v)\| = \max\{\|\mathscr{T}_1(u,v)\|,\|\mathscr{T}_2(u,v)\|\} < r_1 = \|(u,v)\|, \quad \forall (u,v) \in \partial P_{r_1}. \tag{8}$$

On the other hand, by the inequalities in (**H4**), there exists $\varepsilon_0 > 0$ such that $l_1 + \varepsilon_0 > 0$, and also there exists $r_0 > 0$ such that

$$\begin{aligned}
|f_1(t,u,v)| \geqslant &(l_1+\varepsilon_0)u, \quad u \geqslant r_0,\ v \geqslant 0,\ t \in [0,1]; \text{ or} \\
|f_1(t,u,v)| \geqslant &(l_1+\varepsilon_0)v, \quad u \geqslant 0,\ v \geqslant r_0,\ t \in [0,1].
\end{aligned} \tag{9}$$

Choose $r_2 = \max\{3r_1, 3r_0/(2\omega\gamma)\}$. For any $(u,v) \in \partial P_{r_2}$ and $t \in [0,1]$, by the definition of $\|\cdot\|$ and (3), we obtain

$$\begin{aligned}
x(t) \geqslant &\omega t^{\alpha_1-1} r_2 - \rho t^{\alpha_1-1}\left(\lambda_1 \int_0^1 q_1(s)d_qs + \lambda_2 \int_0^1 q_2(s)d_qs\right) \\
= &t^{\alpha_1-1}\left(\omega r_2 - \rho\left(\lambda_1 \int_0^1 q_1(s)d_qs + \lambda_2 \int_0^1 q_2(s)d_qs\right)\right) \\
\geqslant &\gamma\left(\omega r_2 - \rho\left(\lambda_1 \int_0^1 q_1(s)d_qs + \lambda_2 \int_0^1 q_2(s)d_qs\right)\right) \\
\geqslant &\omega\gamma(r_2 - r_1) \geqslant \frac{2\omega\gamma r_2}{3} \geqslant r_0,\ t \in [c,d]
\end{aligned} \tag{10}$$

and

$$\begin{aligned} y(t) \geqslant &\omega t^{\alpha_2-1} r_2 - \rho t^{\alpha_2-1}\left(\lambda_1 \int_0^1 q_1(s) d_q s + \lambda_2 \int_0^1 q_2(s) d_q s\right) \\ =&t^{\alpha_2-1}\left(\omega r_2 - \rho\left(\lambda_1 \int_0^1 q_1(s) d_q s + \lambda_2 \int_0^1 q_2(s) d_q s\right)\right) \\ \geqslant&\gamma\left(\omega r_2 - \rho\left(\lambda_1 \int_0^1 q_1(s) d_q s + \lambda_2 \int_0^1 q_2(s) d_q s\right)\right) \\ \geqslant&\omega\gamma(r_2 - r_1) \geqslant \frac{2\omega\gamma r_2}{3} \geqslant r_0, \quad t \in [c,d]. \end{aligned} \tag{11}$$

Thus, for any $(u,v) \in \partial P_{r_2}$ and $t \in [0,1]$, by (9)-(11), we have

$$f_1(s,[x(s)]^*,[y(s)]^*) \geqslant (l_1+\varepsilon_0)[y(s)]^* \text{ or } f_2(s,[x(s)]^*,[y(s)]^*) \geqslant (l_1+\varepsilon_0)[y(s)]^*, \tag{12}$$

where $t \in [c,d]$. Hence, for any $(u,v) \in \partial P_{r_2}$, by (12) and Lemma 2.4, we conclude that

$$\begin{aligned} \|\mathscr{T}_1(u,v)\| =& \max_{t\in[0,1]}\left|\lambda_1 \int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*) + q_1(s)) d_q s\right. \\ &\left. + \lambda_2 \int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*) + q_2(s)) d_q s\right| \\ \geqslant& \max_{t\in[0,1]} \lambda_1 \int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*) + q_1(s)) d_q s \\ \geqslant& \min_{t\in[c,d]} \lambda_1 \int_c^d \rho t^{\alpha_1-1}\varphi_1(qs) f_1(s,[x(s)]^*,[y(s)]^*) d_q s \\ \geqslant&\lambda_1\rho\gamma \int_c^d \varphi_1(qs)(l_1+\varepsilon_0)[x(s)]^* d_q s \\ \geqslant&\frac{2\lambda_1\rho\gamma^2(l_1+\varepsilon_0)\omega r_2}{3}\int_c^d \varphi_1(qs) d_q s \geqslant r_2 = \|(u,v)\| \end{aligned}$$

or

$$\begin{aligned} \|\mathscr{T}_2(u,v)\| \geqslant& \min_{t\in[c,d]} \lambda_1 \int_c^d \rho t^{\alpha_1-1}\varphi_1(qs) f_1(s,[x(s)]^*,[y(s)]^*) d_q s \\ \geqslant&\frac{2\lambda_1\rho\gamma^2(l_1+\varepsilon_0)\omega r_2}{3}\int_c^d \varphi_1(qs) d_q s \geqslant r_2 = \|(u,v)\|. \end{aligned}$$

Consequently, we have

$$\|\mathscr{T}(u,v)\| = \max\{\|\mathscr{T}_1(u,v)\|, \|\mathscr{T}_2(u,v)\|\} \geqslant r_2 = \|(u,v)\|, \ \ \forall (u,v) \in \partial P_{r_2}. \tag{13}$$

It follows from the above discussion, (8), (13), Lemmas 2.5 and 2.8 that for any fixed $\lambda_1, \lambda_2 \in (0,\infty)$, $\mathscr{T}$ has a fixed point $(u,v) \in P_{[r_1,r_2]}$ and $r_1 \leqslant \|(u,v)\| \leqslant r_2$. Since $\|(u,v)\| \geqslant r_1$, we get $u(t) - \overline{\omega}_1(t) \geqslant \bar{l}\,t^{\alpha_1-1}$ and $v(t) - \overline{\omega}_2(t) \geqslant \bar{l}\,t^{\alpha_2-1}$, $t \in (0,1]$, where $\bar{l} = \omega r_1 - \rho\left(\lambda_1 \int_0^1 q_1(s)d_qs + \lambda_2 \int_0^1 q_2(s)d_qs\right)$. Let $\bar{u}(t) = u(t) - \overline{\omega}_1(t)$ and $\bar{v}(t) = v(t) - \overline{\omega}_2(t)$, then we have $\bar{u}(t) \geqslant \bar{l}t^{\alpha_1-1}$ and $\bar{v}(t) \geqslant \bar{l}t^{\alpha_2-1}$, $t \in (0,1]$.

By Lemma 2.6, we know that for any fixed $\lambda_1, \lambda_2 \in (0,\infty)$, the singular coupled boundary value problem (1) and (2) has at least one positive solution $(\bar{u},\bar{v})$. Moreover, $(\bar{u},\bar{v})$ satisfies $\bar{u}(t) \geqslant \bar{l}t^{\alpha_1-1}$ and $\bar{v}(t) \geqslant \bar{l}t^{\alpha_2-1}$, $t \in [0,1]$. The proof is completed. □

REMARK 3.1. From the proof of Theorem 3.1, we know that the conclusion of Theorem 3.1 is valid if condition (**H4**) is replaced by

(**H4'**) $0 < l_2 \leqslant \liminf_{u\to+\infty} \inf_{t\in[c,d]\subset(0,1),\, v\in[0,\infty)} \frac{f_2(t,u,v)}{u} \leqslant \infty$ or $0 < l_2 \leqslant \liminf_{v\to+\infty} \inf_{t\in[c,d]\subset(0,1),\, u\in[0,\infty)} \frac{f_2(t,u,v)}{v} \leqslant \infty$, where $l_2 = \frac{3}{2}\left(\lambda_2\rho\omega\gamma^2 \int_c^d \varphi_2(qs)d_qs\right)^{-1}$.

THEOREM 3.2. *Assume that* (**H1**) *and* (**H2**) *and for any fixed* $\lambda_1, \lambda_2 \in (0,\infty)$, *the following conditions hold:*

(**H5**) *there exists a constant* $R_1 > \frac{\rho}{\omega}\left(\lambda_1 \int_0^1 q_1(s)d_qs + \lambda_2 \int_0^1 q_2(s)d_qs\right)$, *such that* $f_1(t,u,v) \geqslant R_1/l_1$, *for* $(t,u,v) \in [c,d]\times[0,R_1]^2$;

(**H6**) $0 \leqslant \limsup_{u\to+\infty} \sup_{t\in[0,1],\, v\in[0,\infty)} \frac{h_i(t,u,v)}{u} < L_i$ *or* $0 \leqslant \limsup_{v\to+\infty} \sup_{t\in[0,1],\, u\in[0,\infty)} \frac{h_i(t,u,v)}{v} < L_i$, $i = 1,2$,

where $[c,d] \subset (0,1)$, L_i $(i = 1,2)$ *and* l_1 *are defined as Theorem* 3.1. *Then the singular coupled boundary value problem* (1) *and* (2) *has at least one positive solution* $(\bar{u}^*,\bar{v}^*)$. *Moreover,* $(\bar{u}^*,\bar{v}^*)$ *satisfies* $\bar{u}^*(t) \geqslant \bar{l}^*t^{\alpha_1-1}$ *and* $\bar{v}^*(t) \geqslant \bar{l}^*t^{\alpha_2-1}$, $t \in [0,1]$, *for some positive constant* $\bar{l}^*$.

Proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1 and so we omit it. □

REMARK 3.2. The conclusion of Theorem 3.2 is valid if the condition (**H5**) is replaced by

(**H5'**) $f_2(t,u,v) \geqslant R_1/l_2$, for $(t,u,v) \in [c,d]\times[0,R_1]^2$, where R_1 is defined in Theorem 3.2 and l_2 is defined in Remark 3.1.

THEOREM 3.3. *Assume that* (**H1**) *and* (**H2**) *and the following condition* (**H7**) *hold:*

$$\lim_{u\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ v\in[0,\infty)}} \frac{f_1(t,u,v)}{u} = +\infty \quad or \quad \lim_{v\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ u\in[0,\infty)}} \frac{f_1(t,u,v)}{v} = +\infty.$$

Then there exist $\bar{\lambda}_1 > 0$ *and* $\bar{\lambda}_2 > 0$ *such that the singular coupled boundary value problem* (1) *and* (2) *has at least one positive solution* $(\bar{u}^\star,\bar{v}^\star)$, *provided that* $\lambda_1 \in (0,\bar{\lambda}_1)$ *and* $\lambda_2 \in (0,\bar{\lambda}_2)$. *Moreover,* $(\bar{u}^\star,\bar{v}^\star)$ *satisfies* $\bar{u}^\star(t) \geqslant \bar{l}^\star t^{\alpha_1-1}$ *and* $\bar{v}^\star(t) \geqslant \bar{l}^\star t^{\alpha_2-1}$, $t \in [0,1]$, *for some positive constant* $\bar{l}^\star$.

Proof. Let $S_{iR}=\sup\{h_i(t,u,v): t\in[0,1], u,v\in[0,R]\}$, $i=1,2$, where we choose

$$R>\frac{\rho}{\omega}\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right),\ \ \bar{\lambda}_i=\min\left\{1,\frac{R}{2\rho\int_0^1(p_i(s)S_{iR}+q_i(s))d_qs}\right\},$$

where $i=1,2$, ω and ρ are defined as Remark 2.1 and Lemma 2.5, respectively.

For any $(u,v)\in\partial P_R$ and $s\in[0,1]$, by the definition of $\|\cdot\|$, we know that

$$\begin{aligned}
[x(s)]^* =&[u(s)-\overline{\omega}_1(s)]^*\leqslant u(s)\leqslant\|u\|\leqslant\|(u,v)\|\leqslant R,\\
[y(s)]^* =&[v(s)-\overline{\omega}_2(s)]^*\leqslant v(s)\leqslant\|v\|\leqslant\|(u,v)\|\leqslant R,\ \ s\in[0,1].
\end{aligned}$$

So, for any $(u,v)\in\partial P_R$, $\lambda_1\in(0,\bar{\lambda}_1)$ and $\lambda_2\in(0,\bar{\lambda}_2)$, by Lemma 2.4, we get

$$\begin{aligned}
\|\mathscr{T}_1(u,v)\|=&\max_{t\in[0,1]}\left|\lambda_1\int_0^1 K_1(t,qs)(f_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\right.\\
&\left.+\lambda_2\int_0^1 H_1(t,qs)(f_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\right|\\
\leqslant&\max_{t\in[0,1]}\left|\lambda_1\int_0^1 \rho t^{\alpha_1-1}(p_1(s)h_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs\right.\\
&\left.+\lambda_2\int_0^1 \rho t^{\alpha_1-1}(p_2(s)h_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\right|\\
\leqslant&\rho\lambda_1\int_0^1(p_1(s)S_{1R}+q_1(s))\,d_qs+\rho\lambda_2\int_0^1(p_2(s)S_{2R}+q_2(s))\,d_qs\\
<&R=\|(u,v)\|.
\end{aligned}$$

Similarly, for any $(u,v)\in\partial P_R$, by Lemma 2.5, we also get $\|\mathscr{T}_2(u,v)\|<R=\|(u,v)\|$. Consequently, we have

$$\|\mathscr{T}(u,v)\|=\max\{\|\mathscr{T}_1(u,v)\|,\|\mathscr{T}_2(u,v)\|\}<R=\|(u,v)\|,\ \ \forall(u,v)\in\partial P_R. \tag{14}$$

On the other hand, by the condition (**H7**), choose M_1 such that

$$\lambda_1\rho\gamma^2M_1\omega\int_c^d\varphi_1(qs)d_qs>2\ \text{ or }\ \lambda_1\rho\gamma^2M_1\omega\int_c^d\varphi_2(qs)d_qs>2,$$

where $\gamma=\min\{c^{\alpha_1-1},c^{\alpha_2-1}\}$, ω and ρ are defined as Remark 2.1 and Lemma 2.5, respectively. Then there exists $N^\star>0$ such that

$$\begin{aligned}
f_1(t,u,v)\geqslant&M_1u,\ \ u\geqslant N^\star,\ \ v\geqslant 0,\ \ t\in[c,d]\ \text{ or}\\
f_1(t,u,v)\geqslant&M_1v,\ \ u\geqslant 0,\ \ v\geqslant N^\star,\ \ t\in[c,d].
\end{aligned}\tag{15}$$

Let $R'>\max\{2R,2N^\star/(\gamma\omega)\}$. For any $(u,v)\in\partial P_{R'}$ and $t\in[0,1]$, by the definition of $\|\cdot\|$ and (3), we obtain

$$\begin{aligned} x(t) \geqslant & \omega t^{\alpha_1-1}R' - \rho t^{\alpha_1-1}\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\\ =& t^{\alpha_1-1}\left(\omega R' - \rho\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\right)\\ \geqslant & \gamma\left(\omega R' - \rho\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\right)\\ \geqslant & \omega\gamma(R'-R) \geqslant \frac{2\omega\gamma R'}{3} \geqslant N^\star, \quad t\in[c,d] \end{aligned} \tag{16}$$

and

$$\begin{aligned} y(t) \geqslant & \omega t^{\alpha_2-1}R' - \rho t^{\alpha_2-1}\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\\ =& t^{\alpha_2-1}\left(\omega R' - \rho\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\right)\\ \geqslant & \gamma\left(\omega R' - \rho\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\right)\\ \geqslant & \omega\gamma(R'-R) \geqslant \frac{2\omega\gamma R'}{3} \geqslant N^\star, \quad t\in[c,d]. \end{aligned} \tag{17}$$

Thus, for any $(u,v)\in\partial P_{R'}$ and $t\in[0,1]$, by (15)-(17), we have

$$f_1(s,[x(s)]^*,[y(s)]^*) \geqslant M_1[x(s)]^* \ \text{ or } \ f_1(s,[x(s)]^*,[y(s)]^*) \geqslant M_1[y(s)]^*, \ \ t\in[c,d]. \tag{18}$$

Hence, for any $(u,v)\in\partial P_{R'}$, by (18) and Lemma 2.4, we conclude that

$$\begin{aligned} \|\mathscr{T}_1(u,v)\| \geqslant & \min_{t\in[c,d]} \lambda_1\int_c^d \rho t^{\alpha_1-1}\varphi_1(qs)f_1(s,[x(s)]^*,[y(s)]^*)d_qs\\ \geqslant & \frac{2\lambda_1\rho\gamma^2M_1\omega R'}{3}\int_c^d \varphi_1(qs)d_qs \geqslant R' = \|(u,v)\| \end{aligned}$$

or

$$\begin{aligned} \|\mathscr{T}_1(u,v)\| \geqslant & \min_{t\in[c,d]} \lambda_1\int_c^d \rho t^{\alpha_1-1}\varphi_1(qs)f_1(s,[x(s)]^*,[y(s)]^*)d_qs\\ \geqslant & \frac{2\lambda_1\rho\gamma^2M_1\omega R'}{3}\int_c^d \varphi_1(qs)d_qs \geqslant R' = \|(u,v)\|. \end{aligned}$$

Consequently, we have

$$\|\mathscr{T}(u,v)\| = \max\{\|\mathscr{T}_1(u,v)\|,\|\mathscr{T}_2(u,v)\|\} \geqslant R' = \|(u,v)\|, \quad \forall (u,v)\in\partial P_{R'}. \tag{19}$$

It follows from the above discussion, (14), (19), Lemmas 2.5 and 2.8 that for any $\lambda_1\in(0,\bar{\lambda}_1)$ and $\lambda_2\in(0,\bar{\lambda}_2)$, $\mathscr{T}$ has a fixed point $(u,v)\in P_{[R',R]}$ and $R\leqslant\|(u,v)\|\leqslant R'$.

Since $\|(u,v)\| \geqslant R$, we get $u(t)-\overline{\omega}_1(t) \geqslant \overline{l}^{\star} t^{\alpha_1-1}$ and $v(t)-\overline{\omega}_2(t) \geqslant \overline{l}^{\star} t^{\alpha_2-1}$, for $t \in (0,1]$, where $\overline{l}^{\star} = \omega R - \rho\left(\lambda_1 \int_0^1 q_1(s)d_q s + \lambda_2 \int_0^1 q_2(s)d_q s\right)$. Let $\bar{u}^{\star}(t) = u(t) - \overline{\omega}_1^{\star}(t)$ and $\bar{v}^{\star}(t) = v(t) - \overline{\omega}_2(t)$, then we have $\bar{u}^{\star}(t) \geqslant \overline{l}^{\star} t^{\alpha_1-1}$ and $\bar{v}^{\star}(t) \geqslant \overline{l}^{\star} t^{\alpha_2-1}$, $t \in (0,1]$.

By Lemma 2.6, we know that for any $\lambda_1 \in (0,\bar{\lambda}_1)$ and $\lambda_2 \in (0,\bar{\lambda}_2)$, the singular coupled boundary value problem (1) and (2) has at least one positive solution $(\bar{u}^{\star},\bar{v}^{\star})$, provided that $\lambda_1 \in (0,\bar{\lambda}_1)$ and $\lambda_2 \in (0,\bar{\lambda}_2)$. Moreover, $(\bar{u}^{\star},\bar{v}^{\star})$ satisfies $\bar{u}^{\star}(t) \geqslant \overline{l}^{\star} t^{\alpha_1-1}$ and $\bar{v}^{\star}(t) \geqslant \overline{l}^{*} t^{\alpha_2-1}$, $t \in [0,1]$. The proof is completed. □

REMARK 3.3. From the proof of Theorem 3.3, we know that the conclusion of Theorem 3.3 is valid if the condition (**H7**) is replaced by

(**H7'**) $\lim\limits_{u\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ v\in[0,\infty)}} \frac{f_2(t,u,v)}{u} = +\infty$ or $\lim\limits_{v\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ u\in[0,\infty)}} \frac{f_2(t,u,v)}{v} = +\infty$.

THEOREM 3.4. *Assume that* (**H1**) *and* (**H2**) *hold. And*

(**H8**) $\Lambda < \liminf\limits_{u\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ v\in[0,\infty)}} f_1(t,u,v) = +\infty$ *or* $\Lambda < \liminf\limits_{v\to+\infty} \inf_{\substack{t\in[c,d]\subset(0,1)\\ u\in[0,\infty)}} f_1(t,u,v) = +\infty$, *where* $\Lambda = 4\int_0^1 (q_1(s)+q_2(s))d_q s \Big/ \left(\gamma\omega^2 \max\left\{\int_c^d \varphi_1(qs)d_q s, \int_c^d \varphi_2(qs)d_q s\right\}\right)$ *and* $\gamma = \min\left\{c^{\alpha_1-1}, c^{\alpha_2-1}\right\}$.

(**H9**) $\limsup\limits_{u\to+\infty} \sup_{\substack{t\in[0,1]\\ v\in[0,\infty)}} \frac{h_i(t,u,v)}{u} = 0$ *or* $\limsup\limits_{v\to+\infty} \sup_{\substack{t\in[0,1]\\ v\in[0,\infty)}} \frac{h_i(t,u,v)}{v} = 0$, $i=1,2$.

Then there exist $\widetilde{\lambda}_1 > 0$ *and* $\widetilde{\lambda}_2 > 0$ *such that the singular coupled boundary value problem* (1) *and* (2) *has at least one positive solution* $(\widetilde{u},\widetilde{v})$, *provided that* $\lambda_1 \in (\widetilde{\lambda}_1,\infty)$ *and* $\lambda_2 \in (\widetilde{\lambda}_2,\infty)$. *Moreover,* $(\widetilde{u},\widetilde{v})$ *satisfies* $\widetilde{u}(t) \geqslant \widetilde{l}t^{\alpha_1-1}$ *and* $\widetilde{v}(t) \geqslant \widetilde{l}t^{\alpha_2-1}$, $t \in [0,1]$, *for some positive constant* $\widetilde{l}$.

Proof. It follows from (**H8**) that there exists $\widetilde{N} > 0$ such that

$$f_1(t,u,v) \geqslant \Lambda,\ \ u \geqslant \widetilde{N},\ v \geqslant 0 \ \text{ or } \ f_1(t,u,v) \geqslant \Lambda,\ \ u \geqslant 0,\ v \geqslant \widetilde{N}, \tag{20}$$

where $t \in [c,d]$. Put $\widetilde{\lambda}_i = \widetilde{N} \Big/ \left(2\rho\gamma \int_0^1 q_i(s)d_q s\right)$, $i=1,2$ and $R_1 = \max\left\{(\lambda_1+\lambda_2, 2\lambda_1, 2\lambda_2)\frac{2\rho}{\omega}\int_0^1 (q_1(s)+q_2(s))d_q s\right\}$. For any $(u,v) \in \partial P_{R_1}$ and $t \in [0,1]$, by the definition of $\|\cdot\|$ and (3), we obtain

$$\begin{aligned} x(t) \geqslant & \omega t^{\alpha_1-1} R_1 - \rho t^{\alpha_1-1}\left(\lambda_1 \int_0^1 q_1(s)d_q s + \lambda_2 \int_0^1 q_2(s)d_q s\right) \\ = & t^{\alpha_1-1}\left(\omega R_1 - \rho\left(\lambda_1 \int_0^1 q_1(s)d_q s + \lambda_2 \int_0^1 q_2(s)d_q s\right)\right) \\ \geqslant & \gamma\left(\omega R_1 - \rho\left(\lambda_1 \int_0^1 q_1(s)d_q s + \lambda_2 \int_0^1 q_2(s)d_q s\right)\right) \end{aligned}$$

$$\geqslant \rho\gamma\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\geqslant \tilde{N} \tag{21}$$

and

$$\begin{aligned} y(t) \geqslant& \omega t^{\alpha_2-1}R_1-\rho t^{\alpha_2-1}\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right) \\ =& t^{\alpha_2-1}\left(\omega R_1-\rho\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\right) \\ \geqslant& \gamma\left(\omega R_1-\rho\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\right) \\ \geqslant& \rho\gamma\left(\lambda_1\int_0^1 q_1(s)d_qs+\lambda_2\int_0^1 q_2(s)d_qs\right)\geqslant \tilde{N}, \end{aligned} \tag{22}$$

for $t\in[c,d]$. Thus, for any $(u,v)\in\partial P_{R_1}$ and $t\in[0,1]$, by (20)-(22), we have

$$f_1(s,[u(s)-\overline{\omega}_1(s)]^*,[v(s)-\overline{\omega}_2(s)]^*)\geqslant \Lambda, \quad t\in[c,d]. \tag{23}$$

Hence, for any $(u,v)\in\partial P_{R_1}$, by (23) and Lemma 2.4, we conclude that

$$\begin{aligned} \|\mathscr{T}_1(u,v)\| \geqslant& \min_{t\in[c,d]}\lambda_1\int_c^d \rho t^{\alpha_1-1}\varphi_1(qs)f_1(s,[x(s)]^*,[y(s)]^*)d_qs\geqslant \lambda_1\rho\gamma\Lambda\int_c^d \varphi_1(qs)d_qs \\ \geqslant& R_1=\|(u,v)\|. \end{aligned}$$

Consequently, we have

$$\|\mathscr{T}(u,v)\|=\max\{\|\mathscr{T}_1(u,v)\|,\|\mathscr{T}_2(u,v)\|\}\geqslant R_1=\|(u,v)\|, \quad \forall (u,v)\in\partial P_{R_1}. \tag{24}$$

On the other hand, choose $\varepsilon_i>0$ such that $\varepsilon_i=\left(3\lambda_i\rho\int_0^1 p_i(s)d_qs\right)^{-1}$, $i=1,2$. Then, for the above ε_i, by the first inequality in (**H9**), there exists $\widehat{N}>0$ such that for any $t\in[0,1]$ we have

$$h_i(t,u,v)\leqslant \varepsilon_i u,\ \ u\geqslant \widehat{N},\ v\geqslant 0 \ \text{ or } \ h_i(t,u,v)\leqslant \varepsilon_i v,\ \ u\geqslant 0,\ v\geqslant \widehat{N},$$

where $t\in[0,1]$, $i=1,2$. Then we have

$$\begin{aligned} h_i(t,u,v)\leqslant& \Phi+\varepsilon_i u, \quad u\geqslant \widehat{N},\ v\geqslant 0,\ t\in[0,1] \ \text{ or} \\ h_i(t,u,v)\leqslant& \Phi+\varepsilon_i v, \quad u\geqslant 0,\ v\geqslant \widehat{N},\ t\in[0,1],\ i=1,2, \end{aligned} \tag{25}$$

where $\Phi=\{h_i(t,u,v): 0\leqslant x\leqslant \widehat{N},\ 0\leqslant x\leqslant \widehat{N},\ t\in[0,1],\ i=1,2\}$. Choose

$$R_2=\max\left\{2R_1,\rho(\Phi+1)\left(\lambda_1\int_0^1(p_1(s)+q_1(s))d_qs+\lambda_2\int_0^1(p_2(s)+q_2(s))d_qs\right)\right\}.$$

For any $(u,v)\in\partial P_{R_2}$, by (25) and Lemma 2.5, we get

$$\begin{aligned}\|\mathscr{T}_1(u,v)\| \leqslant \max_{t\in[0,1]}\Big|\lambda_1\int_0^1 \rho t^{\alpha_1-1}(p_1(s)h_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs \\ +\lambda_2\int_0^1 \rho t^{\alpha_1-1}(p_2(s)h_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\Big| \\ \leqslant \rho\lambda_1\int_0^1 (p_1(s)(\Phi+\varepsilon_1[x(s)]^*)+q_1(s))\,d_qs \\ +\rho\lambda_2\int_0^1 (p_2(s)(\Phi+\varepsilon_2[x(s)]^*)+q_2(s))\,d_qs \\ \leqslant \rho(\Phi+1)\left(\lambda_1\int_0^1(p_1(s)+q_1(s))d_qs+\lambda_2\int_0^1(p_2(s)+q_2(s))d_qs\right) \\ +\rho\|u\|\left(\lambda_1\varepsilon_1\int_0^1 p_1(s)d_qs+\lambda_2\varepsilon_2\int_0^1 p_2(s)d_qs\right)\leqslant R_2=\|(u,v)\|\end{aligned}$$

or

$$\begin{aligned}\|\mathscr{T}_1(u,v)\| \leqslant \max_{t\in[0,1]}\Big|\lambda_1\int_0^1 \rho t^{\alpha_1-1}(p_1(s)h_1(s,[x(s)]^*,[y(s)]^*)+q_1(s))d_qs \\ +\lambda_2\int_0^1 \rho t^{\alpha_1-1}(p_2(s)h_2(s,[x(s)]^*,[y(s)]^*)+q_2(s))d_qs\Big| \\ \leqslant \rho\lambda_1\int_0^1 (p_1(s)(\Phi+\varepsilon_1[y(s)]^*)+q_1(s))\,d_qs \\ +\rho\lambda_2\int_0^1 (p_2(s)(\Phi+\varepsilon_2[y(s)]^*)+q_2(s))\,d_qs \\ \leqslant \rho(\Phi+1)\left(\lambda_1\int_0^1(p_1(s)+q_1(s))d_qs+\lambda_2\int_0^1(p_2(s)+q_2(s))d_qs\right) \\ +\rho\|v\|\left(\lambda_1\varepsilon_1\int_0^1 p_1(s)d_qs+\lambda_2\varepsilon_2\int_0^1 p_2(s)d_qs\right)\leqslant R_2=\|(u,v)\|.\end{aligned}$$

Similarly, for any $(u,v)\in\partial P_{R_2}$, by Lemma 2.5, we also get $\|\mathscr{T}_2(u,v)\|<R_2=\|(u,v)\|$. Consequently, we have

$$\|\mathscr{T}(u,v)\|=\max\{\|\mathscr{T}_1(u,v)\|,\|\mathscr{T}_2(u,v)\|\}<R_2=\|(u,v)\|,\ \ \forall(u,v)\in\partial P_{R_2}. \quad (26)$$

It follows from the above discussion, (24), (26), Lemmas 2.5 and 2.8 that for any $\lambda_1\in(\widetilde{\lambda}_1,\infty)$ and $\lambda_2\in(\widetilde{\lambda}_2,\infty)$, $\mathscr{T}$ has a fixed point $(u,v)\in P_{[R_1,R_2]}$ and $R_1\leqslant\|(u,v)\|\leqslant R_2$. Since $\|(u,v)\|\geqslant R_1$, by the same method as Theorem 3.3, we know that for any $\lambda_1\in(\widetilde{\lambda}_1,\infty)$ and $\lambda_2\in(\widetilde{\lambda}_2,\infty)$, the singular coupled boundary value problem (1) and (2) has at least one positive solution $(\widetilde{u},\widetilde{v})$, satisfies $\widetilde{u}(t)\geqslant\widetilde{l}t^{\alpha_1-1}$ and $\widetilde{v}(t)\geqslant\widetilde{l}t^{\alpha_2-1}$, $t\in[0,1]$. The proof is completed. □

REMARK 3.4. From the proof of Theorem 3.4, we know that the conclusion of Theorem 3.4 is valid if the condition **(H8)** is replaced by

(H8') $\Lambda < \liminf_{u\to+\infty} \inf_{t\in[c,d]\subset(0,1), v\in[0,\infty)} f_2(t,u,v) = +\infty$ or $\Lambda < \liminf_{v\to+\infty} \inf_{t\in[c,d]\subset(0,1), u\in[0,\infty)} f_2(t,u,v) = +\infty$, where Λ is defined in Theorem 3.4.

4. Two examples

EXAMPLE 4.1. Consider the following fractional q-difference system with coupled integral boundary conditions

$$\begin{aligned} &D_{0.5}^{2.5}u(t)+\lambda_1 f_1(t,u(t),v(t))=0,\ \ D_{0.5}^{2.5}v(t)+\lambda_2 f_2(t,u(t),v(t))=0, \quad t\in(0,1),\\ &D_{0.5}^{j_1}u(0)=D_{0.5}^{j_2}v(0)=0,\ \ 0\leqslant j_i\leqslant 1,\\ &u(1)=\frac{1}{2}\int_0^1\sqrt{s}v(s)d_qs,\ \ v(1)=\frac{1}{2}\int_0^1\frac{1}{\sqrt{s}}u(s)d_qs, \end{aligned} \tag{27}$$

where λ_1 and λ_2 are two parameters. We have $\nu_1=4/7$, $\nu_2=2/3$, $\kappa=19/21$, which implies (**H1**).

Let $p_1(t)=p_2(t)=1/\sqrt{t(1-t)}$, $q_1(t)=q_2(t)=-\ln t$, $h_1(t,u,v)=u^2+v^2$, $h_2(t,u,v)=1+e^u+e^v$, and

$$f_1(t,u,v)=\frac{u^2+v^2}{\sqrt{t(1-t)}}+\ln t,\ \ f_2(t,u,v)=\frac{1+e^u+e^v}{\sqrt{t(1-t)}}+\ln t,\ \ (t,u,v)\in(0,1)\times[0,+\infty)^2.$$

Then $-q_i(t)\leqslant f_i(t,u,v)\leqslant p_i(t)h_i(t,u,v)$, $(t,u,v)\in(0,1)\times[0,+\infty)^2$, $i=1,2$. By direct calculation, we obtain

$$\int_0^1 p_1(s)d_qs=\int_0^1 p_2(s)d_qs\approx 3.39926,\ \ \int_0^1 q_1(s)d_qs=\int_0^1 q_2(s)d_qs\approx 0.69315,$$

which implies (**H2**). On the other hand, choosing $[1/3,2/3]\subset[0,1]$, we can see that

$$\lim_{u\to+\infty}\inf_{\substack{t\in[1/3,2/3]\subset(0,1)\\ v\in[0,\infty)}}\frac{f_1(t,u,v)}{u}=+\infty \text{ or } \lim_{v\to+\infty}\inf_{\substack{t\in[1/3,2/3]\subset(0,1)\\ u\in[0,\infty)}}\frac{f_1(t,u,v)}{v}=+\infty.$$

So condition (**H7**) of Theorem 3.3 is satisfied. Therefore, by Theorem 3.3, the coupled system (27) has at least one positive solution, provided $\lambda_i>0$ $(i=1,2)$ is small enough.

EXAMPLE 4.2. Consider the fractional q-difference system (27), where

$$f_1(t,u,v)=\frac{\sqrt{2(u+v)}}{\sqrt[3]{t^2(1-t)}(1+t^2(t+1))}-\frac{2}{\sqrt{t}},\ \ f_2(t,u,v)=\frac{\sqrt{2}}{e^{u+v+1}\sqrt[3]{t^2(1-t)}}-\frac{2}{\sqrt{t}},$$

where $(t,u,v)\in(0,1)\times[0,+\infty)^2$. Let $p_1(t)=p_2(t)=\sqrt{2}/\sqrt[3]{t^2(1-t)}$, $q_1(t)=q_2(t)=2/\sqrt{t}$, $h_1(t,u,v)=\sqrt{u+v}/(1+t^2(t+1))$ and $h_2(t,u,v)=1/e^{u+v+1}$, then $-q_i(t)\leqslant f_i(t,u,v)\leqslant p_i(t)h_i(t,u,v)$, $(t,u,v)\in(0,1)\times[0,+\infty)^2$, $i=1,2$. By direct calculation, we obtain

$$\int_0^1 p_1(s)d_qs=\int_0^1 p_2(s)d_qs\approx 6.10606,\ \ \int_0^1 q_1(s)d_qs=\int_0^1 q_2(s)d_qs\approx 3.41530,$$

which implies (**H2**). On the other hand, choosing $[1/3,2/3]\subset[0,1]$, we can see that $\limsup_{u\to+\infty}\sup_{\substack{t\in[0,1]\\ v\in[0,\infty)}}(h_i(t,u,v)/u)=0,\ i=1,2$ and $\liminf_{u\to+\infty}\inf_{\substack{t\in[1/3,2/3]\subset(0,1)\\ v\in[0,\infty)}} f_1(t,u,v)=+\infty$ or $\liminf_{v\to+\infty}\inf_{\substack{t\in[1/3,2/3]\subset(0,1)\\ u\in[0,\infty)}} f_1(t,u,v)=+\infty$. So conditions (**H8**) and (**H9**) of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the coupled system in Example 4.2 has at least one positive solution, provided $\lambda_i>0\ (i=1,2)$ is large enough.

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